Johns Hopkins Math Tournament 2019

Individual Round: General II

February 9, 2019

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. Suppose x and y are one-digit positive integers such that $\frac{1}{x} = 0.\overline{9y}$ (i.e., $\frac{1}{x} = 0.9y9y9y\ldots$) and $\frac{1}{y} = 0.\overline{1x}$. What is $x + y$?

2. Consider three numbers, a, b, c, each of which is picked uniformly at random from the set $\{1, 2, 3, 4, 5\}$ (i.e. the integers between 1 and 9 inclusive). The probability that the quadratic equation $ax^2+bx+c = 0$ has exactly two real roots can be expressed as a common fraction $\frac{m}{n}$. Find $m + n$.

3. Five distinct points are chosen inside or on a square of side length 4. Let m be the smallest possible number such that for any five given points, it is always possible to pick a pair of points from the five such that the two points are less than or equal to m units apart. We can write m in the form $\frac{a\sqrt{b}}{c}$, where $\sqrt{b}$ is in simplest radical form and $\frac{a}{c}$ is a common fraction. What is $a + b + c$?

4. The equation $2^{2x} - 3^{2y} = 55$ has ordered pair solutions (x, y) where x and y are both integers. What is the sum of all x and y for all ordered pair solutions?

5. The infinite series $\frac{1}{10} + \frac{2}{100} + \frac{3}{1000} + \cdots + \frac{n}{10^n} + \cdots$ converges to F. Given that F can be expressed as a common fraction $\frac{a}{b}$, find $a + b$.

6. A set S of positive integers sum to 148. Repeats are allowed within this set. Let P be the largest possible product of all the integers in S. The prime factorization of P will have the form $\prod_{k=1}^{m} a_k^{b_k}$, where $a_1, a_2, \ldots,$ and a_m are all of the distinct prime factors of P. What is the sum of all bases and exponents in the final product when expressed in this form?

7. Two swimmers, starting from opposite ends of a 90 meter long pool, begin continuously swimming across the pool. One swimmer swims at the constant rate of 3 meters per second and the other swims at the constant rate of 2 meters per second. After swimming back and forth for 12 minutes, how many times did the two swimmers pass each other?

8. Among all numbers x that satisfy $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$, find the largest possible value of x^2.

9. Right triangle $\triangle ABC$ has legs $AC = 4$ and $BC = 3$. Points M and N are drawn on hypotenuse $\overline{AB}$ such that $\overline{CM}$ and $\overline{CN}$ trisect angle C. Given that the length of the shorter trisector can be written in the form $\frac{r\sqrt{s}-t}{w}$ where $\sqrt{s}$ is in simplest radical form and the GCD of r, t, and w is 1, find $r+s+t+w$.

10. Nancy has a cube and five distinct colors. For each side of the cube, she chooses a color uniformly at random to paint that side of the cube. The probability that no two adjacent sides of the cube share the same color can be expressed as a common fraction $\frac{m}{n}$. Compute $m + n$.