Johns Hopkins Math Tournament 2020

Proof Round: Partially Ordered Sets

February 8, 2020

Problem	Points	Score
1	3	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	7	
9	10	
10	10	
11	10	
12	15	
13	15	
Total	100	

Instructions

- The exam is worth 100 points; each part's point value is given in brackets next to the part.
- To receive full credit, the presentation must be legible, orderly, clear, and concise.
- If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says justify or prove, then you must prove your answer rigorously.
- Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa.
- Pages submitted for credit should be **numbered in consecutive order at the top of each page** in what your team considers to be proper sequential order.
- **Please write on only one side of the answer papers.**
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Introduction

In this proof round, we will dive into the realms of set theory and explore some properties of *partially ordered sets*. By the end of the round, you will prove a weak version of Dilworth's Theorem, which is useful for characterizing partially ordered sets, and see the theorem in action.

1 Sets

Before we can discuss partially ordered sets, we need to define sets.

Definition 1.1. A *set* is simply a collection of objects that includes no repeated objects. The objects in a set are called its *elements*. We usually name sets using capital English letters, and if we want to list all objects in a set, we wrap the list in curly braces. For example, the set of suits in a standard deck of cards could be written:

$$S_1 = \{\text{hearts, diamonds, spades, clubs}\}.$$

Important note: In the sets defined above, the objects in the set do not have an ordering. As a result, if

$$S_2 = \{\text{diamonds, spades, clubs, hearts}\},$$

then $S_1 = S_2$.

Important note: The elements of a set do not need to be single objects - each element could be a list! Later, we will encounter sets whose elements are *ordered pairs*; that is, lists of two objects. An example of a set whose elements are ordered pairs is the set of playing cards by number and suit:

$$S_3 = \{(2, \text{diamonds}), (4, \text{hearts}), (2, \text{spades}), (9, \text{hearts}), \dots\}.$$

Definition 1.2. We say that a set T is a *subset* of a set S if every element of T is an element of S, and we write $T \subset S$. For example, if

$$T_1 = \{\text{spades, hearts}\},$$

then $T_1 \subset S_1$.

2 Ordered Sets

Before we can discuss ordered sets, we need to define relations on sets.

Definition 2.1. A *relation* is a set of ordered pairs. The objects in the ordered pairs must come from the elements of *ground set*. If the ground set is S, we say we have a relation on S. An example of a relation on the set S_1 above would be:

$$R_1 = \{(\text{diamonds, hearts}), (\text{diamonds, diamonds})\}.$$

Problem 1: (3 points) Is the usual notion of equality for real numbers a relation? Explain.

Yes, elements of R include (x, x) for all x.

For some sets, we can define an *ordering*; intuitively, there are some sets where one element might be "less than" another. An example is the real numbers - we know 1.8 is less than 2.7, for instance. However, we can also define orderings where not every pair of elements is comparable; that is, some elements of a set might be "less than" others, but there also exist elements for which there is no way to decide which is less.

Definition 2.2. A *totally ordered set* is a set S together with a relation R on S which satisfies the following properties. Let a, b, c be distinct elements of S:

1. The relation is reflexive, ie. $(a, a) \in R$ for all a.
2. The relation is antisymmetric, ie. if $(a, b) \in R$, then $(b, a) \notin R$.
3. The relation is transitive, ie. if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
4. All elements of the set are comparable, ie. either $(a, b) \in R$ or $(b, a) \in R$ for all a, b.

Problem 2: (5 points) Let S be the set of integers from 1 to 20, and define a relation R as follows: Let R be the set of all ordered pairs (a, b) from S such that b is divisible by a. Do R and S define a totally ordered set? Hint: Proving a relation is a total ordering involves checking to see if the four conditions given above are true.

No, condition 4 fails, eg. neither $(5, 7)$ nor $(7, 5)$ is in R.

Definition 2.3. A *partially ordered set* is a set together with a relation which satisfies conditions 1 to 3 above, but doesn't necessarily satisfy condition 4. We sometimes abbreviate "partially ordered set" as "poset".

Note: In both partially and totally ordered sets, if $(a, b) \in R$, we write $a \leq b$.

3 Problems on Partially Ordered Sets

Problem 3: (5 points) Let $A_1, A_2 \dots A_n$ be a collection of sets. Prove that the subset relation is a partial order; that is, $A_i \leq A_j$ if $A_i \subset A_j$. Hint: Proving a relation is a partial order involves checking to see if the three conditions given above are true.

We check the three conditions. $A_i \subset A_i$, so condition 1 holds. If $A_i \subset A_j$ and $A_i \neq A_j$, then A_j has at least one element not in A_i and $A_j \not\subset A_i$, so condition 2 holds. If $A_i \subset A_j \subset A_k$ and $x \in A_i$, then $x \in A_j$ and therefore $x \in A_k$, so condition 3 holds.

Problem 4: (5 points) The converse ordering is defined as follows: Given a poset A, let $b \leq_c a$ if $a \leq b$. Prove that the converse ordering is a valid partial ordering. Note: We often use the symbol $\geq$ in place of $\leq_c$.

We check the three conditions. Since $a \leq a$, we have $a \leq_c a$ for any a in the poset and condition 1 holds. If $a \leq b$, then by condition 2 $b \not\leq a$; thus, $b \leq_c a$ and $a \not\leq_c b$, so condition 2 holds. If $c \leq_c b$ and $b \leq_c a$, then $a \leq b \leq c$; by condition 3, $a \leq c$ and $c \leq_c a$, so condition 3 holds.

Problem 5: (5 points) $\mathbb{R}^2$ is defined as all ordered pairs (x_1, x_2), where x_1 and x_2 are real numbers. Define a partial order on $\mathbb{R}^2$. You do not need to prove your partial ordering is valid.

Example: Any norm, such as Euclidean distance from the origin.

Problem 6: (5 points) We use the notation $a < b$ if $a \leq b$ and a and b are not the same element. An element g of a poset A is called the greatest element of the set if $a < g$ for all $a \in A$. Prove that a poset can have at most one greatest element.

Suppose a poset has two greatest elements a and b. Then by the definition of greatest element, $a < b$. However, this means a is not a greatest element, which is a contradiction.

Problem 7: (5 points) Prove that a poset containing n elements need not have a greatest element.

Example: The poset on two elements where they are not compared.

Problem 8: (7 points) An element a of a poset A is called maximal if there does not exist $b \in A$ such that $a < b$. Prove that every poset containing n elements has at least one maximal element.

Suppose a poset does not have a maximal element. Select an element x from the poset. Since x_1 is not maximal, we can find an x_2 such that $x_1 < x_2$. x_2 is also not maximal, so we can find another element x_3 such that $x_1 < x_2 < x_3$. Since the poset only has n elements, we can do this $n-1$ times at most, resulting in an element we will call x_n such that $x_1 < x_2 < \cdots < x_n$. Because of partial order condition 3, we know that $x_i < x_n$ for any i. However, this means x_n is maximal, a contradiction.

Problem 9: (10 points) Let S be a set containing n elements. What is the maximum number of ordered pairs in a partial order on S that is not a total order? Justify your answer.

One fewer than the number of ordered pairs in a total order: $\binom{n}{2}+n-1$. We can always remove exactly one comparison from a total order to obtain a partial order: Remove a comparison between two elements that are adjacent in the total order (ie. there is no element between them). To verify this, we need only check transitivity, since reflexivity and antisymmetry will be unaffected. If we have removed a comparison $a \leq b$, we are looking for a statement of the form "if $a \leq c$ and $c \leq b$, then $a \leq b$." However, by construction there was no c between a and b in the total order, so such a statement cannot be formed and transitivity is unaffected.

Problem 10: (10 points) A subset of a poset is called a chain if all elements in the subset are comparable to each other; that is, for all $x, y \in \hat{A} \subset A$, either $x \leq y$ or $y \leq x$. The height of a poset is the number of elements in the poset's longest chain. Let S be a set containing n elements, and let T be the set of all subsets of S. Define the subset partial order on T as in (1). Calculate the height of the poset, and justify your answer.

If subset a is less than subset b, there is at least one element in b not in a. Thus, we can find a chain of maximal length by adding elements to the subsets one at a time. We begin with the empty set and end with S, so the height is $n+1$.

Problem 11: (10 points) A subset of a poset is called an antichain if none of the elements of the subset are comparable to each other. The width of a poset is the number of elements in its largest antichain. Consider the poset given in (8), and assume n is even. If the width of the poset is w, determine how many antichains of the poset have size w.

One. The largest antichains are those that contain all subsets of a given size. There are the most subsets of size $\frac{n}{2}$; there are $\binom{n}{n/2}$ of them.

Problem 12: (15 points) (Dilworth's Theorem) A *partition of size* h of a set S is a division of the elements of S into h subsets with every element in exactly one subset. Suppose we have a partial order on S with height h. Prove that we can partition S into h separate subsets, each of which forms an antichain in the partial order. Hint: For each element s, let x_s be the height of largest chain whose greatest element is s.

Let $A_i = \{s \in S | x_s = i\}$. Each A_i is an antichain since no two elements of it can belong to a single chain. The largest value of i is h. Every element of S is in exactly one of the A_i. Each A_i is nonempty because of the existence of a chain of height h. Therefore, the A_i together constitute an antichain partition of S into h separate subsets.

Problem 13: (15 points) Suppose we have a poset whose ground set has $mn + 1$ elements. Suppose further that the height of the poset is less than $m + 1$. Prove the width of the poset is at least $n + 1$.

Invoke Dilworth's Theorem. There exists a partition of the ground set into m or fewer antichains. Then (by division of $mn + 1$ by m) at least one antichain must have $n + 1$ or more elements in it.