Time limit: 105 minutes.
Maximum score: 200 points.
Instructions: For this test, you work in teams of four to solve a multi-part, proof-oriented question.

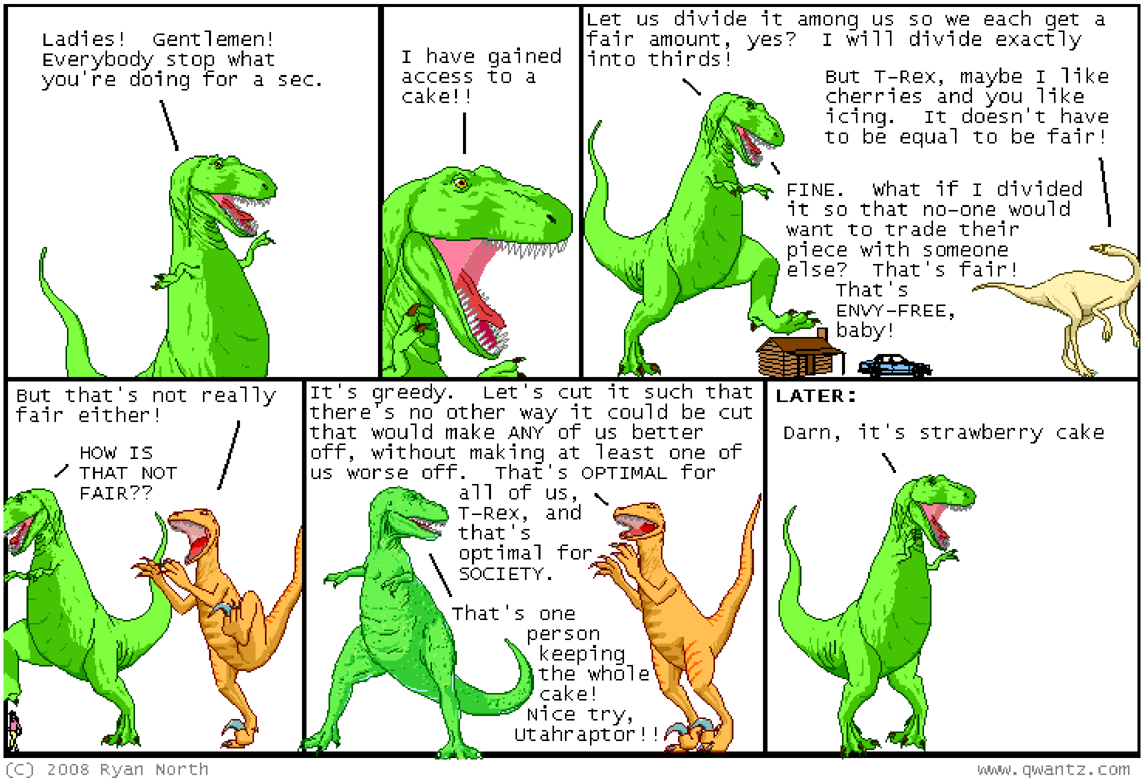
Problems that use the words "compute" or "list" only call for an answer; no explanation or proof is needed. Answers for these problems, unless otherwise stated, should go on the provided answer sheet. Unless otherwise stated, all other questions require explanation or proof. Answers for these problems should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Place a team ID sticker on every submitted page. If you do not have your stickers, you should write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them.

The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.

No calculators.



Introduction

Fair division is the process of dividing a set of goods among several people in a way that is "fair". However, as alluded to in the comic above, what exactly we mean by fairness is deceptively complex. We'll explore different notions of fairness in depth throughout this Power Round.

We'll begin with the canonical motivating example, called the *Cake-Cutting Problem*. Suppose you and your friend wish to split a cake. If the cake is homogeneous (all the same), then it is clearly most fair to split the cake in half, so you each receive an equal share. However, suppose half of the cake contains cherries, while the other half does not, and additionally suppose that you really like cherries, while your friend does not. In this case, splitting the cake equally is no longer the obviously best solution. Since you prefer cherry cake to regular cake while your friend does not, splitting the cake so you get the cherry half and your friend gets the rest is intuitively better than splitting both halves equally. These notions will be formalized shortly.

Definitions

All fair division problems share a few common features. First, they contain a set of n **players** which we will number Player 1 to Player n, and a finite set of **goods**, G, that we wish to divide among the players.

The elements of G may be one of three different types. Some goods, such as a car or a dog, are **indivisible**, which means they can only be assigned to a single player. Other goods, like money or cake are **divisible**, which means they can be divided among multiple players. We further distinguish between **homogeneous** divisible goods like money whose parts are indistinguishable from each other, and **heterogeneous** goods like cake, whose parts may not all be equivalent. In the example above, the half-cherry cake is heterogeneous and divisible, because it can be split among players, but its parts are distinguishable, since some will have more cherries than others.

Define a **division** to be an allocation of goods to players.

We can denote divisions by listing the goods assigned to each player. For example, one possible division of $G = \{A, B, C\}$ among two players is $(1 : (A, B), 2 : (C))$, which denotes that Player 1 receives A and B, while Player 2 receives C. If the goods are divisible and homogeneous, we can add a fraction to the good to denote the portion received by each player. For example, $(1 : (A, B, \frac{1}{3}C), 2 : (\frac{2}{3}C))$ denotes that Player 1 receives A, B, and one third of C, while Player 2 receives the remaining two thirds of C. Note that this notation isn't well-defined for heterogeneous goods, since if A is heterogeneous then "half of A" could mean any half, and they are not all necessarily equivalent.

Next, to model players' preferences, we assume each player has a **value function** that describes how much they value the goods in G. Define v_i to be player i's value function, and let $v_i(A)$ be the value player i places on the goods in set A ($A \subseteq G$). Value functions have the following properties:

i. Values placed on goods by players are always nonnegative real numbers.

ii. For each player i, $v_i(G) = 1$ (that is, all players place the same total value on all of G).

iii. Players' values are **additive**. That is, if $v_i(A) = x$ and $v_i(B) = y$, then $v_i(A \uplus B) = x + y$ for any sets of goods A and B.[1]

iv. Divisible, *homogeneous* goods are valued linearly. For example, if $v_i(g) = x$, and g is homogeneous, then player i would assign value $\frac{x}{2}$ to any half of g, $\frac{x}{5}$ to any fifth of g, etc. Note that this is *not* necessarily the case for heterogeneous goods, for example we've seen that not all cuts of a cake into halves will be valued equally by all players.

Hence, all players assign the same total value to all of G, and a player will never prefer *not* receiving a good g to receiving it. Additionally, the value function of any player can be enumerated by listing the value they place on each element of G. Note, however, that value functions can't necessarily be enumerated at all for all possible divisions of a heterogeneous good, since there may be uncountably many distinguishable divisions.

Define the **total value** that a player receives in a particular division to be the sum of values that player places on all goods they receive. For example, if $v_1(g_a) = 0.1$, $v_1(g_b) = 0.2$, and some division assigns goods g_a and g_b to Player 1, then the total value Player 1 receives from this division will be $v_1(g_a) + v_1(g_b) = 0.1 + 0.2 = 0.3$. Note that total values depend on each player's *own* value functions, so the total value Player 2 receives from g_a and g_b is not necessarily the same as the total value Player 1 receives.

Finally, we assume that players are always **rational** and **honest**. This just means that players will always prefer to maximize the total value they receive, they will never prefer to receive less just to spite another player, and they will never lie about their true preferences until Problem 9.

Proportionality

A division is **proportional** if each of the n players receives a total value of at least $\frac{1}{n}$ according to *their own* value function.

A division is **super proportional** if each player receives total value *greater than* $\frac{1}{n}$ according to their own value function.

[1] Here, $\uplus$ is the *disjoint union*. $A \uplus B$ is the set of all elements in A and B without removing duplicates. Usually, $A \cap B = \{\}$, in which case $A \uplus B = A \cup B$.

1. (a) **[6]** Suppose we wish to divide $G = \{A, B, C\}$, among two players (Player 1 and Player 2), and A, B, and C are indivisible. For each of the following, use the given value functions to find a division that satisfies proportionality. If a proportional division doesn't exist, write "no proportional division".

i.

	v_1	v_2
A	0.4	1.0
B	0.3	0
C	0.3	0

ii.

	v_1	v_2
A	0.6	0.6
B	0.4	0
C	0	0.4

iii.

	v_1	v_2
A	0.3	0.3
B	0.5	0.5
C	0.2	0.2

(b) **[10]** Suppose again that G is continuous. Prove that a super proportional division exists if and only if not all players have the same value function.

Envy-Free Divisions

A division is **envy-free** if every player believes, from their own perspective, that the total value they received is at least as high as any other players'. In other words, no player is envious of any other.

For example, with two players and the value functions given in 1.a.i, the division $(1 : (B), 2 : (A, C))$ is not envy-free, because Player 1 received a total value of 0.3, while Player 1 perceives Player 2 to have received a total value of 0.7. Hence, Player 1 will be envious of Player 2.

2. (a) **[4]** Suppose we wish to divide the set $G = \{A, B, C, D\}$ among three players, and A, B, C, and D are indivisible. For each of the following, use the given value functions to find a division that is envy-free. If an envy-free division doesn't exist, write "no envy-free division".

i.

	v_1	v_2	v_3
A	0.25	0.3	0.5
B	0.25	0.4	0
C	0.25	0.2	0
D	0.25	0.1	0.5

ii.

	v_1	v_2	v_3
A	0.25	0.3	0.1
B	0.25	0.3	0.2
C	0.25	0.2	0.3
D	0.25	0.2	0.4

(b) **[3]** Prove that every envy-free division is also proportional.

(c) **[4]** With two players, is every proportional division also envy-free? Prove, or disprove by finding a counterexample.

(d) [**5**] With three players, is every proportional division also envy-free? Prove, or disprove by finding a counterexample.

Efficiency

A division d_1 **dominates** another division d_2 if at least one player receives greater total value in d_1 than d_2, and no player receives less total value in d_1 than d_2.

A division is **Pareto efficient** if it is not dominated by any other division. Equivalently, a division is Pareto efficient if all other divisions either assign identical total values to each player, or assign a lower total value to at least one player.

3. (a) [**5**] Suppose $G = \{A, B, C\}$, A, B, and C are indivisible, and we wish to divide G among two players.
    i. Choose value functions for the two players, and find a Pareto efficient division that is not proportional.
    ii. Choose value functions for the two players, and find a proportional division that is not Pareto efficient.

   (b) [**12**] If all elements of G are divisible, does a Pareto efficient, envy-free division always exist? Prove or disprove.

The Price of Proportionality

We now consider another metric often used to judge divisions.

Define the **welfare** of a division to be the sum of the total values each player receives. For example, if there are two goods, A and B, and Player 1 values A at 1 (and B at 0) while Player 2 values B at 1 (and A at 0), then giving A to Player 1 and B to Player 2 gives them each a total value of 1, for a welfare of $1 + 1 = 2$, while dividing both goods equally between the players will give them each a total value of 0.5, for a welfare of 1. Hence, while both of these divisions are proportional and envy-free, the former provides the higher welfare (note that the latter is also not efficient).

Maximizing welfare seems like a useful thing to aspire to, but it's often at odds with our other notions of fairness. As we'll seem we often must sacrifice some welfare to achieve a proportional division. This is particularly true for divisions involving a large number of players. For a given scenario (set of goods, players, and player value functions), define the **utilitarian price of proportionality**, or **UPOP**, to be the ratio between the maximal welfare, and the maximal welfare in a proportional division.

4. (a) [**2**] Prove that any division that maximizes welfare is efficient.

   (b) [**5**] Given the value functions from 2.a.i for indivisible goods A, B, C, and D, compute
    i. the division that maximizes welfare.
    ii. the proportional division with the greatest welfare.
    iii. the UPOP of this scenario.

   (c) [**8**] Suppose that we wish to divide a discrete set of indivisible goods among 25 players. Using at most 50 goods, describe a scenario with a UPOP of at least 2.5.

   (d) [**12**] Suppose again that we wish to divide a discrete set of indivisible goods among 25 players. Prove that the UPOP of any such scenario is ≤ 9.

Divide and Choose

One common solution to the cake-cutting problem described in the introduction and other two-player fair division problems is a procedure called **Divide and Choose**. In Divide and Choose, one player cuts the cake into any two pieces that they believe are of equal value. Then the second player chooses which of the pieces they prefer, and finally the first player receives the remaining piece.

More generally, Divide and Choose can be used for many fair division problems with two players. One player chooses a division of G into two sets, each of which would give him equal total value. Then, the other player chooses which set of goods she prefers, and the first player receives the remaining set. Note that the procedure doesn't always work if G contains an indivisible element, since in that case the first player can't necessarily divide G into two halves of equal value.

5. (a) **[4]** Let G be a cake with three homogeneous regions, C, V, and S (corresponding to chocolate, vanilla, and strawberry). Suppose Player 1 and Player 2 have the following value functions:

	v_1	v_2
C	0.2	0.3
V	0.3	0.4
S	0.5	0.3

If both players follow the procedure exactly, which of the following divisions could *not* be the result of Divide and Choose? Note that either player could be the divider. For each case, write whether it is possible or not.

i. $\{1 : (S), 2 : (C, V)\}$
ii. $\{1 : (C, V), 2 : (S)\}$
iii. $\{1 : (V, \frac{1}{2}S), 2 : (C, \frac{1}{2}S)\}$
iv. $\{1 : (\frac{1}{2}V, S), 2 : (C, \frac{1}{2}V)\}$

(b) **[3]** Prove, or disprove by finding a counterexample, that Divide and Choose always results in an envy-free division.

(c) **[3]** Prove, or disprove by finding a counterexample, that Divide and Choose always results in a Pareto efficient division.

(d) **[5]** Consider the following procedure, which attempts to generalize Divide and Choose to three players: Player 1 first creates a division of G of the goods into three sets, each of which would give him equal total value. Then, Player 2 chooses the set from the three that he prefers, Player 3 chooses the set from the remaining two that she most prefers, and finally Player 1 receives the last remaining set. Is this procedure envy-free? Prove, or disprove by finding a counterexample.

Moving Knives

The special case where $G = \{g\}$ contains a single heterogeneous element (for example, if g is a cake) is common, and a number of fair division procedures have been developed specifically for it. One class of such procedures are called the **Moving Knife Procedures**.

For this problem, we make one simplifying assumption. Let the real numbers in the interval $(0, 1)$ correspond to horizontal position across the width of g, which we imagine is similar to a cake. Restrict the cuts we can make in the cake to cuts at positions $x \in (0, 1)$. Then we can describe a division of $G = \{g\}$ by the numeric positions of the cuts we make on g, and the allocation of the resulting pieces to players.

6. (a) **[3]** The two-player Moving Knife Procedure proceeds as follows: A "knife" is held over g at position 0, and slowly swept across so the numeric position of the knife gradually increases. At any point, if either player believes that the portion of g to the left of the knife has half the value of all of g, they call a stop to the procedure, and g is cut at that point. Then, the player who stopped the knife receives the left half and the other player receives the right half. Prove that, like Divide and Choose, this procedure is results in an envy-free division.

   (b) **[5]** A nice advantage of this procedure is that it easily generalizes to more than two people. Consider the following modification for n people: The knife is swept as before, but as soon as any of the n players believe the portion of g to the left of the knife has value $\frac{1}{n}$ of the whole, they call stop. Then, g is cut at that point, and the player who called stop receives the piece to the left of the knife. Then, the entire procedure is repeated with the remainder of g and the remaining $n-1$ players. Prove that this procedure results in a proportional division.

   (c) **[6]** Consider the following variant of the two-player moving knife procedure: Player 1 holds two knives. The first is initially at the left edge of g, and the second is placed at the line that Player 1 believes splits g into two halves of equal value. Then, Player 1 sweeps both knives slowly to the right, such the portion of g between the knives remains exactly half the value of all of g. As soon as Player 2 agrees that the portion between the knives is half the value of all of g, she tells Player 1 to stop. Then, g is cut at the position of the knives, the center piece is given to Player 2, and the remainder is given to Player 1.

      i. Prove that this procedure always terminates.

      ii. Prove that this procedure results in a division where both players believe that they and the other player received a total value of exactly $\frac{1}{2}$. Such a division is called an **exact** division, and is clearly also envy-free and proportional.

Adjusted Winners

To this point we've seen several two-player procedures that ensure proportionality and envy-freeness, but none that also ensure efficiency. However, we proved in Problem 3 that such a division always exists when all goods are divisible. The **Adjusted Winner Procedure** guarantees a division for two players that is both envy-free and efficient when G contains divisible, homogeneous elements.

The adjusted winner procedure proceeds as follows:

i. Suppose $G = \{g_1, g_2, \ldots, g_k\}$ is a set of k goods to be divided, and Players 1 and 2 have value functions v_1 and v_2, respectively.

ii. Let $G_1 = \{g_i \mid v_1(g_i) > v_2(g_i)\}$, let $G_2 = \{g_i \mid v_2(g_i) > v_1(g_i)\}$, and let G_r be the remainder, namely $\{g_i \mid v_1(g_i) = v_2(g_i)\}$. Without loss of generality suppose $v_1(G_1) \geq v_2(G_2)$.

iii. Tentatively assign the goods in $G_1 \cup G_r$ to Player 1, and the goods in G_2 to Player 2, so each player gets the goods they value higher than the other player, and Player 1 gets all goods for which the values were tied.

iv. List the goods on an order $g_{i_1}, g_{i_2}, \ldots g_{i_k}$ such that

$$\frac{v_1(g_{i_1})}{v_2(g_{i_1})} \geq \frac{v_1(g_{i_2})}{v_2(g_{i_2})} \geq \cdots \geq \frac{v_1(g_{i_k})}{v_2(g_{i_k})}$$

v. Then, Player 1 has been assigned goods $g_{i_1} \dots g_{i_r}$ for some r (in particular, all the goods with $\frac{v_1(g_{i_1})}{v_2(g_{i_1})} \geq 1$), and Player 2 has been assigned goods $g_{i_{r+1}} \dots g_{i_k}$.

vi. At this point, Player 1 has more total value than Player 2. To fix this, Player 1 gives goods or fractions of goods to Player 2 in the order $g_{i_r}, g_{i_{r-1}}, g_{i_{r-2}}, \dots$ as necessary until both players have the same total value (based on their own value function).

In the following problem, we'll prove the fairness of the Adjusted Winner Procedure.

7. (a) **[4]** Suppose that Player 1 values g_i at least as much as Player 2 does, and Player 2 values g_j at least as much as Player 1 does. Additionally, suppose Player 1 possesses a fraction s $(0 < s \leq 1)$ of g_i, and Player 2 possesses a fraction t $(0 < t \leq 1)$ of g_j in some division. Prove that if a trade of Player 1's fraction of g_i for Player 2's fraction of g_j increases either player's total value, then the other player's total value must decrease.

   (b) **[5]** Suppose, as in (a), that Player 1 possesses a fraction s $(0 < s \leq 1)$ of g_i, and Player 2 possesses a fraction t $(0 < t \leq 1)$ of g_j in some division, and suppose $\frac{v_1(g_j)}{v_2(g_j)} \leq \frac{v_1(g_i)}{v_2(g_i)}$. Again, prove that if a trade of Player 1's fraction of g_i for Player 2's fraction of g_j increases either player's total value, then the other player's total value must decrease.

   (c) **[5]** Prove that if a given division is not efficient, then there are goods g_i and g_j such that a trade of part of g_i for part of g_j yields a division that dominates[2] the given one.

   (d) **[6]** Using the results from (a), (b), and (c), prove that the Adjusted Winner Procedure yields a division that is both envy-free and efficient.

Three-player Procedures

The Selfridge-Conway Procedure, discovered in 1960, was the first discrete fair division procedure for three players that is envy-free. It's quite a bit more complex than the others we've seen. The procedure is as follows:

i. Player 1 divides G into three sets of what he believes are equal value.

ii. Let A be the set that Player 2 most prefers, let B be the next set, and let C be the set Player 2 least prefers. Player 2 partitions A into A_1 and A_2, so that A_1 has the same value to her as B, and A_2 (possibly empty) is the remainder.

iii. Player 3 chooses the set among A_1, B, and C that he most prefers.

iv. Player 2 chooses the set among the remainder that she most prefers, with the restriction that she must choose A_1 if it's available.

v. Player 1 chooses the last set from those three, leaving just A_2 to be divided (if it exists).

vi. Either Player 2 or Player 3 must have chosen A_1. Without loss of generality suppose it's Player 2 (otherwise swap the roles of players 2 and 3 in the following steps).

vii. Player 3 divides A_2 into three sets of equal value.

viii. Player 2 chooses her favorite one of these subsets.

[2]See Problem 3 for a definition of "dominates".

ix. Player 1 chooses his favorite among the remaining two.

x. Player 3 chooses the final subset of A_2.

8. (a) **[6]** Prove that the Selfridge-Conway Procedure always produces an envy-free division when all elements of G are divisible.

   (b) **[10]** Note that Selfridge-Conway requires up to 5 "cuts" to be made: two to create the initial three regions, one to create A_1 and A_2, and two more to divide A_2. By relaxing our notion of fairness, we can reduce the number of cuts required.

   i. Design a new algorithm for three players, based on Selfridge-Conway and possibly others we've seen, that requires at most 3 cuts, and always produces a *proportional* division when all elements of G are divisible.

   ii. Prove that your algorithm satisfies these conditions. Namely, prove that it requires at most three cuts, and that it always produces a proportional division.

Lying for Fun and Profit

To this point, we've assumed that the players are always honest, and accurately represent their true preferences. However, what happens when we remove this assumption?

9. (a) **[5]** Let $G = \{A, B, C\}$, where A, B, and C are each *divisible*. Additionally, suppose Player 1 and Player 2's true value functions are as follows:

	v_1	v_2
A	0.2	0.3
B	0.3	0.4
C	0.5	0.3

   If Player 1 and Player 2 divide G using the Divide and Choose method, and Player 1 knows Player 2's value function (as well as his own), then Player 1 can guarantee himself a total value of $\frac{x}{1000}$ for many integers x by carefully selecting the initial division (knowing that Player 2 will then choose whichever half has greatest total value to her). Compute the largest such integer x.

   (b) **[3]** Suppose a division is chosen by a third party that maximizes welfare. For example, using the values given in part (a), the maximal division is $\{1 : (C), 2 : (A, B)\}$, which yields a total value of 0.5 to Player 1. If Player 1 lied about his value function to the third party but Player 2 told the truth, what is the maximal true total value (using his original value function from (a)) that Player 1 could achieve?