Homework 2 Solutions

1. A certain strain of bacteria that reproduces asexually triples its size every 45 days. If after 180 days there are 1620 bacteria, how many bacteria were there originally?

Solution: Let N_0 be the number of bacteria there are initially, let t denote days, in multiples of 45 (i.e. when $t = 1$, 45 days have passed, when $t = 2$, 90 days have passed, when $t = 2.5$, $2.5 \cdot 45$ days have passed, etc.), and let $N(t)$ denote the number of bacteria at time t. Then we have that $N(0) = N_0$, $N(1) = 3 \cdot N_0$, $N(2) = 3^2 \cdot N_0$ and in general $N(t) = 3^t \cdot N_0$. We are given that after 180 days, there are 1620 bacteria. Since $180 = 45 * 4$, this is equivalent to the equality $N(4) = 1620$. That is,

$$3^4 N_0 = N(4) = 1620$$

Thus, $N_0 = 1620/3^4 = 1620/81 = 20$, so there were 20 bacteria originally.

2. A picture supposedly painted by Vermeer (1632-1675) contains 99.5% of its carbon-14. It is known that carbon-14 has a half life of 5730 years. From this information, can you decide whether or not the picture is a fake? Please explain your answer.

Solution: Let t denote time in years. Then $W(t) = W_0 e^{-\lambda t}$ denotes the amount of carbon-14 there is after t years, where W_0 is the amount of carbon-14 there was initially and λ is some constant. Let's start by figuring out what λ is. The half life is the time t at which the amount of carbon that there is, $W(t)$, is half of what there was initially, $W_0/2$. Since we are given that the half-life is 5730 years, we get the equation

$$W(5730) = W_0/2$$

That is,

$$W_0 e^{-\lambda \cdot 5730} = W_0/2$$

The factor of W_0 cancels, and we get

$$e^{-5730\lambda} = \frac{1}{2}$$

Thus,

$$\lambda = \frac{\ln 2}{5730}$$

We need to figure out how long ago the picture was painted. If t denotes the present time (i.e. the number of years since the picture was painted), then the fact that the picture contains 99.5% of its carbon-14 means that $W(t) = 0.995 \cdot W_0$. Let's solve this equation for t (we will plug in for λ only at the end to keep things from getting messy)

$$\begin{aligned}
W_0 e^{-\lambda t} &= 0.995 W_0 \\
e^{-\lambda t} &= 0.995 \\
-\lambda t &= \ln(0.995) \\
t &= -\frac{\ln(0.995)}{\lambda} \\
t &= -\frac{5730 \cdot \ln(0.995)}{\ln(2)}
\end{aligned}$$

Plugging this value of t into a calculator, we see that $t \approx 41$. That is, it has been about 41 years since the picture was painted. Since Vermeer was not alive 41 years ago, he did not paint the picture. So the picture is a fake.

3. Determine the values of the sequence $\{a_n\}$ for $n = 0, 1, 2, 3, 4$ in the following sequences:

   (a)
$$a_n = e^{\cos(\pi n)}$$

   Solution: $a_0 = e^{\cos(0)} = e^1 = e$, $a_1 = e^{\cos(\pi)} = e^{-1}$, $a_2 = e^{\cos(2\pi)} = e$,
   $a_3 = e^{\cos(3\pi)} = e^{-1}$, $a_4 = e^{\cos(4\pi)} = e$

   (b)
$$a_n = \frac{n^2}{n+1}$$

   Solution: $a_0 = \frac{0^2}{0+1} = 0$, $a_1 = \frac{1^2}{1+1} = \frac{1}{2}$, $a_2 = \frac{2^2}{2+1} = \frac{4}{3}$
   $a_3 = \frac{3^2}{3+1} = \frac{9}{4}$, $a_4 = \frac{4^2}{4+1} = \frac{16}{5}$

   (c)
$$a_n = \sqrt{n+4}$$

   Solution: $a_0 = \sqrt{0+4} = 2$, $a_1 = \sqrt{1+4} = \sqrt{5}$, $a_2 = \sqrt{6}$,
   $a_3 = \sqrt{7}$, $a_4 = \sqrt{8} = 2\sqrt{2}$

   (d)
$$a_n = \sin\left(\frac{\pi n}{2}\right)$$

   Solution: $a_0 = \sin(0) = 0$, $a_1 = \sin(\pi/2) = 1$, $a_2 = \sin(\pi) = 0$,
   $a_3 = \sin(3\pi/2) = -1$, $a_4 = \sin(2\pi) = 0$

   (e) The recursively defined sequence with $a_0 = 128$ and $a_{n+1} = \sqrt{a_n}$.

   Solution: $a_0 = 128$, $a_1 = \sqrt{a_0} = \sqrt{128} = 8\sqrt{2} = 2^{\frac{7}{2}}$, $a_2 = \sqrt{a_1} = \sqrt{8\sqrt{2}} = 2\sqrt{2\sqrt{2}} = 2\sqrt{2}\sqrt[4]{2} = 2^{\frac{7}{4}}$,
   $a_3 = \sqrt{a_2} = \sqrt{2\sqrt{2}\sqrt[4]{2}} = \sqrt{2}\sqrt[4]{2}\sqrt[8]{2} = 2^{\frac{7}{8}}$, $a_4 = \sqrt{a_3} = \sqrt[4]{2}\sqrt[8]{2}\sqrt[16]{2} = 2^{\frac{7}{16}}$

4. Determine if the following sequences converge or diverge. In case they converge, find their limit.

(a)
$$a_n = \sin(\pi n)$$

Solution: Notice that $\sin(\pi n) = 0$ for any integer n. Thus, for any n, $a_n = 0$. Since $\lim_{n\to\infty} 0 = 0$, we have that the sequence $\{a_n\}$ converges to 0.

(b)
$$b_n = \sin(2\pi n)$$

Solution: Since $\sin(2\pi n)$ is also equal to zero for any integer n, we have that $b_n = 0$ for all n. As above, the sequence thus converges to zero.

(c)
$$c_n = e^{-2n}$$

Solution: First, rewrite $c_n = e^{-2n} = (e^{-2})^n$. Notice that $0 < e^{-2} < 1$. From Example 12 in section 2.2 of the textbook in the case that $a_0 = 1$ and $R = e^{-2}$, we see that the sequence c_n converges to zero. Intuitively, this is true because $c_n = 1/e^{2n}$ and e^{2n} grows without bound while the numerator stays constant, so we can make c_n as small as we want by choosing n large enough.

(d)
$$d_n = \frac{n+1}{n^2}$$

Solution: We have
$$d_n = \frac{n+1}{n^2} = \frac{n}{n^2} + \frac{1}{n^2} = \frac{1}{n} + \frac{1}{n^2}$$
Since $\lim_{n\to\infty} \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n^2}$ both exist, we may distribute the limit over the sum, i.e.

$$\begin{aligned}\lim_{n\to\infty} d_n &= \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n^2}\right)\\ &= \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n^2} = 0 + 0 = 0\end{aligned}$$

so the sequence converges to zero.

(e)
$$e_n = \frac{n^2+1}{n}$$

Solution: We have $e_n = \frac{n^2+1}{n} = n + \frac{1}{n} > n$. Since n increases without bound, so does e_n. Thus, $\lim_{n\to\infty} e_n$ does not exist. Intuitively, what this means is that since e_n is always bounded below by a sequence that gets arbitrarily large, e_n must get arbitrarily large as well. Thus, e_n cannot converge to any specific number.