Middle School JHMT 2020

Individual Round: Long

February 22, 2020

Instructions

- **<u>DO NOT</u> TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 30 questions to be solved individually in 40 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. For Halloween, Elvin buys 8 Kat-Kitz bars and 17 Silky Way bars. What percent of Elvin's candy bars are Silky Ways?
2. What is the largest prime factor of 2020?
3. The volume of a cube is 27. What is its surface area?
4. Steve is currently twice as old as his sister. His dad is 30 years old. After 6 years, the sum of Steve's age and his sister's age will be equal to his dad's age. What is Steve's current age?
5. Randy fails the exams of 3 of his 5 classes, and gets a failing homework grade in 2 of his 5 classes. To pass a class, he must pass both the exam and the homework. What is the difference between the maximum number of classes he can pass this semester and the minimum number of classes he can pass this semester?
6. Aaron and Tim are both at one corner of a rectangular lawn that is 8 meters wide and 6 meters tall. Aaron decides to get to the opposite corner by taking the shortest path through the lawn. Tim doesn't want to ruin the grass, so he decides to take the shortest path walking around the lawn. What is the absolute difference between the distance traveled by Aaron and Tim in meters?
7. The measure of the complement of an angle is 40° greater than the measure of the angle itself. What is the degree measure of the supplement of the angle?
8. Compute the following sum: $-1 + 2 - 3 + 4 - 5 + 6 - ... + 2020$.
9. Let a, b be positive integers and $a^2 + b^2 = 17^2$. Find the value of $a + b$.
10. Jarro Lightfeather found an artifact which doubles his gold every 2 days. He currently has 300 gold. How many days does he have to wait to amass his fortune to at least $1,000,000$ gold?
11. For what value of a does the system of equations below not have any solutions?

$$\begin{cases} x - 2y = 20 \\ ax + 6y = 15 \end{cases}$$

12. What is the area of an isosceles triangle that has two sides of lengths 18 and 41?
13. A regular hexagon has side length 8, and an equilateral triangle has side length 16. The ratio of the hexagon's area to the triangle's area is $\frac{a}{b}$, where $\frac{a}{b}$ is a fraction expressed in simplest form. What is $a + b$?
14. If $\frac{2a^2 - 5a + 3}{2a^2 - a - 3} = 2020$, what is the sum of the digits of $\frac{a^2 - 2a + 1}{a^2 + 2a + 1}$?
15. Harry throws a dart at a dartboard. There is a 20% chance he misses completely. If he does indeed hit the dartboard, the dart will land on the dartboard in a random location. The bulls-eye of the dartboard is a circle of radius 1 and the dartboard as a whole is a circle of radius 3. The probability that Harry hits the dartboard, but misses the bulls-eye can expressed as $\frac{a}{b}$, where $\frac{a}{b}$ is a fraction expressed in simplest form. What is $a + b$?
16. If $\frac{4}{11 + b} = \frac{1}{11} + \frac{1}{b}$, what is the sum of all possible values of b?
17. Four points lie on a circle of area 2π in such a way that the distance between the two closest points along the circle is as large as possible. What is the area of the quadrilateral that has these four points as vertices?
18. What is the smallest positive integer with exactly 20 factors?

19. A bucket contains oil and water. The liquid in the bucket is 95% water by mass and the total mass of the liquid is 2kg. How many kilograms of water need to be added to the bucket so that the liquid in the bucket becomes 99% water by mass?

20. Three fair 6 sided die are thrown. The probability that the product of three numbers rolled is 36 is $\frac{a}{b}$, where $\frac{a}{b}$ is a fraction expressed in simplest form. What is $a+b$?

21. In Fillip, there are two square districts named Desmonida and Erida, whose side lengths are 10 miles and 15 miles, respectively. One of the vertices of Erida is located on the center of Desmonida, and Erida is rotated 30° counterclockwise, with respect to Desmonida. What is the area of the region belonging to both Desmonida and Erida?

22. After selling all his Pokémon cards, Ryan brags that he has 46 bills in his wallet, whose combined value is \$111. Out of jealousy, Aaron steals a \$50 bill out of Ryan's wallet. However, after noticing you watching him, Aaron puts the bill back. If Aaron comes back later and takes a random bill out of Ryan's wallet, the probability that it is a \$5 bill is $\frac{a}{b}$, where $\frac{a}{b}$ is a fraction expressed in simplest form. What is $a+b$? Ryan only uses the standard \$1, \$5, \$10, \$20, and \$50 bills.

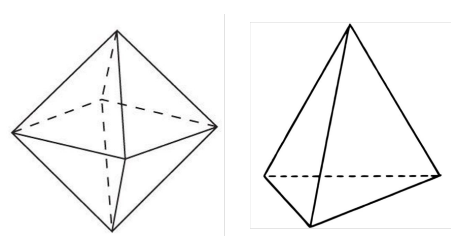
23. Suppose five non-parallel lines lie in a plane such that no three lines intersect at a point. How many regions do these lines divide the plane into?

24. What is the sum of the integers between $-4\sqrt{5}$ and $3\sqrt[3]{9}$?

25. a and b are positive integers such that $a \times b = 25!$, and $\frac{a}{b}$ is a fraction expressed in simplest form. How many pairs (a, b) satisfy these conditions? Note that $\frac{x}{1}$ for any integer x is considered a valid fraction in simplest form.

26. The number $1a4b4c$ is divisible by 132 for certain values of a, b, and c. What is the largest possible value of $a \times b \times c$?

27. An octahedron (shown below on the left) and a tetrahedron (shown below on the right) are connected externally such that they share exactly one vertex. Rob the robot starts at the vertex that belongs to both the octahedron and the tetrahedron. If each edge of the octahedron and each edge of the tetrahedron has length 1, and Rob can only travel along edges, what is the minimum distance Rob must travel in order to traverse each of the 18 edges in the figure at least once?



28. How many ordered triplets (a, b, c) satisfy the equation $abc + ab + ac + bc + a + b + c = 1000$ if a, b, and c are all positive integers?

29. For some positive integer n, 441_b is equal to n^2_{10}, and 351_b is equal to $(n-2)^2_{10}$. What is the value of b_{10}? Note that a_b represents a number a when written in base b.

30. How many distinct multiples of 3 can be created using only the digits 1, 2, 3, and 4 each at most twice if the multiples can have at most 6 digits?

Individual Round: Long Solutions

1. Elvin has 17 Silky Ways, and $8 + 17 = 25$ total candy bars. Thus, as a fraction, $\frac{17}{25}$ of his candy bars are Silky Ways. If we multiply the numerator and the denominator of this fraction by 4, we find that $\frac{68}{100} = \boxed{68\%}$ of his candy bars are Silky Ways.

2. $\boxed{101}$. This can be determined by drawing the following prime factorization tree.

```
2020
 / \
2   1010
    /  \
   2    505
        /  \
       5    101
```

3. The volume of a cube with side length s can be found with s^3. Thus, in this problem, we have that $s^3 = 27$, which means that the side length of the cube is 3. Thus, the surface area is $6 \times 3^2 = \boxed{54}$.

4. Suppose Steve's sister's current age is x, which means that Steve's current age is $2x$. Thus, in 6 years, Steve's age will be $2x + 6$, Steve's sister's age will be $x + 6$, and Steve's dad's age will be 36. Thus, we can set up the equation $(2x + 6) + (x + 6) = 36$. If we solve the equation, we find that $x = 8$, which means that Steve's age is $2x = 2 \times 8 = \boxed{16}$.

5. Randy passes the maximum number of classes if in two of his classes, he failed both the exam and the homework (this way his failures are more concentrated and we are left with more classes that he passes!). In this case, he fails 3 of his classes, and passes 2 of them. Randy fails the maximum number of classes if his exam and homework failures are completely spread out, i.e. if in each of his 5 classes, he fails either the exam or the homework. In this case, Randy fails all 5 of his classes, and passes 0 classes. Thus, the difference is $2 - 0 = \boxed{2}$.

6. The distance between the corners of the lawn when going through the lawn can be calculated using the Pythagorean Theorem: $6^2 + 8^2 = 10^2$, thus Aaron travels 10 meters. The distance going around the lawn is simply the sum of two sides of the lawn, thus Tim travels $6 + 8 = 14$ meters. Thus, the difference between the distance they travel is $14 - 10 = \boxed{4}$ meters.

7. Suppose the angle in question measures x°. Thus,the complement of the angle will be $90^\circ - x$. Thus, we know that $x + 40 = 90 - x$, which means that $x = 25$. Thus, the supplement of the angle will be $180^\circ - 25^\circ = \boxed{155^\circ}$.

8. If we group the terms in the sum into pairs using the Associative Property of Addition, we are left with $(-1 + 2) + (-3 + 4) + (-5 + 6) - ... + (-2019 + 2020)$. Doing this, we find that each pair sums to exactly 1, and we have $\frac{2020}{2} = 1010$ pairs. Thus, the total sum will be $1 \times 1010 = \boxed{1010}$.

9. We have to find solution to $a^2 + b^2 = 17^2$. Without loss of generality, say $a \le b$. Since $2 \times 13^2 > 17^2$, 13^2 is already greater than half of 17^2, and since a must be smaller than b, it is not possible for $a \ge 13$. Thus, we only need to check if there is a solution for $1 \le a \le 12$. Furthermore, note that b can at most be 16, and since $17^2 - 16^2 = 33$, a^2 must be at least 33, which means $a \ge 6$ must be true. Thus, in the range $6 \le a \le 12$, we find only one solution namely $(a, b) = (8, 15)$. So the answer is $a + b = \boxed{23}$.

10. If we divide $1,000,000$ by 300 (this is equivalent to dividing $10,000$ by 3, which is computationally easier), we are left with ≈ 3333. Thus, we want to find the smallest power of 2 greater than 3333. From here, we find that $2^{12} = 4096$, thus Jarro needs his gold to double 12 times to amass over 1 million. Since his gold doubles every 2 days, he needs $\boxed{24}$ days.

11. We are interested in finding a, which is a coefficient in front of the x variable, so a reasonable way to start is by eliminating y. We can do this by multiplying the first equation by 3, and adding it to the second equation. After doing this, we are left with $3x + ax = 75$. In order for this equation to have no solution, we must now completely eliminate the x variable from the left hand side. The only way to do this is to set $a = \boxed{-3}$. Doing this causes the last equation to become $0 = 75$, which of course has no solutions.

12. Since the triangle must be isosceles, two sides of the triangle must have equal length. Thus, there are two possible triangles we can create. In the first case, the triangle has side lengths 18, 18, and 41. However, note that $18 + 18 < 41$, so by the triangle inequality, this triangle is not possible! Thus, the second case where the triangle has side lengths 18, 41, and 41 is the only valid case. Suppose the side of length 18 is the base of the triangle. If we draw the height of the triangle, we split our triangle into 2 right triangles. Using the Pythagorean Theorem, we find the height to be $41^2 - 9^2 = 40^2$, i.e. the height is 40. Thus, the area of the triangle is $\frac{40 \times 18}{2} = \boxed{360}$.

13. First, let's find the area of the hexagon. A regular hexagon is made up of 6 equilateral triangles that are created by drawing lines from the center of the hexagon to each vertex. Each of these 6 equilateral triangles has side length 8, so their height is $4\sqrt{3}$, and their area is $\frac{8 \times 4\sqrt{3}}{2} = 16\sqrt{3}$. Thus, the total area of the hexagon is $96\sqrt{3}$.

    Now, the area of the equilateral triangle given in the problem can be found in similar fashion. It has side length 16, so its height is $8\sqrt{3}$, and its area is $\frac{16 \times 8\sqrt{3}}{2} = 64\sqrt{3}$. The ratio of their areas is thus $\frac{96\sqrt{3}}{64\sqrt{3}} = \frac{3}{2}$. The final answer is $3 + 2 = \boxed{5}$

14. If we factor the left-hand side of the initial equation, we find that $\frac{2a^2 - 5a + 3}{2a^2 - a - 3} = \frac{(a-1)(2a-3)}{(a+1)(2a-3)}$. Now, if we cancel $(2a - 3)$ out of the numerator and demoninator of the fraction, we are left with $\frac{a-1}{a+1} = 2020$. Now, we notice that $\frac{a^2 - 2a + 1}{a^2 + 2a + 1} = (\frac{a-1}{a+1})^2$, which means that $\frac{a^2 - 2a + 1}{a^2 + 2a + 1} = 2020^2$. Since $2020^2 = 4080400$, the answer is $\boxed{16}$.

15. First, let's assume that Harry hits the dartboard. Since his dart will land on the board at a random location, we can find the likelihood it misses the bulls-eye by computing the fraction of the dartboard that is not part of the bulls-eye. The entire dartboard has radius 3, so its area is $3^2\pi = 9\pi$. The bulls-eye has radius 1, so its area is $1^2\pi = \pi$. Thus, $\frac{9\pi - \pi}{9\pi} = \frac{8}{9}$ of the board is non bulls-eye area. This means that if Harry hits the dartboard, he has a $\frac{8}{9}$ chance of missing the bulls-eye. Since there is a 20% chance Harry misses the dartboard completely, there is a $100\% - 20\% = \frac{4}{5}$ chance Harry hits the dartboard, so the overall probability that he his the dartboard but misses the bulls-eye is $\frac{4}{5} \times \frac{8}{9} = \frac{32}{45}$. Thus, the answer is $32 + 45 = \boxed{77}$.

16. In order to get rid of all fractions, we will multiply both sides by $(11 + b) \times b \times 11$. After, we get the quadratic $b^2 - 22b + 121 = 0$. We find that the only solution to this quadratic is $b = 11$, so the solution is $\boxed{11}$.

17. The way to maximize the distance between the two closest points would be to make all points equidistant from each other (convince yourself that this is true by drawing a few examples!), which means the resulting quadrilateral is a square. Since the area of the circle is 2π, we know that the radius of the circle is $\sqrt{2}$. The side length of the square formed will be 2 by Pythagorean Theorem, and therefore its area will be $\boxed{4}$

18. If you already are comfortable with using the prime factorization of a number to find how many factors it has, skip to the paragraph below :) Now, before we directly tackle this problem, let's first discuss how we know how many factors an integer has in general. For example, consider the number 60. 60 is small enough that we could just list out every factor, but let's say we want a more rigorous way of finding how many factors it has. The first step is to determine its prime factorization. In this case, $60 = 2^2 \times 3^1 \times 5^1$. Note that we explicitly wrote out the exponent of each prime factor, even though some of them are 1 and can be ignored. This is because the next step is to add 1 to each exponent. Thus, we now have three numbers: 3, 2, and 2. The last step is to multiply these numbers together to get 12, which is the number of factors of 60. Why does this method work? Once we have the prime factorization of 60, in order to create a factor of 60, we have to choose some (maybe none, maybe all) of its prime factors. Since 60 is composed of 2^2, we can either choose zero 2's, one 2, or two 2's when creating a factor of 60. Similarly, we can either choose zero 3's, or one 3. This is why we add 1 to each exponent before multiplying them together - to account for the zero case. From here, we are basically left with a combinatorics problem, so we multiply the exponent plus one terms together. This is because as a whole, when creating a factor of 60, we have 3 choices of how many 2's to pick (zero, one, or two), 2 choices of how many 3's to pick (zero or one), and 5 choices of how many 5's to pick (zero or one). With that being said, let's approach the problem now.

    We are trying to find an integer with 20 factors. In order to do this, let's figure out what types of prime factorization such a number would have. In general, numbers with 20 factors have a prime factorization with the form $p_1^{q_1} \times p_2^{q_2} \times ...$ such that $(q_1+1) \times (q_2+1) \times ... = 20$. Thus, the exponents of integers with 20 factors have a product of 20 after 1 is added to each. This means that the exponents could be, for example, 9 and 1, since $(9+1) \times (1+1) = 20$. However, since we are looking for the smallest possible integer, we want to keep these exponents as small as possible. Thus, we notice that $20 = 2 \times 2 \times 5$, which means we can use 1, 1, and 4 as our exponents. Now, we want to choose the smallest primes possible as our 3 prime bases, so we choose 2, 3, and 5. Finally, we want to give 2 the exponent of 4 in order to minimize the final integer. Thus, we find that the answer is $2^4 \times 3^1 \times 5^1 = \boxed{240}$.

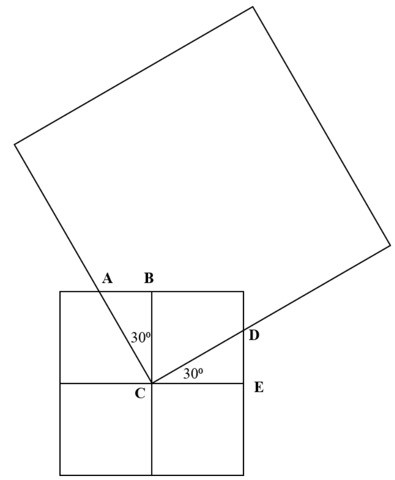
19. Since the bucket is 2 kg, when it starts off at 95% water, it contains $2 \times .95 = 1.9$ kg of water. Now, we want to add x kg more water in order to make the bucket 99% water. We can now set up the equation $\frac{1.9+x}{2+x} = 0.99$. If we multiple both sides by $(2+x)$ and combine like terms, we are left with $0.01x = 0.08$. Thus, $x = \boxed{8}$ kg.

20. First, note that there are $6^3 = 216$ outcomes when we roll three die. Now, we need to find triplets of integers between 1 and 6 inclusive whose product is 36. $36 = 2^4 \times 3^2$, so we can use the prime factorization to guide our casework. The specific cases are listed below:

    - One such triplet is $(1, 6, 6)$. This triplet corresponds to 3 distinct outcomes, depending on which dice gets the 1.
    - Another triplet is $(2, 3, 6)$. This triplet corresponds to $3! = 6$ distinct outcomes.
    - The final triplet is $(4, 3, 3)$. This triplet corresponds to 3 distinct outcomes.

    Thus, all in all, we have $3 + 6 + 3 = 12$ outcomes where the product of the three die is 36, so the final probability is $\frac{12}{216} = \frac{1}{18}$, so the answer is $1 + 18 = \boxed{19}$.

21. Refer to the diagram below. Notice that $\triangle ABC$ is congruent to $\triangle DEC$, as both are $30 - 60 - 90$ triangles, and $BC = EC$. Thus, to find the area of the overlapping region, we can imagine shifting $\triangle ABC$ so that it now occupies the area of $\triangle DEC$. Afterwards, all we have to do is the find the area of a square of side length 5. Thus, the answer is $5^2 = \boxed{25}$.



Note that no matter how much we rotate Erida by, the shared region doesn't change, thus the answer is always 25 regardless of how much Erida is rotated with respect to Desmonida.

22. We know that one of the bills in Ryan's wallet is a 50. Thus, the total value of the other 45 bills is \$61. Now, suppose Ryan has another \$50 bill in his wallet. If this were true, then the last 44 bills would amount to \$11. We see that this is not possible, as the smallest bill is \$1, so there is no way 44 bills could amount to less than \$44. Instead suppose that Ryan has a \$20 bill in his wallet. If this were true, then the last 44 bills would amount to \$41. Again, this also isn't possible. Thus, we now know that these 45 bills amounting to \$61 can only be composed of \$10, \$5, and \$1 bills.

    Now, suppose Ryan has a non-zero amount of \$10 bills. Firstly, we find that if Ryan has two \$10 bills, he will have 43 remaining bills amounting to \$41, which isn't possible. Thus, if Ryan has \$10 bills, he can only have 1. This means that Ryan has 44 \$5 and \$1 bills that amount to \$51. Unfortunately, this isn't possible (we won't write out a full explanation here, but try setting up two linear equations with two unknowns to prove it to yourself). Thus, as a whole, we now know that Ryan also cannot have any \$10 bills.

    Finally, we can conclude that Ryan has 45 \$5 and \$1 bills that amount to \$61. If we set up another system of two linear equations in two unknowns, we find that Ryan must have 4 \$5 bills and and 41 \$1 bills, in addition to the \$50 bill we were told about in the problem. Thus, of the 46 total bills, 4 of them are \$5 bills, which means that Aaron has a $\frac{4}{46} = \frac{2}{23}$ chance of pulling a \$5 bill. Thus, the answer is $2 + 23 = \boxed{25}$.

23. Since none of the lines can be parallel, whenever we add a new line, it must intersect every existing line once. Thus, if there are n such lines in a plane, then adding the next line will create exactly $n+1$ additional regions (if this doesn't intuitively make sense, try drawing a few examples!). Finally, since we start with 1 region even with no lines, we must add that in as well. Therefore, for 5 lines, there are $1 + 1 + 2 + 3 + 4 + 5 = \boxed{16}$ regions that the plane is divided into.

24. Notice that $-4\sqrt{5} = -\sqrt{80}$, and since $\sqrt{81} = 9$, $-4\sqrt{5}$ lies between -8 and -9. Next, we find that $3\sqrt[3]{9} = \sqrt[3]{243}$, and since $6^3 = 216$ and $7^3 = 343$, $3\sqrt[3]{9}$ lies between 6 and 7. Hence, the sum of integers between these numbers is $-8+-7+-6+-5+-4+-3+-2+-1+1+2+3+4+5+6 = -8+-7 = \boxed{-15}$.

25. Note 25! by definition contains all numbers between 1 and 25 as factors, and we need to make sure that a has no prime factors in common with b in order for $\frac{a}{b}$ to be in simplest form. Note that the prime factors between 1 and 25 are $2, 3, 5, 7, 11, 13, 17, 19, 23$. We essentially need to split these prime factors into the numerator and denominators, or into two sets. Thus, for each of the 9 prime factors, we must choose one of the two sets. Thus, there are $2^9 = \boxed{512}$ ordered pairs we can make.

26. The number must be divisible by 4, 3, and 11. To be divisible by 11, we need the alternating sum of digits to be divisible by 11, which means $1 - a + 4 - b + 4 - c = 9 - (a + b + c)$ is a multiple of 11. Since the maximum value of $a + b + c$ is 27, this limits the value of $a + b + c$ to 9 or 20. To be divisible by 3, we need the sum of digits to be divisible by 3, so $1 + 4 + 4 + a + b + c$ is a multiple of 3, which means $a + b + c$ is a multiple of 3. The only possible value of $a + b + c$ is therefore 9. To be divisible by 4, we need $4c$ to be divisible by 4. This means $c = 4$ or 8. If $c = 8$, then $a + b = 1$, which means either a or b must be 0, so the product $a \times b \times c$ will be 0.

    In the next case, $c = 4$, and thus $a + b = 5$. The maximal value of $a \times b$ is $2 \times 3 = 6$. Therefore the maximum value of $a \times b \times c$ is $2 \times 3 \times 4 = \boxed{24}$.

    For proofs of the divisibility rules discussed above, take a look at this link: `https://artofproblemsolving.com/wiki/index.php/Divisibility_rules`.

27. We can divide this problem into two sub-problems: traversing the 12 edges of the octahedron and traversing the 6 edges of the tetrahedron. If we start at the vertex of the octahedron tangent to the tetrahedron, then we can traverse all 12 edges of the octahedron and return to the same vertex without repeating any edges. One way to to do this is to break the 12 edges into 2 sets of 4 diagonal edges and another set of 4 edges which form a square in the middle. It's easy to see that one can traverse 4 diagonal edges and return to the same vertex without repeating any edges, and traverse another 3 edges on the second set of diagonal edges, before traversing the four edges which form a square, and finally complete the full cycle with the last diagonal edge.

    For the tetrahedron, one needs to see that it is impossible to traverse all 6 edges without repetition. This is because each vertex of the tetrahedron has an odd degree, meaning that there is an odd number of edges connected to it. Since every edge passing through a vertex must enter on one side and exit on the other side, a path visiting every edge with no repetitions can have at most 2 vertices of odd degrees, in which case one is the starting point and one is the ending point. Therefore, the minimum distance Rob has to travel on the tetrahedron is 7 edges, which means the final answer is $7 + 12 = \boxed{19}$.

28. We can factor the left hand side of the equation to be $(a + 1)(b + 1)(c + 1) - 1 = 1000$. Thus, we find that $(a + 1)(b + 1)(c + 1) = 1001$. Now, we need to find the factors of 1001. The prime factorization of 1001 is $7 \times 11 \times 13$. Thus, it must be the case that one of a, b, c is 6, another is 10, and the last is 12. This means that there are a total of $3! = \boxed{6}$ solutions. Note that none of $(a + 1)$, $(b + 1)$, and $(c + 1)$ can be 1, as that would make a, b, or c non-positive.

29. We have the following equations: $441_b = n^2$, and $351_b = (n - 2)^2$. From $441_b = n^2$, we can expand to find $4b^2 + 4b + 1 = n^2$. Thus, $(2b + 1)^2 = n^2$, which means that $2b + 1 = n$. Now, in order to substitute into the second equation, we need to find a different expression for $(n - 2)^2$. Using the fact that $2b + 1 = n$, we find that $(n - 2)^2 = (2b - 1)^2$. Thus, we can set up the equation $3b^2 + 5b + 1 = (2b - 1)^2$. This quadratic has two solutions: 0 and 9. However, since base 0 isn't possible, the answer must be $\boxed{9}$.

30. Note a number is a multiple of 3 if and only if the sum of the digits of the number is a multiple of 3. Thus, let's split into cases based on the sum of the digits of the multiples of 3 we wish to count.

- **Case 1:** The digits sum to 3.
  - $\{1, 2\} \Rightarrow 2$ multiples.
  - $\{3\} \Rightarrow 1$ multiple.
- **Case 2:** The digits sum to 6.
  - $\{4, 2\} \Rightarrow 2$ multiples.
  - $\{4, 1, 1\} \Rightarrow 3$ multiples.
  - $\{3, 3\} \Rightarrow 1$ multiple.
  - $\{3, 2, 1\} \Rightarrow 6$ multiples.
  - $\{2, 2, 1, 1\} \Rightarrow 6$ multiples.
- **Case 3:** The digits sum to 9.
  - $\{4, 4, 1\} \Rightarrow 3$ multiples.
  - $\{4, 3, 2\} \Rightarrow 6$ multiples.
  - $\{4, 3, 1, 1\} \Rightarrow 12$ multiples.
  - $\{4, 2, 2, 1\} \Rightarrow 12$ multiples.
  - $\{3, 3, 2, 1\} \Rightarrow 12$ multiples
  - $\{3, 2, 2, 1, 1\} \Rightarrow 30$ multiples
- **Case 4:** The digits sum to 12.
  - $\{4, 4, 3, 1\} \Rightarrow 12$ multiples.
  - $\{4, 4, 2, 2\} \Rightarrow 6$ multiples.
  - $\{4, 4, 2, 1, 1\} \Rightarrow 30$ multiples.
  - $\{4, 3, 3, 2\} \Rightarrow 12$ multiples.
  - $\{4, 3, 3, 1, 1\} \Rightarrow 30$ multiples.
  - $\{4, 3, 2, 2, 1\} \Rightarrow 60$ multiples.
  - $\{3, 3, 2, 2, 1, 1\} \Rightarrow 90$ multiples
- **Case 5:** The digits sum to 15.
  - $\{4, 4, 3, 3, 1\} \Rightarrow 30$ multiples.
  - $\{4, 4, 3, 2, 1, 1\} \Rightarrow 180$ multiples.
  - $\{4, 4, 3, 2, 2\} \Rightarrow 30$ multiples.
  - $\{4, 3, 3, 2, 2, 1\} \Rightarrow 180$ multiples.
- **Case 6:** The digits sum to 18.
  - $\{4, 4, 3, 3, 2, 2\} \Rightarrow 90$ multiples.
  - $\{4, 4, 3, 3, 2, 1, 1\} \Rightarrow 630$ multiples.

In the table above, we ignore the part of the problem that states multiples of 3 only count if they have 6 or less digits. Considering this allows us to skip the second sub-case of Case 6. If we add up all the valid sub-cases, we have a total of $2+1+2+3+1+6+6+3+6+12+12+12+30+12+6+30+12+30+60+90+30+180+30+180+90 = \boxed{846}$ total multiples.