Johns Hopkins Math Tournament 2019

Individual Round: General I

February 9, 2019

Instructions

- **<u>DO NOT</u> TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. A quadratic equation P exists such that $P(1) = 1$, $P(2) = 5$ and $P(3) = 2019$. What is the sum of the coefficients of P?

2. Suppose Kevin is at the state fair and wants to eat a corndog. He then lines up at a corndog stand to buy his corndog. Suppose that there are a total of 5 people in line (including Kevin) and that each person in line will decide to buy either one or two corndogs uniformly at random. Suppose further that Kevin's position within the line of 5 people is random (with uniform distribution). If the stand initially has 5 corndogs, what is the probability that the corndogs are sold out by the time he goes to purchase one?

3. In how many ways can three distinct increasing integers be chosen from the range $[1, 100]$ such that they form an arithmetic progression?

4. Let x be a real number chosen randomly between 100 and 200. If $\lfloor \sqrt{x} \rfloor = 12$, then the probability that $\lfloor \sqrt{100x} \rfloor = 120$ can be written as the fraction $\frac{a}{b}$ when expressed in simplest form. What is $a + b$?

5. Paul and George decide to run around a perfectly circular track. They start on opposite ends of the track and run at constant speeds in opposite directions. If they start running at the same time, they first pass each other after George has run 100 meters, and then they pass each other a second time 60 meters before Paul finishes his first lap. What is the circumference of the track in meters?

6. Find the number of positive integers that are divisors of at least one of 10^{10}, 15^7, 18^{11}.

7. A subset of the first 100 positive integers has the property that none of the subset's members is exactly 3 times any other member. What is the largest number of members of such a subset?

8. Consider a 5 by 5 square lattice, i.e., a grid with 5 vertices on each side. The probability that three points that are chosen from this lattice uniformly at random are collinear is $\frac{m}{n}$, where m and n are coprime. Compute $m + n$.

9. Eighteen pairs of people go to a square dance. The 36 total people are randomly paired up as dancing partners. The expected number of people who are paired up with the person they came with can be expressed as a fraction written in simplest form, $\frac{p}{q}$. Find $p + q$.

10. Find the smallest positive integer n such that there exist four distinct ordered pairs (x, y) of positive integers such that $x^2 - y^2 = n$.