Solutions

1. Note that $1/x = (90+y)/99$ and that $1/y = (10+x)/99$. So $90x + xy = 99$ and $10y + xy = 99$. Subtracting the two equations gives that $90x = 10y$, so $y = 9x$. Thus $y = 9$ and $x = 1$, so $x + y = \boxed{10}$.

2. Note that via the quadratic formula, it suffices to consider a, b, c s.t. $b^2 > 4ac$. For each value of b, we find the number of choices for a, c which will satisfy the inequality. We add $0 + 5 + 14 + 21 + 24 = 64$, which corresponds to the number of combinations of a, c for which $b^2 > 4ac$ for each $b \in \{1, 2, 3, 4, 5\}$. We divide this by 125,since that is the number of possible (a, b, c), and obtain $\frac{64}{125}$, yielding $\boxed{189}$ as our answer.

3. Divide the square into four smaller squares of side length 1. Since we are trying to space out our 5 point as much as possible, place 1 point in each of the four smaller squares. By the Pidgeonhole Principle, this means that the 5th point must go in the same square that already contains a point. The furthest apart any two points in the same small square can be is $2\sqrt{2}$. Thus the answer is $\boxed{5}$.

4. We rewrite the equation as $55 = (2^x)^2 - (3^y)^2 = (2^x + 3^y)(2^x - 3^y)$. The only factorizations of 55 are $55 \cdot 1$ and $11 \cdot 5$, and only the latter has factors whose average fits the form 2^x (the average is $8 = 2^3$). Hence, the only solution is $(x, y) = (3, 1)$, so the answer is $\boxed{4}$.

5. Let $S(a) = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ for $|a| < 1$. Then,
$$\frac{dS(a)}{da} = \frac{1}{(1-a)^2} = \sum_{n=0}^{\infty} na^{n-1},$$
so $\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2}$. Plugging in $a = \frac{1}{10}$ makes the sum $\frac{10}{81}$, so $\boxed{91}$ is our answer.

6. To begin, suppose want to maximize the product of two positive integers x and y that sum to some constant F. We can quickly see that the product is maximized when x and y are as close as possible to $\frac{F}{2}$. Thus, in order for the product of all values in S to attain P, any two values in S must be at most 1 apart (if they are more than 1 apart, then we can bring them closer to their average and increase their product). This forces the optimal set S to contain only two distinct values. Suppose we have a fixed partial sum $F \leq 148$ and want to reach this sum by adding up $\frac{F}{v}$ copies of just one value v. The product of these repeated values is $v^{(F/v)}$, or $\sqrt[v]{v}^F$. A well-known result of calculus is that, if v is allowed to be any positive real number, then $\sqrt[v]{v}$ is maximized at $v = e \approx 2.718$ and that the function $w = \sqrt[v]{v}$ has no other local maxima. We care about integer values of v, so the maximal product $\sqrt[v]{v}^F$ is attained either at $v = 3$ or $v = 2$. Because $2^3 < 3^2$, we have $\sqrt{2} = \sqrt[6]{2^3} < \sqrt[6]{3^2} = \sqrt[3]{3}$, so $v = 3$ is the best value to repeat. Of course, this only works when F is a multiple of 3, so we try $F = 147$. In this case, S contains 49 "3"s and one "1". But this violates the property that any two values in S are at most 1 apart. Therefore, we replace a "3" and a "1" with two "2"s, so the partial product $3 \cdot 1$ increases to $2 \cdot 2$. This gives us the optimal set S, so
$$P = 2^2 \cdot 3^{46},$$
and thus the answer is $2 + 2 + 3 + 46 = \boxed{53}$.

7. Swimmer A (the faster swimmer) swims one length of the pool in 30 seconds while swimmer B (the slower simmer) swims one length of the pool in 45 seconds. Notice that after 90 seconds, the two swimmers are at the same spot on one end of the pool and after another 90 seconds, both swimmers are back at their starting positions. During this 180 second interval, the swimmers pass each other exactly 5 times (twice before meeting at the wall, once when they meet at the wall, and twice after meeting at the wall). Since this 180 second interval cycles 4 times over the entire 12 minute duration, the swimmers will pass each other a total of $\boxed{20}$ times.

8. Let $u = \sqrt[3]{x-9}$. Rearrange the original equation as $\sqrt[3]{x+9} = 3 + \sqrt[3]{x-9}$. Cubing this equation yields

$$x + 9 = 27 + (x-9) + 27u + 9u^2 \implies 0 = 9 + 27u + 9u^2 \implies u^2 + 3u + 1 = 0 \implies u = -\frac{3 \pm \sqrt{5}}{2}.$$

Then, $x - 9 = u^3 = -9 \pm 4\sqrt{5}$, so $x = \pm 4\sqrt{5}$ and therefore $x^2 = \boxed{80}$.

9. First, notice that the shorter trisector will be the one closer to BC. Let this shorter trisector intersect hypotenuse BA at point N. Draw a line from N to side BC such that the line is perpendicular to BC. Suppose this line intersects BC at point M. Now, let length MN be called x. Thus, since triangle MNC is a 30-60-90 triangle, MC has length $x\sqrt{3}$ and MB has length $3 - x\sqrt{3}$. Now, since triangle NBM is similar to triangle ABC, we can create a ratio relating the two triangles. Specifically, we see that $\frac{x}{3-x\sqrt{3}} = \frac{4}{3}$. If we solve this equation, we find that $x = \frac{16\sqrt{3}-12}{13}$. Since we are looking for the length of NC which is the hypotenuse of our 30-60-90 triangle, the final length is $\frac{32\sqrt{3}-24}{13}$. Thus the solution is $\boxed{72}$.

10. We count the number of valid colorings directly (we define a valid coloring to have no adjacent faces with the same color). If we use all five colors, then two faces will have the same color. These faces must be opposite each other, and there are three ways to select this pair of opposite faces. There are five ways to select this color that gets repeated. There are 4! ways to permute the remaining four colors over the four unpainted faces. Thus, our count for the five color case is $3 \cdot 5 \cdot 4! = 360$.

If we use four colors, then there must be two pairs of faces with common color. Like before, each pair must contain opposite faces, and there are $\binom{3}{2} = 3$ ways to choose these two pairs. There are $5 \cdot 4 = 20$ ways to select the colors that go on these pairs. The remaining two unpainted faces can have any two distinct colors of the three unused colors, so there are $3 \cdot 2 = 6$ distinct ways to accomplish this last step. Thus, our count for the four color case is $3 \cdot 20 \cdot 6 = 360$.

If we use three colors, then the three colors we choose must be assigned (in some order) to the three pairs of opposite faces. There are $\binom{5}{3} = 10$ ways to select the colors, and there are 3! ways to permute them over the three face pairs. Hence, our count for the three color case is $10 \cdot 3! = 60$.

Observe that we cannot use just one or two colors to paint the cube because there will be a color that gets repeated on at least three faces, and no set of three faces on the cube has no adjacent pairs. Altogether, there are $360 + 360 + 60 = 780$ valid colorings. Since there are $5^6 = 15625$ total colorings, the success rate is $\frac{780}{15625} = \frac{156}{3125}$, so the answer is $\boxed{3281}$.