Johns Hopkins Math Tournament 2020

Individual Round: General

February 8, 2020

Instructions

- **<u>DO NOT</u> TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 12 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. Simplify $(1+\frac{1}{2020})\times(1+\frac{1}{2019})\times(1+\frac{1}{2018})\times ... \times(1+\frac{1}{2})(1+\frac{1}{1})$.

2. There are 12 students in a math class. For Pi Day, the class decides that at least 2 students will wear a red shirt, and at least 8 students will wear a blue shirt. How many different ways can the 12 students choose their shirt colors?

3. The sum of the 12 edges lengths of a rectangular prism is equal to 24, and the sum of the areas of the 6 faces of the rectangular prism is equal to 12. What is the sum of squares of its 12 edge lengths?

4. Two letters have fallen off a sign that reads $HOMEWOOD$. A child notices this, but since the child is still learning how to spell, they simply put the letters back into the two empty positions at random. The probability that the sign still reads $HOMEWOOD$ afterwards can be expressed as $\frac{a}{b}$, where a and b are co-prime. What is $a+b$?

5. Quadrilateral $EFGH$ has the following properties: $\angle E = 90°$, $\angle F = 60°$, $\overline{EF} = \overline{FG} = \overline{EH}$. Find $\angle H$ in degrees.

6. x is a positive integer such that $\frac{x}{9}$ is a two-digit integer and $x + 110$ is a four-digit integer. Find x.

7. Alice and Bob are on **opposite sides** a circular track field of circumference 400m. They both run along the track towards each other (i.e. in opposite directions) with constant but different speed. They first cross each other when Alice is 80m away from her start. When they meet again, what is the difference in the total distances they traveled in meters?

8. A regular hexagon is inscribed in a circle. Consider all triangles formed by picking 3 vertices of the hexagon to be vertices of the triangle. Of all such triangles, how many contain at least $\frac{3\sqrt{3}}{4\pi}$ of the area of the circle?

9. Suppose a right triangle has side lengths 1, x, ,and x^2. Assume $x > 1$. The value x can be expressed in the form $\sqrt{\frac{a+\sqrt{b}}{c}}$ where b is expressed in simplest radical form. What is $a+b+c$?

10. Ashley and Rebecca each choose a number uniformly at random between 0 and 1. The probability that Rebecca's number is at least 0.3 more than Ashley's can be expressed as $\frac{p}{q}$, where p and q are co-prime. What is $p+q$?

11. Suppose we have some positive integer m. Denote its unique prime factorization as $m = {p_1}^{n_1} \times p_2^{n_2} \times p_3^{n_3} \ldots$, where $p_1, p_2, \ldots$ are unique prime factors of m and $n_1, n_2, \ldots$ are positive integers. How many choices of m exist such that $n_1 \times p_1 + n_2 \times p_2 + n_3 \times p_3 \cdots = 16$?

12. Let there be a right triangle ABC where $AB = 5$, $CA = 12$, and $BC = 13$. Let S be the center of the circle that is inscribed inside the triangle. The circle intersects $\overline{BC}$ at point P. Let the continuation of $\overline{AS}$ intersect with $\overline{BC}$ on point K. Find the value of $17\overline{PK}$.

General Solutions

1. If we combine the terms inside each pair of parentheses, we are left with $\frac{2021}{2020} \times \frac{2020}{2019} \times ... \times \frac{3}{2} \times \frac{2}{1}$. Notice that this product is equivalent to $\frac{2021!}{2020!} = \boxed{2021}$.

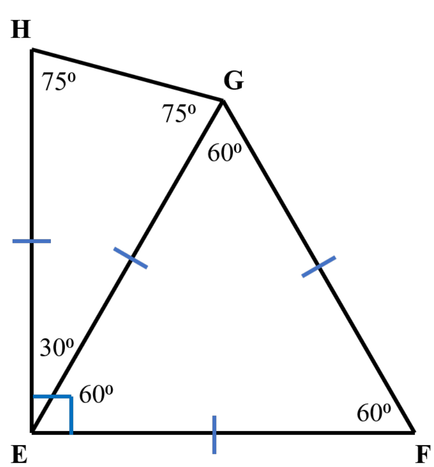
2. There are 3 cases to consider, based on how many total students wear a blue/red shirt. In the first case, 2 students wear a red shirt, and 10 wear a blue shirt. There are $\binom{12}{2}$ ways of assigning the shirts to students in this first case. In the second case, 3 students wear a red shirt, and 9 wear a blue shirt. There are $\binom{12}{3}$ ways of assigning the shirts to students in the second case. In the final case, 4 students wear a red shirt, and 8 students wear a blue shirt. There are $\binom{12}{4}$ ways of assigning the shirts to students in the final case. If we add these three cases together, we get the total count: $\binom{12}{2} + \binom{12}{3} + \binom{12}{4} = 66 + 220 + 495 = \boxed{781}$.

3. Let the height, width, and length of the rectangular prism be denoted by the variables h, w, and l respectively. Since we are told the sum of the 12 edges is 24, we can create the equation $4h+4w+4l = 24$, or $h + w + l = 6$. Furthermore, since the sum of the area of the faces is 12, we can create the equation $2hw + 2hl + 2wl = 12$. Now, we are asked to find the sum of the squares of the 12 edge lengths, which can be expressed as $4h^2 + 4w^2 + 4l^2$. To do this, we first square both sides of the first equation, and see that $(h + w + l)^2 = 6^2$. Simplify to yield the equation $h^2 + w^2 + l^2 + 2hw + 2hl + 2wl = 36$. Now, we can subtract the second equation we created to be left with only the squares of edge lengths. In other words, we can create the equation $h^2 + w^2 + l^2 = 24$. Now, we can multiply this equation by 4 to get the desired answer expression, which is $24 \times 4 = \boxed{96}$.

4. Answer: Note there are $\binom{8}{2} = \frac{8!}{6!2!} = 28$ ways to choose two letters from the word $HOMEWOOD$. In the $\binom{3}{2} = 3$ cases in which 2 of the O's fall off, putting the letters back in any order will result in a correct sign. In the other 25 cases, there will be a $\frac{1}{2}$ chance the letters are put back correctly. We add these probabilities up, $\frac{3}{28} \times 1 + \frac{25}{28} \times \frac{1}{2} = \frac{31}{56}$. Thus, the answer is $31 + 56 = \boxed{87}$.

5. Since $\overline{FE} = \overline{FG}$, $\triangle FGE$ is an isosceles triangle, meaning $\angle FGE = \angle GEF$. Since the 3 angles of any triangle must add up to 180° and $\angle F = 60°$, for $\triangle FGE$, we can set up the equation $60° + \angle FGE + \angle GEF = 60° + 2 \times \angle GEF = 180°$. Thus, we find that $\angle GEF = 60°$.

   We now find that $\angle GEH = 90° - 60° = 30°$. Then, as $\overline{GE} = \overline{EH}$, $\triangle GEH$ is an isosceles triangle, which means that $\angle H = \angle G$. Again, since the 3 angles of any triangle must add up to 180°, for $\triangle GEH$, we can set up the equation $30° + \angle H + \angle G = 30° + 2 \times \angle H = 180°$, and thus $\angle H = \boxed{75°}$. Refer to the figure below for more clarity.

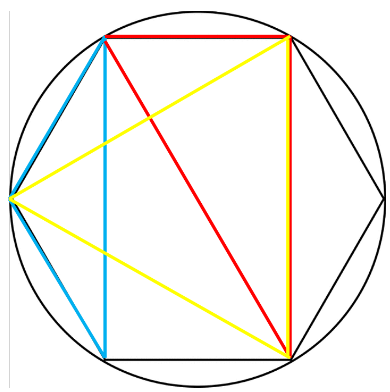


6. Answer: Note that saying $\frac{x}{9}$ is a 2 digit number is equivalent to saying $10 \leq \frac{x}{9} \leq 99$. Saying $x + 110$ is a four digit number is equivalent to saying $1000 \leq x + 110 \leq 9999$. Putting this in terms of x gives us $90 \leq x \leq 891$ and $890 \leq x \leq 9889$. Combining the right side of the first inequality with the left side of the second inequality gives us $890 \leq x \leq 891$, meaning that x is either 890 or 891. Since $\frac{x}{9}$ is an integer, x is divisible by 9, thus $x = \boxed{891}$.

7. Since Alice and bob start on opposite sides of a 400 meter track, at the start, they are 200 meters apart. Thus, when they first meet, Alice has traveled 80 meters while Bob has traveled $200 - 80 = 120$ meters. Thus, the speed ratio for Alice to Bob is $80 : 120 \to 2 : 3$.

   After meeting the first time, they will need to travel a combined distance of 400 meters in order to meet again. Suppose a is the distance Alice travels after the first meeting, and b is the distance Bob travels after the first meeting. We find that $\frac{a}{b} = \frac{2}{3}$ and $a + b = 400$. If we solve this system of linear equations, we see that $a = 160$ meters and $b = 240$ meters. Thus, the total distance Alice has traveled is $80 + 160 = 240$ meters, and the total distance bob has traveled is $120 + 240 = 360$ meters. Thus the difference between their traveled distance is $360 - 240 = \boxed{120}$ meters.

8. There are 3 different possible triangle shapes, as shown in the figure below in red, blue, and yellow. Note that all other triangles we can make by choosing vertices of the hexagon are equivalent to one of these 3 triangle shapes. Now, let us calculate the area of the largest triangle (show in yellow). As one edge's length is r, the height is $\frac{\sqrt{3}r}{2}$. Thus the area is $\frac{\sqrt{3}r^2}{4}$. As the area of the circle is πr^2, the equilateral triangle is $\frac{\sqrt{3}}{4\pi}$ of the area of the circle. Thus the only triangle that contains at least $\frac{\sqrt{3^3}}{4\pi}$ of the circle is this largest equilateral triangle. We see that there are only 2 possible equilateral triangles that can be made, thus the answer is $\boxed{2}$.

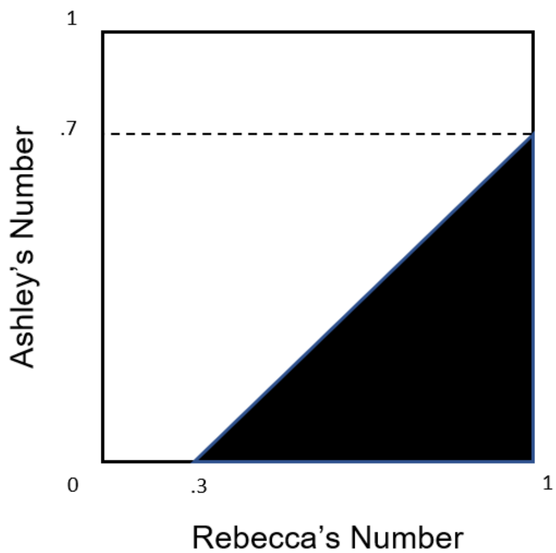


9. There are 3 cases to consider, depending on which of these values we choose to be the hypotenuse length. In the first case, suppose 1 is the length of the hypotenuse, and x and x^2 are the lengths of the two legs. Thus, using the Pythagorean Theorem, we can set up the equation $x^4 + x^2 = 1$. Now, let $a = x^2$. By substitute, we can turn these equation into $a^2 + a - 1 = 0$, which is a quadratic that we know how to solve. Using the quadratic formula, we find that the positive solution for a is $a = \frac{-1+\sqrt{5}}{2}$, meaning that $x = \sqrt{\frac{-1+\sqrt{5}}{2}}$. However, since $-1 + \sqrt{5} < 2$, we see that this value will be less than 1, so it does not satisfy the conditions given in the problem as a possible solution.

   In the second case, suppose x is the length of the hypotenuse, and 1 and x^2 are the lengths of the two legs. Thus, using the same substitution trick as above, we can now create the equation $a^2 - a + 1 = 0$. Unfortunately, when we try to solve this equation, we find that no real solutions for a exist, so these case comes up at a dead end as well.

   In the final case, suppose that x^2 is the length of the hypotenuse, and 1 and x are the lengths of the two legs. Thus, again using the same substitution trick as above, we create the equation $a^2 - a - 1 = 0$. Using the quadratic formula again, we find the positive solution for a is $a = \frac{1+\sqrt{5}}{2}$, meaning $x = \sqrt{\frac{1+\sqrt{5}}{2}}$. Fortunately, we see that since $1+\sqrt{5} > 2$, this solution for x is indeed greater than 1, so it is a valid solution. Thus, the answer is $1+5+2 = \boxed{8}$.

10. We can solve this problem using geometric probability. In the diagram below, for each point (x, y) in the unit square, the x coordinate denotes the number Rebecca picks and the y coordinate denotes the number Ashley picks. Notice that the unit square will thus contain every possible pair of numbers Rebecca and Ashley can choose. Now, to actually solve the problem, we need to figure out which points in this unit square denote cases where Rebecca's number is at least 0.3 more than Ashley's. The shaded region in the figure below shows precisely these points. Finally, the probability of this occurring would be given by the area of the shaded region divided by the total area of 1. Thus the probability is $\left(\frac{7}{10}\right)^2 \times \frac{1}{2} = \frac{49}{200}$, meaning the answer is $49+200 = \boxed{249}$.



11. The problem is equivalent to asking the number of non-negative integer solutions to $2n_1 + 3n_2 + 5n_3 + 7n_4 + 11n_5 + 13n_6 = 16$, where we are filling in p_1, p_2, p_3, and p_4 with 2, 3, 5, 7, 11, and 13 respectively. Note that we do not consider primes greater than 13 because the next smallest prime is 17, and as 17 is already greater than 16, it must be the case that $n_i = 0$ for any $i > 6$.

    Now, we must actually count the solutions to the equation above. One such method is to consider cases based on the largest prime factor of m. Since the largest prime factor may not be greater than 16, we start with 13. The only way to add a prime to 13 and get 16 is $13+3$, so we have 1 solution in the first case. Next, if the greatest prime is 11, then we may add either $2+3$ or 5 to it to get 16, so we have 2 solutions in this case. Next, if the greatest prime is 7 then we may add $7+2$, $5+2+2$, $3+3+3$, or $3+2+2+2$, so there are 4 solutions in this case. Next, if the greatest prime is 5, we may add $5+3+3$, $5+2+2+2$, $3+3+3+2$, or $3+2+2+2+2$, so there are 4 solutions in this case. Next, if the greatest prime is 3, then we may add $3+3+3+2+2$, or $3+2+2+2+2+2$, so there are 2 solutions in this case. Finally, if the greatest prime is 2, then we can only have eight 2's to get to 16, so there is 1 solution in the final case. This totals to $1+2+4+4+2+1 = \boxed{14}$ total solutions, and therefore 14 choices of m. Note that each of the 14 solutions results in a unique m because by definition, each solution represents a unique prime factorization, and we know from the Fundamental Law of Arithmetic that no two differing prime factorizations can yield the same number. Each of the 14 solutions is also listed below for your convenience :)

- $3 \times 13 = 39$
- $5 \times 11 = 55$
- $2 \times 3 \times 11 = 66$
- $7^2 \times 2 = 98$
- $7 \times 5 \times 2^2 = 140$
- $7 \times 3^3 = 189$
- $7 \times 3 \times 2^3 = 168$
- $5^2 \times 3^2 = 225$
- $5^2 \times 2^3 = 200$
- $5 \times 3^3 \times 2 = 270$
- $5 \times 3 \times 2^4 = 240$
- $3^4 \times 2^2 = 324$
- $3^2 \times 2^5 = 288$
- $2^8 = 256$

12. First, recall that the radius of a circle inscribed inside a triangle can be found using the formula $r \times (\text{perimeter of } \triangle ABC) = 2 \times (\text{area of } \triangle ABC)$. Furthermore, from the given length of edges, $\triangle ABC$ is a right triangle. So if the radius of the inscribed circle is r, $(5 + 12 + 13)r = 5 \times 12$, thus $r = 2$. Let the intersection of the inscribed circle and segment $\overline{AB}$ be Q. We see that $AQ = 2 \implies BQ = 3 \implies BP = 3$. Furthermore, $\overline{AK}$ bisects $\angle BAC$, so by the angle bisector theorem, $BK = 13 \cdot \frac{5}{17} = \frac{65}{17}$. Thus, $PK = BK - BP = \frac{65}{17} - \frac{51}{17} = \frac{14}{17}$. Thus, $17PK = \boxed{14}$.