Middle School JHMT 2020

Individual Round: Short

February 22, 2020

Instructions

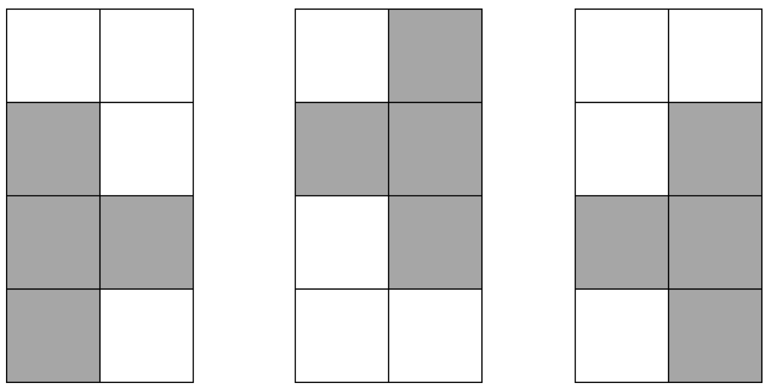
- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 12 questions to be solved individually in 30 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. Sungwon is running out of dining dollars. He realizes that he can at most order 3 salmon pokes that cost \$12.95 each. He also realizes that he can at most order 5 chicken curries that cost \$8.75 each. Finally, he notices that he can at most order 6 tuna rolls that cost \$8.25 each. If Sungwon has a whole number amount of dollars, what is the maximum amount of dining dollars he can have?

2. In a school, every student speaks at least one of English, Chinese, and French. 35 students speak English, 38 students speak Chinese, and 18 students speak French. Moreover, 17 students speak both English and Chinese, 13 students speak English and French, and 6 students speak both Chinese and French. Finally, 5 students speak all 3 languages. How many total students are in this school?

3. There are two opaque, unlabeled bags. One bag contains 2 gold and 8 red marbles and the other contains 3 gold and 7 red marbles. Alice chooses a bag at random and picks a marble out of the bag she chose. The probability that Alice picks a gold marble can be expressed as $\frac{a}{b}$ where $\frac{a}{b}$ is a fraction expressed in simplest form. What is $a + b$?

4. In a 2×4 grid, how many different designs can be created by filling exactly 2 squares? Note that designs that are the same after the **entire grid** is rotated or flipped are considered the same. Thus, if we were instead filling exactly 4 squares, then the three designs shown below would all be considered the same.



5. There are 40 switches and 40 robots labeled 1 through 40 in a factory. Each switch has two states - it is either on or off, with each switch initially off. Robot #1 flips every switch, then robot #2 flips every other switch, then robot #3 flips every 3rd switch, and so on until all 40 robots have gone. Afterwards, how many switches are on?

6. Let $ABCD$ be a symmetric trapezoid with area 600 and base lengths $\overline{AB} = 18$ and $\overline{DC} = 32$. Let P and Q be the midpoints of AB and DC respectively, and O be the midpoint of PQ. What is the perimeter of $\triangle DOA$?

7. Sphere A is inscribed in cube B, which itself is inscribed in hemisphere C. The surface area of sphere A is 16π. If the surface area of hemisphere C can be expressed as $k\pi$, what is k? Note that a hemisphere is a sphere cut in half by a plane passing through its center, and that the surface area of a hemisphere includes the area of the circle at its base.

8. Two congruent circles have their centers on the same diagonal of a square, such that the two circles are externally tangent to each other and both circles are tangent to two different sides of the square. Let the length of a side of the square be 12. If the radius of one circle can be expressed as $a - b\sqrt{c}$ where $b\sqrt{c}$ is in simplest radical form, what is $a + b + c$?

9. Steve wants to buy 7 candies. Zoe's candy shop has an infinite amount of peach flavored candies, strawberry flavored candies, and raspberry flavored candies. How many different combinations of candies can Steve buy from Zoe's shop?

10. Elvin decides to spend 5 hours studying and doing his chores. If Elvin spends x hours studying, he will gain $f(x)$ units of stress, where $f(x) = x^2 + 2x$. If he spends y hours doing his chores, he will gain $g(y)$ units of stress, where $g(y) = 3y - 14$. Elvin must spend all 5 hours either studying or doing chores, and he can only do one at a time. In order to minimize his stress, Elvin should spend $\frac{a}{b}$ hours doing chores, where $\frac{a}{b}$ is a fraction expressed in simplest form. What is $a + b$?

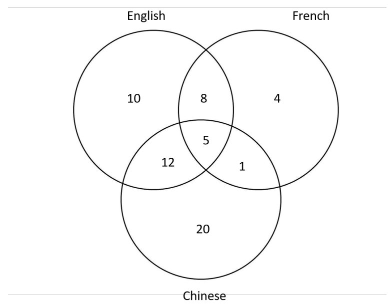
11. Freddy wants to rearrange the digits of 123456789 to form "fresh" numbers. A number is "fresh" if none of its digits are in the same spot as they were in the original number, and every prime digit is in a spot that was also occupied by a prime digit in the original number. How many "fresh" numbers can Freddy form?

12. $\triangle ABC$ has a right angle at A, with $AB = 12$ and $AC = 6$. X is a point on $\overline{AC}$, Y is a point on $\overline{BC}$, and Z is a point on $\overline{AB}$ such that $AXYZ$ is a square. Furthermore, P is the center of a circle inscribed in $\triangle XCY$, and Q is the center of a circle inscribed in $\triangle ZYB$. The value $(PQ)^2$ can be expressed as $f - g\sqrt{j}$, where $g\sqrt{j}$ is expressed in simplest radical form. What is $f + g + j$?

Individual Round: Short Solutions

1. Let the amount of dining dollars Sungwon have be x. From the three conditions, we know that $3 \times 12.95 = 38.85 \leq x \leq 4 \times 12.95 = 51.80$, $5 \times 8.75 = 43.75 \leq x \leq 6 \times 8.75 = 52.50$, and $6 \times 8.25 = 49.50 \leq x \leq 7 \times 8.25 = 57.75$. The answer is the largest whole number x that satisfies these 3 conditions, which is $\boxed{51}$.

2. **Solution 1:** We can solve this problem by making a triple Venn diagram! A filled in solution is shown below. Using it, we find that the answer is $\boxed{60}$.



Solution 2: We can also solve this problem using the Principle of Inclusion-Exclusion. When we have three sets A, B, and C, this principle states that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. To see why this principle holds, try visualizing what this formula is doing using the Venn Diagram above! Now, using this formula, we find that the total number of students will be $35 + 38 + 18 - 17 - 13 - 6 + 5 = \boxed{60}$.

3. To find the probability, we need to consider two different cases based on which bag Alice picks at random. There is a $\frac{1}{2}$ chance she will pick the first bag, and in this case she will have a $\frac{2}{10}$ chance to pick a gold marble. There is also a $\frac{1}{2}$ chance she will pick the second bag, and in this case she will have a $\frac{3}{10}$ chance of picking a gold marble. We then add these cases together to find $\frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{3}{10} = \frac{5}{20}$ or $\frac{1}{4}$. Thus, the answer is $1 + 4 = \boxed{5}$.

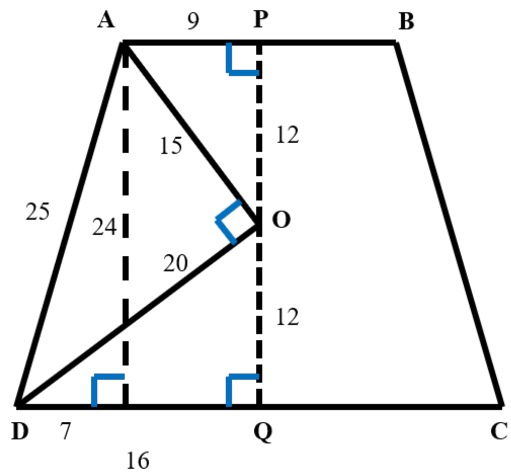
4. We can count the possible outcomes by splitting the solutions into 2 different cases. In the first case, we fill a corner square. In this case, suppose the left top square is colored. Then we see that there are 7 different designs, as we can fill in any of the other 7 squares to get a different design. Note that we do not need to also consider cases where we fill a corner besides the left top, because all these other cases are identical after rotations/reflections.

   Now, the second case is when no vertex squares are colored. Thus, suppose the left square on the 2nd row is colored. We then see that there are 3 different designs, as we can fill in any non-corner square to get a different design. Again, note that we do not need to also consider cases where we fill a non-corner besides this one because all these other cases are identical after rotations/reflections. Thus, there are a total of $7 + 3 = \boxed{10}$.

5. The key insight to make is that only switches with an odd number of factors will be on at the end. This is because switches are flipped by robots whose numbers are factors of the switch number, and switches need to be flipped an odd number of times to be on at the end, as all switches are initially off. The only numbers with an odd number of factors are perfect squares, since their square root doesn't have a different factor to "pair up" with. Thus, the answer is $\boxed{6}$ because there are 6 perfect squares between 1 and 40, inclusive.

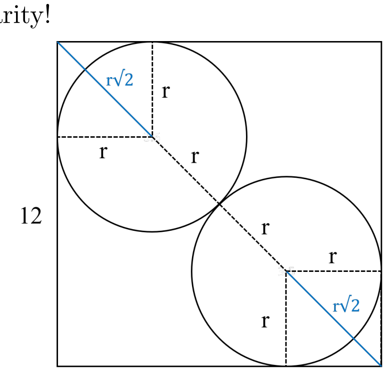
6. Refer to the diagram below! Using the area and base lengths of the trapezoid, we can calculate the height of the trapezoid to be $\frac{600}{\frac{18+32}{2}} = 24$. Therefore $\overline{PO} = \overline{OQ} = 12$. By Pythagorean theorem, we

have $\overline{AO} = 15$ and $\overline{DO} = 20$. To find the length of DA, draw a line from A down to DC such that the line is perpendicular to DC. Suppose this line intersects DC at point X. We find that DX has length $(32 - 18)/2 = 7$. Furthermore, since $\triangle DXA$ is a right triangle, we can use the pythagorean theorem to find that the length of DA is 25. Thus, the final perimeter is $15 + 20 + 25 = \boxed{60}$.



7. Recall that the surface area of a sphere with radius r is $4\pi r^2$. Thus, we have that $16\pi = 4\pi r_a^2$, so the radius of sphere A is 2. Thus, the side length of cube B is 4. Now, notice that the radius of hemisphere C is the hypotenuse of a triangle with one leg a side of cube B, and another leg half the diagonal of a face of cube B. Thus, $r_c^2 = 4^2 + (2\sqrt{2})^2$, so the radius of hemisphere C is $\sqrt{24}$. Finally, to find the surface area of hemisphere C, we must add the surface area of the curved part of the hemisphere with the area of the flat part of the hemisphere. The curved surface area will just be half the surface area of a full sphere, so we find it to be $\frac{4\pi(\sqrt{24})^2}{2} = 48\pi$. The flat part of the hemisphere is simply a circle with radius $\sqrt{24}$, so this area is 24π. Thus, the total surface area of the hemisphere will be $48\pi + 24\pi = 72\pi$, which means the answer is $\boxed{72}$.

8. The diagonal of the square will have length $12\sqrt{2}$. Furthermore, by drawing the diagonal going through the centers of the circles, we see that the length of the diagonal equals the sum of the 2 circle radii and twice the length of $\sqrt{2}$ times the radius (draw a $45 - 45 - 90$ triangle with the radii as legs). Thus, we can solve the equation $2r + 2\sqrt{2}r = 12\sqrt{2}$ to find the radius. If we factor our r from the left hand side, we get $r(2 + 2\sqrt{2}) = 12\sqrt{2}$. Then, if we divide both sides by $(2 + 2\sqrt{2})$ and simplify the radicals on the denominator of the ensuing fraction, we find that $r = 12 - 6\sqrt{2}$ so the answer is $12 + 6 + 2 = \boxed{20}$. Refer to the diagram below for clarity!



9. Let a, b, c be the number of peach, strawberry, and raspberry candies respectively. Note one needs to count the number of ways that $a + b + c = 7$ with a, b, c non-negative. This is equivalent to a "bars and stars" problem where 2 bars need to be placed among 7 stars. There are $\binom{3-1+7}{3-1} = \binom{9}{2} = \boxed{36}$

choices. Here is a link with more background on this counting technique: `https://brilliant.org/wiki/integer-equations-star-and-bars/`.

10. Note that one requires that $x+y=5$. We can rewrite this as $y=5-x$ and so $g(x)=3(5-x)-14=15-3x-14=-3x+1$. Adding this to $f(x)$ gives $f(x)+g(x)=x^2-x+1$ and we need to minimize this, which is the total amount of stress. The vertex of this parabola is $\frac{1}{2}$ and so $x=\frac{1}{2}$ is the minimum. So Elvin spends $\frac{9}{2}$ hours doing chores and so $a+b=\boxed{11}$.

11. Note that there are 4 numbers here that are prime. We are trying to count the ways to derange 4 and 5 objects. This problem has a general solution. If we denote the number of ways to derange n objects $!n$ (this is often known as the subfactorial of n), then $!n=(n-1)[!(n-1)+!(n-2)]$. To see why this is true, note that we can break all the possible derangements down into two cases. In order to derange n objects, the first object has $(n-1)$ places to go to. Say it occupies the original position of object k. Then if object k occupies the original position of object 1, we have the two objects swapping positions and are left with the task of deranging $n-2$ objects. On the other hand, if object k doesn't occupy the original position of object 1, then we are left with a task of deranging $n-1$ objects, which proves the formula.

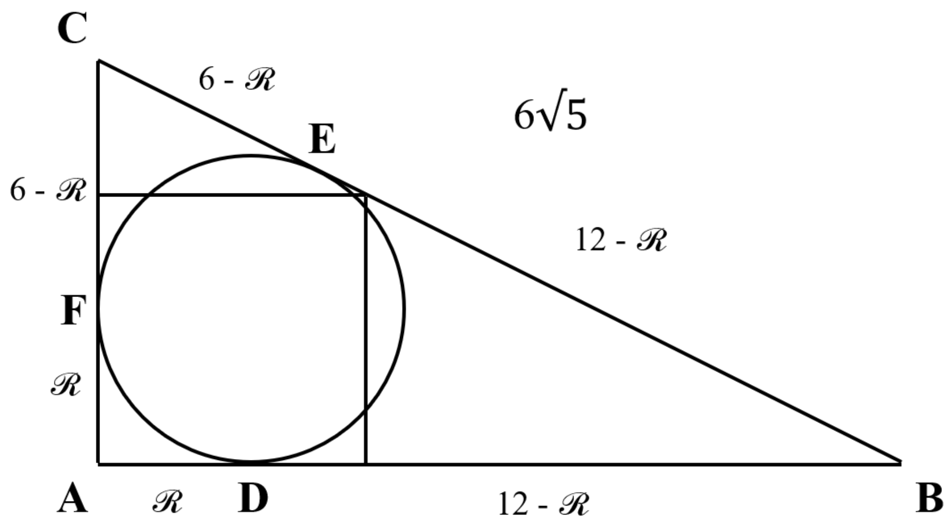
    Starting from $!1=0$ and $!2=1$, we see that there are 9 derangement of 4 objects and 44 derangement of 5 objects. Hence, there are $9\times 44=\boxed{396}$ fresh numbers Freddy can form.

12. Refer to the two diagrams below for clarity throughout this solution. $\triangle CXY$ is similar to $\triangle CAB$. Let the side length of the square $AXYZ$ be k. Then as $CX=6-k$ and $XY=k$, $6-k:k=1:2$. Solving this proportion, We find that the side length is 4. Now, there are two methods through which we can determine the radius r of circle P, and the radius R of circle Q.

    Method #1 to find r and R:
    Let the radius of a circle that inscribes $\triangle ABC$ be $\mathcal{R}$. First, we see that the length of the hypotenuse is $\sqrt{6^2+12^2}=6\sqrt{5}$. Let that circle intersect AB at D, BC at E, and AC at F. Then by the external tangent theorem, $BD=BE=12-\mathcal{R}$, and $CF=CE=6-\mathcal{R}$. Therefore $BE+CE=18-2\mathcal{R}=BC=6\sqrt{5}$, so $\mathcal{R}=9-3\sqrt{5}$. We can scale this radius to find the radius of the inscribed circle in $\triangle ZYB$ and $\triangle XCY$, since they are both similar to $\triangle ABC$. The radii are $R=6-2\sqrt{5}$ and $r=3-\sqrt{5}$ respectively.

    Method #2 to find r and R:
    First, note that the radius of a circle inscribed in a triangle can be calculated with the formula $\dfrac{2\times A}{p}$, where A is the area of the triangle and p is the perimeter of the triangle. To see why this formula works, try splitting the triangle into 3 smaller triangles by drawing lines from the center of the inscribed circle to the vertices of the triangle, and then calculating the areas of the 3 smaller triangles. Now, we will use this formula to find r and R. We see that $r=\dfrac{2\times 4}{6+2\sqrt{5}}=3-\sqrt{5}$. Similarly, we find that $R=\dfrac{2\times 16}{12+4\sqrt{5}}=6-2\sqrt{5}$.

    Using r and R to finish the problem:
    Let the point created by the intersection of the line drawn straight down from P and the line drawn horizontal to the left from Q be I. Then by Pythagorean theorem, $(PQ)^2=(PI)^2+(QI)^2=(4-R+r)^2+(4-r+R)^2=(4-3+\sqrt{5})^2+(4+3-\sqrt{5})^2=(1+\sqrt{5})^2+(7-\sqrt{5})^2=60-12\sqrt{5}$. Thus, the answer is $60+12+5=\boxed{77}$.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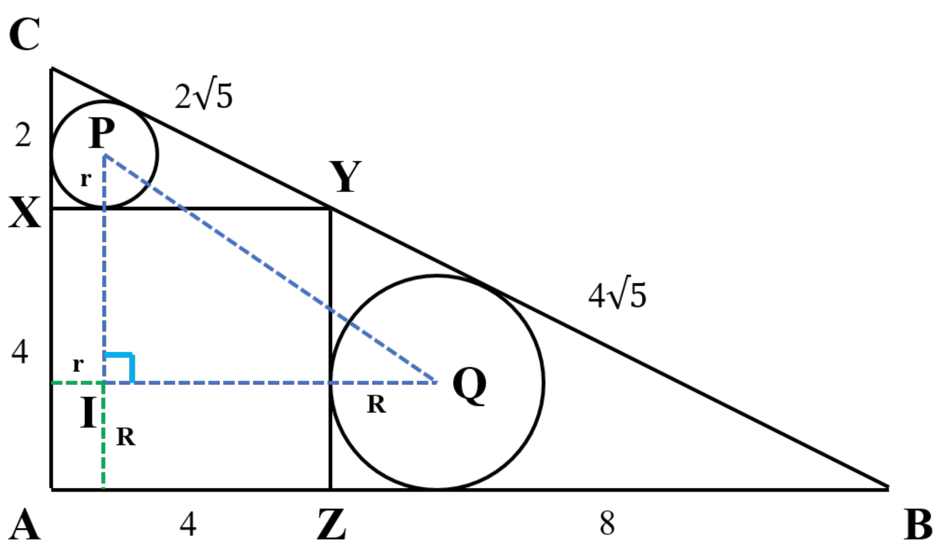
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