Johns Hopkins Math Tournament 2019

Individual Round: Calculus

February 9, 2019

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

Note: If necessary, recall that Euler's constant is $e \approx 2.718$. You will not need any more decimal places.

1. Evaluate the definite integral
$$\int_{20}^{19} dx.$$

2. Compute the greatest integer less than or equal to the limit $\lim_{x \to 0^+} (\cos(x))^{\ln x}$.

3. Determine the greatest integer less than or equal to
$$100 \sum_{n=0}^{\infty} \frac{1}{(n+3) \cdot n!}.$$

4. The value of the following series
$$\sum_{n=2}^{\infty} \frac{3n^2+3n+1}{(n^2+n)^3}$$
can be written in the form $\frac{m}{n}$, where m and n are coprime. Compute $m+n$.

5. Given
$$4 \int_{\ln 3}^{\ln 5} \frac{e^{3x}}{e^{2x} - 2e^x + 1} \, dx = a + b \ln 2,$$
where a and b are integers, what is the value of $a+b$?

6. The *double factorial* of a positive integer n is denoted $n!!$ and equals the product $n(n-2)(n-4)\cdots\left(n - 2\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\right)$; we further specify that $0!! = 1$. What is the greatest integer q such that
$$\sqrt[4]{q} < \sum_{n=0}^{\infty} \frac{1}{(2n)!!}?$$

7. Let e be Euler's constant. For all real x greater than e, let $f(x)$ be the unique positive real value y satisfying $y < x$ and $x^y = y^x$. Over $x \in (e, \infty)$, the function $y = f(x)$ is differentiable, and the value of $f'(4)$ can be expressed as $\frac{1}{a} - \frac{1}{b - \ln c}$ for positive integers a, b, and c. Compute the value of $a+b+c$.

8. A circle of radius 4 is tangent to the parabola $y = x^2$ at two distinct points and is centered at some point on the y-axis. The distance between the center of the circle and the origin $(x, y) = (0, 0)$ can be expressed as $\frac{p}{q}$, for relatively prime positive integers p and q. Compute $p+q$.

9. A cylinder of radius 6 rests on the Euclidean plane, with the center of its base at the origin. One end of a string of length 6π is attached to the cylinder at the point $(6, 0)$. Assume that the string's width is negligible. The area of the region on the plane that can be reached by the free end of the string can be written as $m\pi^3$ for a natural number m. Find the value of m.

10. In the Euclidean plane, vertices $A(-1, 0)$, $B(1, 0)$, and $C(x, y)$ form a triangle with perimeter 12. What is the largest possible integer value of $x+y$?