Johns Hopkins Math Tournament 2019

Individual Round: Geometry

February 9, 2019

Instructions

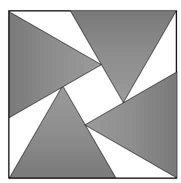
- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. Phillip is trying to make a two-dimensional donut, but in a fun way: He is trying to make a donut shaped in a way that the inner circle of the donut is inscribed inside a pentagon, and the outer circle of the donut circumscribes the same pentagon. This pentagon has a side length of 6. The area of Phillip's donut is of the form $a\pi$. Find a. (Note that $\sin 54^\circ = \frac{\sqrt{5}+1}{4}$)

2. JHMT Pizzeria messed up ordering boxes to put their 10 inch pizza in. (10 inch pizza means the diameter of the pizza is 10 inches) They accidentally ordered an 8in. $\times$ 8in. box and they immediately need to deliver 10 inch pizzas to customers, and they decide to cut the pizza minimally so that the most part of the pizza fits in to the 8in. $\times$ 8in. box. The area they need to cut out from the original 10 inch pizza can be written in form of $\alpha \cdot \arccos(\beta) - \gamma$. Find the value of $\alpha\beta\gamma$, where α and γ are integers and β is a rational number strictly between $\frac{1}{2}$ and 1.

3. Square $ABCD$ has side length of 2. Quarter-circle arcs $\widehat{BD}$ (centered at C) and $\widehat{AC}$ (centered at D) divide $ABCD$ into four sections. The area of the smallest of the four sections that are formed can be expressed as $a - \frac{b\pi}{c} - \sqrt{d}$. Find $abcd$, where a, b, c and d are integers, $\sqrt{d}$ is a written in simplest radical form, and $\frac{b}{c}$ is written in simplest form.

4. Let there be a unit square initially tiled with four congruent shaded equilateral triangles, as seen below. The total area of all of the shaded regions can be expressed in the form $\frac{a-b\sqrt{c}}{d}$, where a, b, c, and d are positive integers and c is not divisible by the square of any prime. Compute $a + b + c + d$.



5. Triangle $\triangle ABC$ has $AB = 8$, $BC = 12$, and $AC = 16$. Point M is on $\overline{AC}$ so that $AM = MC$. Then, $\overline{BM}$ has length x. Find x^2.

6. Circles C_1 and C_2 intersect at exactly two points I_1 and I_2. A point J on C_1 outside of C_2 is chosen such that $\overline{JI_2}$ is tangent to C_2 and $\overline{JI_2} = 3$. A line segment is drawn from J through I_1 and intersects C_2 at point K and $\overline{JK} = 6$. $\angle JI_2I_1 = \angle I_2KI_1 = \frac{1}{2}\angle I_1I_2K$. Let $\overline{I_1I_2} = a$, and let a equal the fraction $\frac{m\sqrt{p}}{n}$, where m and n are coprime and p is a positive integer not divisible by the square of any prime. Find $100m + 10p + n$.

7. Regular hexagon $ABCDEF$ has side length α. Line l intersects A and bisects $\overline{CD}$ (and the point of intersection is M), line m intersects C and E, and line n intersects B and E. Lines n and l intersect at a point G, and lines m and l intersect at a point H. $[\triangle CHM] : [\triangle GHE] : [\triangle ABG] = a : b : c$ where $[\triangle ABC]$ is the area of $\triangle ABC$. Find $a + b + c$.

8. In $\triangle ABC$, $\text{m}\angle A = 90^\circ$, $\text{m}\angle B = 45^\circ$, and $\text{m}\angle C = 45^\circ$. Point P inside $\triangle ABC$ satisfies $\text{m}\angle BPC = 135^\circ$. Given that $\triangle PAC$ is isosceles, the largest possible value of $\tan\angle PAC$ can be expressed as $s + t\sqrt{u}$, where s and t are integers and u is a positive integer not divisible by the square of any prime. Compute $100s + 10t + u$.

9. Triangle $\triangle ABC$ is inscribed in circle O and has sides $AB = 47$, $BC = 69$, and $CA = 34$. Let E be the point on O such that $\overline{AE}$ and $\overline{BC}$ intersect inside O, 8 units away from B. Let P and Q be the points on $\overleftrightarrow{BE}$ and $\overleftrightarrow{CE}$, respectively, such that $\angle EPA$ and $\angle EQA$ are right angles. Suppose lines $\overleftrightarrow{AP}$ and $\overleftrightarrow{AQ}$ respectively intersect O again at X and Y. Compute the distance XY.

10. In $\triangle PQR$, $\angle Q + 10^\circ = \angle R$. Let M be the midpoint of $\overline{QR}$. If $\text{m}\angle PMQ = 100^\circ$, then find the measure of $\angle Q$ in degrees.