Homework 4 Solutions

(1)

Let $f(x) = \cos(x) - \frac{2x}{\pi}$. f is a continuous function, $f(0) = \cos(0) - 0 = 1 > 0$, and $f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - 1 = -1 < 0$. By the intermediate value theorem, there exists some c in the interval $\left[0, \frac{\pi}{2}\right]$ for which $f(c) = 0$. Therefore, there exists some c in the interval $\left[0, \frac{\pi}{2}\right]$ for which $\cos(c) = \frac{2c}{\pi}$.

(2)

All polynomials are continuous functions. In particular, the polynomial $p(x) = x^4 - x^2 - 10x + 1$ is a continuous function. $p(0) = 1 > 0$ and $p(1) = 1 - 1 - 10 + 1 = -9 < 0$. By the intermediate value theorem, there exists some c in the interval $[0, 1]$ for which $p(c) = 0$. Therefore, p has at least one root (a number c for which $p(c) = 0$).

(3)

Let $w(x) = \sqrt{x+1}$. Then

$$\begin{aligned}
\lim_{h \to 0} \frac{w(h+1) - w(1)}{h} &= \lim_{h \to 0} \frac{\sqrt{(h+1)+1} - \sqrt{(1)+1}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{h+2} - \sqrt{2}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{h+2} + \sqrt{2}}{\sqrt{h+2} + \sqrt{2}} \\
&= \lim_{h \to 0} \frac{h + 2 - 2}{h\left[\sqrt{h+2} + \sqrt{2}\right]} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{h+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.
\end{aligned}$$

Note that a key step in computing the limit is to multiply and divide by the conjugate of $\sqrt{h+2} - \sqrt{2}$. Also note that the exercise amounts to computing $w'(1)$ using the limit definition of the derivative.

(4)(a)

Let $g(x) = x^2 + 1$. Then

$$\begin{aligned}
g'(1) &= \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} \\
&= \lim_{h \to 0} \frac{\left[(1+h)^2 + 1\right] - \left[(1)^2 + 1\right]}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 2h + 2 - 2}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 2h}{h} \\
&= \lim_{h \to 0} [h + 2] = 2.
\end{aligned}$$

(4)(b)

The slope of the tangent line at $(1,2)$ is given by the derivative at $x = 1$. Thus, $m = g'(1) = 2$. Using the point-slope formula

$$y - y_0 = m(x - x_0),$$

we determine that the equation of the tangent line to the graph of $g(x)$ at the point $(1,2)$ is

$$y - 2 = 2(x - 1).$$

(4)(More Generally)

$$\begin{aligned}
g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{\left[(x_0+h)^2 + 1\right] - \left[(x_0)^2 + 1\right]}{h} \\
&= \lim_{h\to 0} \frac{x_0^2 + 2x_0 h + h^2 + 1 - x_0^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{2x_0 h + h^2}{h} \\
&= \lim_{h\to 0} \left[2x_0 + h\right] = 2x_0.
\end{aligned}$$

The slope of the tangent line at $(x_0, g(x_0))$ is given by the derivative at $x = x_0$. Thus $m = g'(x_0) = 2x_0$. We determine that the equation of the tangent line to the graph of $g(x)$ at the point $(x_0, g(x_0))$ is

$$y - g(x_0) = 2x_0(x - x_0).$$

(5)(a)
Define

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

To show that f is continuous at $x = 0$, we should first compute the limit of f as x tends to 0. To do this, we will use the sandwich theorem (or squeeze theorem). We know that $\left|\sin\theta\right| \leq 1$, and so $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$. Then $\left|x\sin\left(\frac{1}{x}\right)\right| = \left|x\right|\left|\sin\left(\frac{1}{x}\right)\right| \leq \left|x\right| \cdot 1 = \left|x\right|$ for all $x \in \mathbb{R}$. Thus

$$-\left|x\right| \leq x\sin\left(\frac{1}{x}\right) \leq \left|x\right|.$$

Then since$\left|x\right| \to 0$ as $x \to 0$ and $-\left|x\right| \to 0$ as $x \to 0$, then $x\sin\left(\frac{1}{x}\right) \to 0$ as $x \to 0$ by the sandwich theorem. Therefore

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0 = f(0)$$

and by definition of being continuous at a point, we conclude f is continuous at $x = 0$.

(5)(b)

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h\sin\left(\frac{1}{h}\right)-0}{h} \\ &= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \end{aligned}$$

The limit of $\sin\left(\frac{1}{h}\right)$ as $h \to 0$ is the same as the limit of $\sin u$ as $u \to \infty$, which does not exist. We conclude f is not differentiable at $x = 0$.