(1)(a)

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{|h|-|0|}{h} \\
&= \lim_{h\to 0^-} \frac{-h-0}{h} \\
&= \lim_{h\to 0^-} -1 = -1
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{|h|-|0|}{h} \\
&= \lim_{h\to 0^+} \frac{h-0}{h} \\
&= \lim_{h\to 0^+} 1 = 1
\end{aligned}$$

(1)(b)

Because the limit from below and limit from the right are not equal, the limit as h tends to zero of the difference quotient does not exist. Thus, by definition of a derivative at a point, $f'(0)$ does not exist.

(2)

These problems heavily use the following rules:

Constant rule:

(c is a constant, that is, it doesn't depend on x)

$$(cf)'(x) = cf'(x)$$

Summation rule:

$$(f+g)'(x) = f'(x) + g'(x)$$

Product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient rule:

(if $g(x) \neq 0$)

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

(2)(a)

First note that $\cos\left(\frac{\pi}{2}\right) = 0$.

$$
\begin{aligned}
a'(x) &= \frac{d}{dx}\left[cx^3\right] \\
&= c\frac{d}{dx}x^3 \\
&= c \cdot 3x^2 = 3cx^2
\end{aligned}
$$

(2)(b)

Note that the variable of integration is t. I also change the square root symbols into rational exponents to make use of the power rule.

$$
\frac{d}{dx}x^a = ax^{a-1}
$$

$$
\begin{aligned}
b'(t) &= \frac{d}{dt}\sqrt{t} + \frac{d}{dt}\frac{1}{\sqrt{t}} \\
&= \frac{d}{dt}t^{1/2} + \frac{d}{dt}t^{-1/2} \\
&= \frac{1}{2}t^{-1/2} - \frac{1}{2}t^{-3/2}
\end{aligned}
$$

(2)(c)

A method of organization is to use brackets around the quantity you want to differentiate, as I've shown below.

$$
\begin{aligned}
c'(x) &= \frac{(x^2-2)\frac{d}{dx}\left[x+1\right] - (x+1)\frac{d}{dx}\left[x^2-2\right]}{(x^2-2)^2} \\
&= \frac{(x^2-2) - (x+1)2x}{(x^2-2)^2}
\end{aligned}
$$

(2)(d)

$$
\begin{aligned}
d'(x) &= (x^2+1)\frac{d}{dx}\sqrt{x} + \sqrt{x}\frac{d}{dx}(x^2+1) \\
&= (x^2+1)\frac{d}{dx}x^{1/2} + \sqrt{x}(2x) \\
&= (x^2+1)\frac{1}{2}x^{-1/2} + 2x\sqrt{x}
\end{aligned}
$$

(2)(e)

$$
\begin{aligned}
e(s) &= \frac{\frac{d}{ds}\left[s^{1/2} - s^{-1/2}\right](s^2+1) - (s^{1/2} - s^{-1/2})\frac{d}{ds}(s^2+1)}{(s^2+1)^2} \\
&= \frac{\left(\frac{1}{2}s^{-1/2} + \frac{1}{2}s^{-3/2}\right)(s^2+1) - (s^{1/2} - s^{-1/2})2s}{(s^2+1)^2}
\end{aligned}
$$

(3)

The problem asks us to find points where the tangent line is horizontal. This is the same as asking where $\frac{dy}{dx} = 0$. Thus, let's compute the derivative of the equation $y = 3x^2 - 6x - 2$ and solve for when the derivative equals zero.

$$\frac{dy}{dx} = 6x - 6$$

$6x - 6 = 0$ when $x = 1$. Thus the point in the parable where the tangent line is horizontal is $(1, -5)$. Note that the problem asks for the point on the parabola, not for the point on the x-axis, for which I would answer $x = 1$.

(4)

A line that is tangent to $y = 4x - 2$ has a slope of 4. Thus this problem is similar to problem number 3, except we want to solve $\frac{dg}{dx} = 4$.

$$\frac{dg}{dx} = 2x^2 + 5x + 1$$

Then we have the equation

$$2x^2 + 5x + 1 = 4$$

This gives us

$$2x^2 + 5x - 3 = 0$$

We can apply the quadratic formula to obtain

$$x = \frac{-5 \pm \sqrt{25 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm 7}{4}$$

Thus $x = -3$ or $x = \frac{1}{2}$.

As such, we have $(-3, g(-3))$ and $(\frac{1}{2}, g(\frac{1}{2}))$ are two points that satisfy the conditions of the problem.

(5)

First note that $(0, 0)$ is not a point on the graph of $h(x)$. The simplest way to do this problem is to find the tangent line at any point (x_0, y_0) on the graph first. Such a point is of the form $(x_0, x_0^2 + 1)$. And the derivative at that point is $2x_0$. Thus the equation of the tangent line to that point is

$$y - (x_0^2 + 1) = 2x_0(x - x_0)$$

Now if the line passes through $(0, 0)$, then we have

$$0 - (x_0^2 + 1) = 2x_0(0 - x_0)$$

or more simply

$$-(x_0^2 + 1) = -2x_0^2$$

Solving we get that $x_0^2 = 1$. That is $x_0 = 1$ or $x_0 = -1$. Thus the points we are looking for are $(1, 2)$ and $(-1, 2)$.

Looking at the graph of $h(x)$, these answers make sense. It often helps to figure out ways to check if our answers make sense.

(6)

We can use some important facts in this problem:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\lim_{s\to 0} \frac{\sin(s)}{s} = 1$$

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \cos(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1} - \sin(x) \cdot 1 \\
&= \cos(x) \lim_{h\to 0} \frac{\cos^2(h) - 1}{h(1 + \cos(h))} - \sin(x) \\
&= \cos(x) \lim_{h\to 0} \frac{-\sin^2(h)}{h(1 + \cos(h))} - \sin(x) \\
&= \cos(x) \lim_{h\to 0} \frac{-\sin(h)}{(1 + \cos(h))} \lim_{h\to 0} \frac{\sin(h)}{h} - \sin(x) \\
&= \cos(x) \cdot 0 \cdot 1 - \sin(x) \\
&= -\sin(x)
\end{aligned}$$

(7)(a)

The ball hits the ground ($s = 0$) when $t = 5$, so to find H we have

$$0 = s(5) = H + 16(5) - 16(5)^2 = H - 320$$

Thus the building is 320 feet tall.

(7)(b)

Theoretically, we know we just need to calculate the derivative of the function and find where the derivative is equal to zero. But we should understand the physical interpretation for a physical problem. The derivative of the position of the ball gives us the velocity of the ball. Thus the ball reaches its maximum height when it stops moving up and starts moving down. There will be a short moment when the ball has zero velocity. That is why we want

$$\frac{ds}{dt}(t) = v(t) = 0$$

$$0 = \frac{ds}{dt}(t) = 16 - 32t$$

Thus the ball reaches its maximum height at half a second.
(7)(c)
This maximum height is

$$s(\frac{1}{2}) = 320 + 16(\frac{1}{2}) - 16\left(\frac{1}{2}\right)^2 = 320 + 8 - 4 = 324$$

(324 feet)
(7)(d)
The velocity of the ball at 5 seconds is

$$\frac{ds}{dt}(5) = 16 - 32(5) = -144$$

(144 feet per second downwards)

Note that I specified the ball is moving downwards. This comes from the negative sign. If the answer was positive, I'd conclude the ball was moving upwards. We should be careful to make the correct physical interpretations.

The accelaration, as reminded by the problem, is the derivative velocity, or the second derivative of position.

$$a(t) = \frac{dv}{dt}(t) = \frac{d^2s}{dt^2}(t) = -32$$

(32 feet per second per second downwards)

Acceleration in this problem is independent of time. The interpretation is that gravity, for this problem, is assumed to apply constant acceleration. Notice the units are in ft/s^2.

(8)

$$\begin{aligned} f'(x) &= 2\cos(x)\,[-\sin(x)] \\ &= -2\cos(x)\sin(x) \\ &= -\sin(2x) = 0 \end{aligned}$$

Then $2x = k\pi$ for any integer k.
$x = \frac{k\pi}{2}$ for any integer k.
But we also want that the solution satisfy $0 \leq x_0 \leq 2\pi$.
Thus x can belongs to the set

$$\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$$