Homework 3 Solutions

1. Determine if the following limits exist. When they do find their value

   (a) $\lim_{x\to\ 2}\ e^{x+2}$
   Solution:

$$\begin{aligned}\lim_{x\to\ 2}\ e^{x+2} &= e^{2+2}\\ &= e^4\end{aligned}$$

   (b) $\lim_{x\to\ 3^+}\frac{x^3-27}{x-3}$
   Solution:

$$\begin{aligned}\lim_{x\to\ 3^+}\frac{x^3-27}{x-3} &= \lim_{x\to\ 3^+}\frac{(x-3)(x^2+3x+9)}{x-3}\\ &= \lim_{x\to\ 3^+}(x^2+3x+9)\\ &= 27\end{aligned}$$

   (c) $\lim_{x\to\ 0}\frac{e^x+1}{3x+1}$
   Solution:

$$\begin{aligned}\lim_{x\to\ 0}\frac{e^x+1}{3x+1} &= \frac{e^{(0)}+1}{3(0)+1}\\ &= \frac{1+1}{1} = 2\end{aligned}$$

   (d) $\lim_{x\to\ 2^+}\frac{1}{x-2}$
   Solution:

$$\lim_{x\to\ 2^+}\frac{1}{x-2} = \infty$$

   As x approaches 2 from the right, the limit blows up to arbitrarily large numbers for $\frac{1}{x-2}$. So the limit does not exist!

   (e) $\lim_{x\to\ 0}\frac{\sqrt{x+1}-1}{x}$
   Solution:

$$\begin{aligned}\lim_{x\to\ 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to\ 0}\frac{\sqrt{x+1}-1}{x}\times\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\\ &= \lim_{x\to\ 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to\ 0}\frac{x}{x(\sqrt{x+1}+1)}\\ &= \lim_{x\to\ 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{2}\end{aligned}$$

2. Consider the function $f(x)$ defined below

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x > 1 \\ 0 & \text{if } x = 0 \\ x^3 - x & \text{if } x < 0 \end{cases}$$

Determine if the following quantities exist. In case they exist, find their values.

(a) $\lim_{x\to\ 1^+} f(x)$.

Solution:

$$\lim_{x\to\ 1^+} f(x) = \lim_{x\to\ 1^+} \frac{1}{x^2} = 1$$

(b) $\lim_{x\to\ 1^-} f(x)$.

Solution:

$$\lim_{x\to\ 1^-} f(x) = \lim_{x\to\ 1^-} (x^3 - x) = (1)^3 - 1 = 0$$

(c) $\lim_{x\to\ 1} f(x)$.

Solution: From (a) and (b), it is clear that $\lim_{x\to\ 1^-} f(x) \neq \lim_{x\to\ 1^+} f(x)$. Therefore, because the one-sided limits do not coincide, $\lim_{x\to\ 1} f(x)$ does not exist.

3. Determine if the function $g(x)$ defined below is continuous at $x = -4$.

$$g(x) = \begin{cases} \frac{x^2-16}{x+4} & \text{if } x \neq -4 \\ -8 & \text{if } x = -4 \end{cases}$$

Solution:

(i) First, we determine whether $g(-4)$ exists. $g(-4) = -8$. Thus, $g(x)$ is defined at $x = -4$.
(ii) Next, we determine whether $\lim_{x\to\ -4} g(x)$ exists.

$$\begin{aligned} \lim_{x\to\ -4} g(x) &= \lim_{x\to\ -4} \frac{x^2-16}{x+4} \\ &= \lim_{x\to\ -4} \frac{(x+4)(x-4)}{x+4} \\ &= \lim_{x\to\ -4} (x-4) = -8 \end{aligned}$$

(iii) Last, we determine whether the limit of g as x approaches -4 is equal to the value of g at $x = -4$. From (i) and (ii), it is easy to see that:

$$\lim_{x\to\ -4} g(x) = -8 = g(-4)$$

. Therefore, the function $g(x)$ is continuous at $x = -4$.

4. The function $h(x)$ is defined as follows

$$h(x) = \begin{cases} \frac{\sqrt{x}+2}{x+1} & \text{if } x > 0 \\ -2 & \text{if } x = 0 \\ e^{x+c} & \text{if } x < 0 \end{cases}$$

(a) Find the value of c such that $\lim_{x\to\ 0} h(x)$ exists. Please explain your answer.

Solution: In order for the limit to exist, the one-sided limits must be equal.

$$\lim_{x\to\ 0^+} h(x) = \lim_{x\to\ 0^+} \frac{\sqrt{x}+2}{x+1} = \frac{\sqrt{0}+2}{(0)+1} = 2$$

$$\lim_{x\to\ 0^-} h(x) = \lim_{x\to\ 0^-} e^{x+c} = e^{(0)+c} = e^c.$$

In order for the limit to exist, $\lim_{x\to\ 0^-} h(x) = \lim_{x\to\ 0^+} h(x)$.

$$\implies e^c = 2 \implies c = ln(2)$$

(b) Suppose now that c is the number you found in part (a). Is the function $h(x)$ continuous at $x = 0$. Please explain your answer.

Solution:

(i) First, we determine whether $h(0)$ exists. $h(0) = -2$. Thus, $h(x)$ is defined at $x = 0$.

(ii) Next, we determine whether $\lim_{x\to\ -4} g(x)$ exists. From (a), we know

$$\lim_{x\to\ 0} h(x) = \lim_{x\to\ 0^-} h(x) = \lim_{x\to\ 0^+} h(x) = 2$$

(iii) Last, we determine whether the limit of g as x approaches -4 is equal to the value of g at $x = -4$. From (i) and (ii), it is easy to see that:

$$\lim_{x\to\ 0} h(x) = 2 \neq -2 = h(0) \implies \lim_{x\to\ 0} h(x) \neq h(0)$$

Therefore, the function $h(x)$ is not continuous at $x = 0$.