

Homework 1 Solutions

1. The functions f and g are defined below

$$f(x) = \sqrt{x^2 - 1}$$
$$g(x) = \frac{1}{x}$$

Find explicit descriptions and the domains of the following functions.

- (a) $f \circ g$

Solution:

$$f \circ g(x) = f(g(x)) = f(1/x) = \sqrt{\frac{1}{x^2} - 1}$$

As for the domain of $f \circ g$, first notice that the domain of the square root is all nonnegative numbers, so we must have

$$\frac{1}{x^2} - 1 \geq 0$$

This implies that $x^2 \leq 1$, so we must have that $-1 \leq x \leq 1$. We also cannot have $x = 0$ since 0 is not in the domain of g and hence not in the domain of $f \circ g$. Thus, the domain of $f \circ g$ is all nonzero x with $-1 \leq x \leq 1$.

- (b) $g \circ f$

Solution:

$$g \circ f(x) = g(f(x)) = g(\sqrt{x^2 - 1}) = \frac{1}{\sqrt{x^2 - 1}}$$

The square root in the denominator is defined whenever $x^2 - 1 \geq 0$ or equivalently, when $x \leq -1$ or $x \geq 1$. However, if $x^2 - 1 = 0$, then the denominator will be zero and $g \circ f$ will still be undefined. So we cannot have $x^2 - 1 = 0$, or equivalently, we cannot have $x = \pm 1$. Thus, the domain of $g \circ f$ is all real numbers x with $x < -1$ or $x > 1$.

- (c) $f \circ f$

Solution:

$$f \circ f(x) = f(f(x)) = f(\sqrt{x^2 - 1}) = \sqrt{(\sqrt{x^2 - 1})^2 - 1}$$

The first of the two nested square roots is defined whenever $x^2 - 1 \geq 0$, or equivalently when $-1 \leq x \leq 1$. The second of the two nested square roots is defined whenever $(\sqrt{x^2 - 1})^2 - 1 \geq 0$. That is, we must have $x^2 - 1 \geq 1$ or equivalently, $x^2 \geq 2$. So the outer square root is defined whenever $x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$. Thus, the domain of $f \circ f$ is all real numbers **not** in the interval $(-\sqrt{2}, \sqrt{2})$.

- (d) $g \circ g$

Solution:

$$g \circ g(x) = g\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}}$$

The only restriction on the domain of $g \circ g$ is that x cannot be zero. Thus, the domain is all nonzero real numbers. Note that $g \circ g$ is NOT the same function as $h(x) = x$ since $g \circ g$ is not defined when $x = 0$ whereas $h(x)$ is. They do, however, have the same value at all nonzero real numbers.

2. Consider the functions

$$p(x) = \frac{x^3 - x}{x + 1}$$
$$q(x) = x^2 - x$$

(a) Find the domains of $p(x)$ and $q(x)$.

Solution: $p(x)$ is a rational function, so its domain is all real numbers except those for which its denominator is zero. That is, the domain of $p(x)$ is all real numbers not equal to -1 . $q(x)$ is a polynomial function, and so its domain is all real numbers.

(b) Show that $p(x) = q(x)$ for all $x \neq -1$.

Solution: When $x \neq -1$, we have

$$p(x) = \frac{x^3 - x}{x + 1} = \frac{x(x^2 - 1)}{x + 1} = \frac{x(x - 1)(x + 1)}{x + 1} = \frac{x(x - 1)}{1} = x^2 - x = q(x)$$

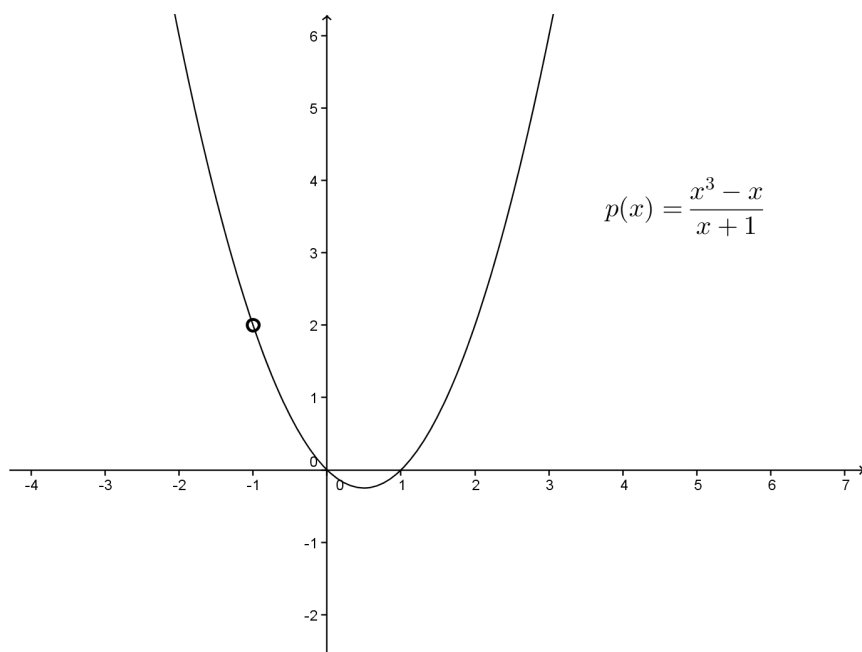
Note that when $x = -1$, we cannot speak of equality of $p(x)$ and $q(x)$ since the former is not defined when $x = -1$.

(c) Are $p(x)$ and $q(x)$ the same functions? Please explain your answer.

Solution: No, they are not. From the definition in the textbook, we have that two functions are the same if and only if they have the same domains and they have the same value at each number in their domains. Since $p(x)$ and $q(x)$ have different domains, they are not the same.

(d) Provide a sketch for the graph of $p(x)$.

Solution: Since $p(x)$ agrees with $q(x)$ for all $x \neq -1$, at all such x the graph of $p(x)$ is just the graph of the parabola $x^2 - x$. When $x = -1$, there is a hole in the graph since $p(x)$ is not defined there.



3. Let h be the function defined by

$$h(x) = \frac{x-1}{x+1}$$

(a) Show that h is a one to one function.

Solution: By definition, h is one to one if and only if whenever we have $h(x) = h(y)$, we must also have $x = y$. (In other words, whenever h sends two numbers to the same number, the two numbers must have been the same in the first place, or equivalently, each value of h is unique.) So, suppose that $h(x) = h(y)$. That is,

$$\frac{x-1}{x+1} = \frac{y-1}{y+1}$$

Cross multiplying, we have

$$(x-1)(y+1) = (y-1)(x+1)$$

So

$$\begin{aligned} xy + x - y - 1 &= xy - x + y - 1 \\ x - y &= -x + y \\ 2x &= 2y \end{aligned}$$

Thus, $x = y$. Since we showed this for ALL x and y (we didn't assume anything about them), this proves that each value of $h(x)$ is unique. Thus, h is one to one.

(b) Find the inverse function of h .

Solution: We can find the inverse function of h by writing $y = h(x)$, interchanging y and x , and then solving for y in terms of x . So we have

$$\begin{aligned} x &= \frac{y-1}{y+1} \\ x(y+1) &= y-1 \\ xy + x &= y-1 \\ xy - y &= -x-1 \\ y(x-1) &= -(x+1) \\ y &= -\frac{x+1}{x-1} \end{aligned}$$

Thus,

$$h^{-1}(x) = -\frac{x+1}{x-1}$$

4. Solve the following equations:

(a)

$$\ln(x^2 - 1) - \ln(x + 1) = 1$$

Solution: We first rewrite the left hand side as

$$\ln(x^2 - 1) - \ln(x + 1) = \ln\left(\frac{x^2 - 1}{x + 1}\right)$$

Now, whenever $x \neq -1$, we have

$$\ln\left(\frac{x^2 - 1}{x + 1}\right) = \ln(x - 1)$$

and our equation is

$$\ln(x - 1) = 1$$

Raising e to the power of both sides, we get (whenever $x - 1$ is nonnegative)

$$x - 1 = e^{\ln(x-1)} = e^1 = e$$

Thus, $x = e + 1$. Since $e + 1$ is in the domain of all of the functions involved in the original equation, it is a solution to the equation. When $x = -1$, the $\ln(x + 1)$ term is not defined and the equation does not make sense. So the only solution is $x = e + 1$.

(b)

$$e^{\ln(x+1) - \ln(x-1)} = x$$

Solution: We begin by simplifying the left hand side of the equation. Whenever $x + 1$ and $x - 1$ are nonnegative, we have

$$e^{\ln(x+1) - \ln(x-1)} = e^{\ln\left(\frac{x+1}{x-1}\right)} = \frac{x+1}{x-1}$$

Thus, for such x we solve the equation

$$\frac{x+1}{x-1} = x$$

We get

$$\begin{aligned} x + 1 &= x(x - 1) \\ x + 1 &= x^2 - x \\ 0 &= x^2 - 2x - 1 \end{aligned}$$

From the quadratic formula, we see that there are two real solutions to this equation: $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$. When $x + 1$ or $x - 1$ are negative, the left hand side of the original equation is not defined. This happens when $x = 1 - \sqrt{2}$, so the only real solution to the original equation is $x = 1 + \sqrt{2}$.

5. Let a be the function defined by $a(x) = \cos(\frac{\pi x}{4})$. Assume also that b is a one to one function such that

$$b(0) = -2, \quad b(1) = 0, \quad b(2) = 4, \quad b(3) = 5$$

and let c be the function whose graph may be found in the homework assignment. Find the following quantities:

(a) $a \circ b^{-1}(0)$

Solution: Since b is one to one, it has an inverse. Since $b(1) = 0$, we have $b^{-1}(0) = 1$ and $a \circ b^{-1}(0) = a(1) = \cos(\pi/4) = \sqrt{2}/2$.

(b) $b \circ c(2)$.

Solution: From the graph of $c(x)$ we see that $c(2) = 0$. Thus, $b \circ c(2) = b(0) = -2$.

(c) $c \circ a(4)$

Solution: We have $a(4) = \cos(\frac{4\pi}{4}) = \cos(\pi) = -1$. From the graph of c , we see that $c(-1) = 0$.

6. Radioactive isotopes are used in real life to determine the absolute age of fossils. The amount of a given radioactive isotope $W(t)$ present in a fossil after t years is given by the equation

$$W(t) = W_0 e^{-\lambda t}$$

In this case $\lambda = 0.0023104$ and W_0 is the initial quantity of the radioactive isotope. Find the half life of this isotope.

Solution: The half life is the time t at which there is half of the amount that there was initially. This is represented by the equation

$$W(t) = \frac{W(0)}{2} = \frac{W_0}{2}$$

We solve the equation (notice how it doesn't matter what W_0 is since it cancels!)

$$\begin{aligned} W_0 e^{-\lambda t} &= \frac{W_0}{2} \\ e^{-\lambda t} &= \frac{1}{2} \\ -\lambda t &= \ln\left(\frac{1}{2}\right) \\ t &= -\frac{\ln\left(\frac{1}{2}\right)}{\lambda} \end{aligned}$$

This is the exact value of the half life (note that since $\ln \frac{1}{2}$ is negative and we have a minus sign up front, the answer is actually positive, as the half life should be). A decimal approximation is given by $t \approx 300.012$.