Johns Hopkins Math Tournament 2020

Individual Round: Geometry

February 8, 2020

Instructions

- <u>DO NOT</u> **TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 12 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. In a country named Fillip, there are three major cities called Alenda, Breda, Chenida. This country uses the unit of "FP". The distance between Alenda and Chenida is 100 FP. Breda is 70 FP from Alenda and 30 FP from Chenida. Let us say that we take a road trip from Alenda to Chenida. After 2 hours of driving, we are currently at 50 FP away from Alenda and 50 FP away from Chenida. How many FP are we away from Breda?

2. Let $\triangle XYZ$ be a triangle such that $\angle X = 70°$. There exists a point F inside triangle $\triangle XYZ$ such that YF bisects $\angle XYZ$ and ZF bisects $\angle XZY$. What is the measure of $\angle YFZ$?

3. Consider a right cylinder with height $5\sqrt{3}$. A plane intersects each of the bases of the cylinder at exactly one point, and the cylindric section (the intersection of the plane and the cylinder) forms an ellipse. Find the product of the sum and the difference of the lengths of the major and minor axes of this ellipse.

   Note: An ellipse is a regular oval shape resulting when a cone is cut by an oblique plane which does not intersect the base. The major axis is the longer diameter and the minor axis the shorter.

4. Quadrilateral ABCD is inscribed in a circle of radius 6. If $\angle BDA = 40°$ and $AD = 6$, what is the measure of $\angle BAD$ in degrees?

5. Let A and B be fixed points in the Euclidean plane with $AB = 6$. Let $\mathcal{R}$ be the region of points in the plane such that, for each $P \in \mathcal{R}$, there exists a point C such that $AC = 3$ and P does not lie outside $\triangle ABC$. Compute the greatest integer less than or equal to the area of $\mathcal{R}$.

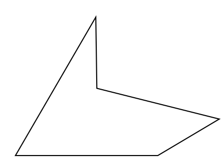
6. Triangle ABC has $\angle A = 60°$, $\angle B = 45$, and $AC = 6$. Let D be on $\overline{AB}$ such that $AD = 3$. There is exactly one point E on BC such that $\overline{DE}$ divides ABC into two cyclic polygons. Compute DE^2.

7. Quadrilateral $ABCD$ is cyclic and has positive integer side lengths. Suppose $AC \cdot BD = 53$ and $CD < DA$. The value of $\frac{AB/BC}{AD/DC}$ can be expressed as a common fraction $\frac{p}{q}$, where p and q ae relatively prime. Compute $p + q$.

8. Triangle $\triangle ABC$ has $\text{m}\angle C = 135°$, and D is the foot of the altitude from C to $\overline{AB}$. We are told that $CD = 2$ and that AD and BD are finite positive integers. What is the sum of all distinct possible values of AB?

9. Two points J and H lie 26 units apart on a given plane. Let M be the locus of points T on this plane such that $JT^2 + HT^2 = 2020$. Then, M encloses a region on the plane with area a and perimeter p. If q and r are coprime positive integers and $\frac{a}{p} = \frac{q}{r}$, then compute $q + r$.

10. Concave pentagon $ABCDE$ has a reflex angle at D, with $\text{m}\angle EDC = 255°$. We are also told that $BC = DE$, $\text{m}\angle BCD = 45°$, $CD = 13$, $AB + AE = 29$, and $\text{m}\angle BAE = 60°$. The area of $ABCDE$ can be expressed in simplest radical form as $a\sqrt{b}$. Compute $a + b$.



11. The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ satisfies the property $\varphi^2 = \varphi + 1$. Point P lies inside equilateral triangle $\triangle ABC$ such that $PA = \varphi$, $PB = 2$, and angle $\angle APC$ measures 150 degrees. What is the measure of $\angle BPC$ in degrees?

12. Circle O is inscribed inside a non-isosceles trapezoid $JHMT$, tangent to all four of its sides. The longer of the two parallel sides of $JHMT$ is $\overline{JH}$ and has a length of 24 units. Let P be the point where O is tangent to $\overline{JH}$, and let Q be the point where O is tangent to $\overline{MT}$. The circumcircle of $\triangle JQH$ intersects O a second time at point R. $\overleftrightarrow{QR}$ intersects $\overleftrightarrow{JH}$ at point S, 35 units away from P. The points inside $JHMT$ at which $\overline{JQ}$ and $\overline{HQ}$ intersect O lie $\frac{63}{4}$ units apart. The area of O can be expressed as $\frac{m\pi}{n}$, where $\frac{m}{n}$ is a common fraction. Compute $m + n$.

Geometry Solutions

1. The cities fall in a straight line, and we are at the midpoint of Alenda and Chenida. The distance from the midpoint to Breda is $|50-30| = |50-70| = \boxed{20}$ FP.

2. Since the measure of $\angle X$ is $70°$, we know that the sum of $\angle Y$ and $\angle Z$ is $110°$. Since YF and ZF bisect $\angle Y$ and $\angle Z$ respectively, we also know that the sum of $\angle FYZ$ and $\angle FZY$ is $110°/2 = 55°$. Thus, the measure of $\angle YFZ$ will be $180° - 55° = \boxed{125°}$.

3. Let the major and minor axes of this ellipse be denoted m and n. We wish to find $(m+n)(m-n) = m^2 - n^2$. The minor axis is equal to the diameter of each base, so $n = 2r$. By the Pythagorean Theorem, we have $m^2 = (2r)^2 + h^2$. Squaring the first equation and subtracting it from the second, we get $m^2 - n^2 = (2r)^2 + h^2 - (2r)^2 = h^2 = \boxed{75}$.

4. Label the center of the circle as Point O. Observe that the chord AD has the same length as the radius of the circle. Therefore, we can draw an equilateral triangle AOD by connecting points A and D to point O. Therefore, $\angle AOD = 60°$, resulting in our minor arc of $AD = 60°$. Similarly, since $\angle ADB = 40°$, minor arc $AB = 80°$. Therefore, the major arc of $BD = 360° - 80° - 60° = 220°$, resulting in $\angle BAD = 220°/2 = \boxed{110°}$.

5. Consider the set of points P corresponding to each possible location for the C. We wish to find the union of all of these points P. Let O be the circle of radius 3 centered around A. C must lie on O, and the possible locations for P are the interior and edges of $\triangle ABC$. Segment $\overline{AC}$ traces out the interior of O. Outside of O, $\triangle ABC$ also contains the area bounded by $\overline{AB}$, $\overline{BC}$, and the edge of O. This area increases strictly with $\angle ABC$, which is maximized when $\overline{BC}$ is tangent to O. Call the two points of tangency D and E (for the two possible tangents). $\angle ABC$ cannot be larger than $\angle ADB$ (or $\angle AEB$), or else $AC > 3$. Notice that this tangent case results in two $30-60-90$ triangles with side lengths 3, $3\sqrt{3}$, and 6. The total area of these two triangles and the circle is $\frac{2}{3}\pi 3^2 + 2 * 3 * 3\sqrt{3}/2 = 6\pi + 9\sqrt{3}$. Approximating $6\pi + 9\sqrt{3}$ as $6 * 3.14 + 9 * 1.73 = 34.41$ (the actual value is about 34.438), our answer is $\boxed{34}$.

6. $\overline{DE}$ divides $\triangle ABC$ into a quadrilateral and a triangle, the latter of which is automatically cyclic. Since quadrilateral ACED must be cyclic, $\angle CED = 180° - 60° = 120°$. Observing that $\triangle ACD$ is a $30° - 60° - 90°$ triangle, we know that $CD = 3\sqrt{3}$. Construct point F on the midpoint of $\overline{AC}$ so that $AF = CF = 3$. $\triangle ADF$ is an equilateral triangle, so $DF = 3$ and $\angle AFD = 60°$. Because $CF = DF$, $\triangle CDF$ is an isosceles triangle. $\angle CFD = 180° - \angle AFD = 120°$. Therefore, $\angle CDF = \angle DCF = 30°$. $\angle ECD = \angle BCA - \angle DCF = 45°$. Using the law of sines on $\triangle CED$, we get

$$\frac{3\sqrt{3}}{\sin(120°)} = \frac{DE}{\sin(45°)}$$

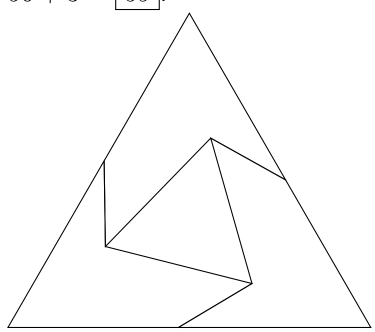
Solving, we get $DE = 3\sqrt{2}$, so $DE^2 = \boxed{18}$.

7. Rewrite $\frac{AB/BC}{AD/DC}$ as $\frac{AB \cdot CD}{AD \cdot BC}$. Because $ABCD$ is cyclic, Ptolemy's theorem states that $AB \cdot CD + AD \cdot BC = AC \cdot BD = 53$. Because $ABCD$ has positive integer side lengths and because 53 is prime, $AB \cdot CD$ and $AD \cdot BC$ must be relatively prime positive integers. Then, if we let $p = AB \cdot CD$ and $q = AD \cdot BC$, then $\frac{p}{q}$ is the common fraction representation of $\frac{AB/BC}{AD/DC}$, so the answer is $p + q = AB \cdot CD + AD \cdot BC = AC \cdot BD = \boxed{53}$.

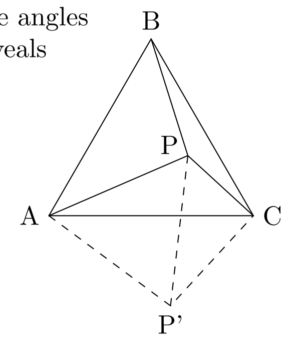
8. Let $\angle BAC = \theta$ and $\angle ABC = 45° - \theta$. We have $\tan(\theta) = \frac{2}{AD}$ and $\tan(45° - \theta) = \frac{2}{BD}$, so $AD = \frac{2}{\tan(\theta)}$ and $BD = 2\cot(45° - \theta) = 2(\frac{1+\tan(45°)\tan(\theta)}{\tan(45°)-\tan(\theta)}) = 2(\frac{1+\tan(\theta)}{1-\tan(\theta)})$. Since $\theta \in (0, 45°)$, we know $\tan(\theta) \in (0, 1)$. Since $\tan(\theta) = \frac{2}{AD}$ and AD must be an integer, $\tan(\theta)$ must take the form $\frac{2}{x}$ for some integer x. Since $\tan(\theta) \in (0, 1)$, x must be at least 3. From here, we can test values of $\tan(\theta)$. Without loss of generality, assume $AD \leq BD$. Plugging $x = 3$ and $x = 4$ into our equations for $\tan(\theta)$, AD, and BD give $AD = 3, BD = 10$ and $AD = 4, BD = 6$. When $\tan(\theta) \geq 5$, $AD > BD$. Thus, adding up the two possibilities of AB, we get a total of $(3 + 10) + (4 + 6) = \boxed{23}$.

9. Without loss of generality, let J have the coordinates $(-13, 0)$ and H have the coordinates $(13, 0)$. Let T have the coordinates (x, y). By the Pythagorean Theorem, $JT^2 = (x+13)^2 + y^2$ and $HT^2 = (x-13)^2 + y^2$. We know that $JT^2 + HT^2 = (x+13)^2 + y^2 + (x-13)^2 + y^2 = 2x^2 + 2y^2 + 2 * 13^2 = 2020$, so $x^2 + y^2 = 841$. This is the equation of a circle with radius 29 centered around the origin. Thus, M has area $a = 841\pi$ and perimeter 58π. $\frac{a}{p} = \frac{841\pi}{58\pi} = \frac{29}{2}$, so our answer is $29 + 2 = \boxed{31}$.

10. Observe the pentagon as it fits in the following figure. The area we are looking for is one-third of the difference in areas of the equilateral triangles, which is $\frac{29^2\sqrt{3}/4 - 13^2\sqrt{3}/4}{3} = 56\sqrt{3}$. The answer is $56 + 3 = \boxed{59}$.



11. A more general formulation of the problem is to let $PA = a$, $PB = b$, $PC = c$, $\angle BPC = \alpha$, $\angle CPA = \beta$, and $\angle APB = \gamma$. We will prove an elegant result of this generalization: as long as P is inside $\triangle ABC$, there exists a (unique) triangle with sides of length a, b, and c whose opposite angles measure $\alpha - 60°$, $\beta - 60°$, and $\gamma - 60°$, respectively. The clever trick that reveals this result is a $60°$ rotation of $\triangle APB$ around point B that sends A to C, as shown in the diagram. Let P' be the location of P after the rotation. Then, $P'C = PA = a$, and $P'B = PB = b$. Because $\angle PBP'$ measures 60 degrees, $\triangle PBP'$ is equilateral, meaning $PP' = b$. Therefore, $\triangle PCP'$ has sides $P'C = a$, $PP' = b$, and $PC = c$. To compute the angle measures of $\triangle PCP'$, note that $\text{m}\angle CP'P = \text{m}\angle CP'B - \text{m}\angle BP'P = \text{m}\angle APB - 60° = \gamma - 60°$. Next, observe that $\text{m}\angle CPP' = \text{m}\angle CPB - \text{m}\angle BPP' = \alpha - 60°$. Finally, because $\alpha + \beta + \gamma = 360°$, the remaining angle $\angle PCP'$ in $\triangle PCP'$ must measure $\beta - 60°$ in order for the three vertex angles to sum to $180°$.



The Law of Sines on $\triangle PCP'$ tells us that

$$\frac{a}{\sin(\alpha - 60°)} = \frac{b}{\sin(\beta - 60°)} = \frac{c}{\sin(\gamma - 60°)},$$

from which we can solve for angle measures or side lengths. In our particular problem, we are given $a = \varphi$, $b = 2$, and $\beta = 150°$, and we want to solve for α. Substitute the given values into the law of sines equation above:

$$\frac{\varphi}{\sin(\alpha - 60°)} = \frac{2}{\sin(150° - 60°)} = \frac{2}{1} \implies \alpha = 60° + \arcsin\left(\frac{\varphi}{2}\right).$$

What remains is for us to compute $\arcsin(\varphi/2)$. A well known result in geometry states that a regular pentagon with sides of length s has diagonals of length $\varphi \cdot s$. Because the vertex angle of a regular pentagon measures $108°$, an isosceles triangle with side lengths s, s, and $\varphi \cdot s$ has angles measuring $108°$, $36°$, and $36°$. This triangle may be halved into two congruent $36°$-$54°$-$90°$ right triangles, revealing that $\sin 54° = \frac{\varphi}{2}$. Hence, $\alpha = 60° + 54° = \boxed{114°}$. In general, when a trigonometric ratio involves φ, it is a good idea to think of angles related to regular pentagons, specifically integer multiples of $18°$.

12. Let $\overline{JQ}$ intersect O again at X, and let $\overline{HQ}$ intersect O again at Y. Because $\overline{JH}$ and $\overline{MT}$ are parallel sides, $\overline{PQ}$ is a diameter of O, and we will let its length be d. By the Pythagorean Theorem, $JQ^2 - JP^2 = HQ^2 - HP^2 = d^2$. We apply power of a point to J and H with respect to circle O:

$$JX \cdot JQ = JP^2 \implies (JQ - QX)JQ = JP^2 \implies QX \cdot QJ = JQ^2 - JP^2 = d^2;$$

$$HY \cdot HQ = HP^2 \implies (HQ - QX)HQ = HP^2 \implies QY \cdot QH = HQ^2 - HP^2 = d^2.$$

Because $QX \cdot QJ = QY \cdot QH$, quadrilateral $JHYX$ is cyclic, so we will let Ω be its circumcircle. Let Γ be the circumcircle of $\triangle JQH$, which intersects O again at R (by definition of R). Observe that the three pairs of circles from the triple (O, Ω, Γ) produce the three radical axes $\overleftrightarrow{JH}$, $\overleftrightarrow{XY}$, and $\overleftrightarrow{QR}$. We are told that $\overleftrightarrow{JH}$ and $\overleftrightarrow{QR}$ intersect at S and that $SP = 35$. By the Radical Axis theorem, $\overleftrightarrow{XY}$ also passes through S. From this point onward, we do not need to use R, M, T, or Γ. Because S lies on the radical axis of O and Ω, the power of point S with respect to these two circles is the same. Because S is on a line that is tangent to O at point P, the power of S equals SP^2, or 35^2. Without loss of generality, suppose S lies closer to J than H. Then, $SH - SJ = 24$ and $SY - SX = \frac{63}{4}$. Because the power of S with respect to both O and Ω is 35^2,

$$35^2 = SJ \cdot SH = SJ(SJ + 24) \implies SJ = 25, \text{ and}$$

$$35^2 = SX \cdot SY = SX\left(SX + \frac{63}{4}\right) \implies SX = 28.$$

The first result further yields $JP = 10$ and $HP = 14$. We apply Menelaus' theorem to $\triangle SYH$ and the transversal $\overleftrightarrow{JXQ}$:

$$\frac{HQ}{QY} \cdot \frac{63/4}{28} \cdot \frac{25}{24} = 1 \implies \frac{HQ}{QY} = \frac{128}{75}.$$

The power of point H with respect to circle O is $HP^2 = 14^2$; this power is equal to $HY \cdot HQ$, so

$$14^2 = (HQ - QY)HQ = HQ^2\left(1 - \frac{75}{128}\right) \implies HQ^2 = \frac{14^2 \cdot 128}{53}.$$

By the Pythagorean theorem, $d^2 = HQ^2 - HP^2 = 14^2\left(\frac{128}{53} - 1\right) = \frac{14^2 \cdot 3 \cdot 5^2}{53}$. The area of O is $\frac{\pi d^2}{4} = \frac{3 \cdot 35^2 \cdot \pi}{53} = \frac{3675\pi}{53}$, so the answer is $3675 + 53 = \boxed{3728}$.