Solutions

1. $\boxed{47}$.

2. Multiply all terms by $4ab$ to yield the equation $ab - 4a - b = 0$. Then, use Simon's favorite factoring trick to yield the equation $16 = (a-4)(b-4)$. Since 16 has 3 pairs of factors (1&16, 2&8, 4&4), there are $\boxed{5}$ such ordered pairs.

3. This is equivalent to the condition $a \equiv -1 (\text{mod} 60)$. Since $60 \times 33 = 1980$, and $2019 - 1980 = 39 < 59$, there are $\boxed{33}$ such numbers.

4. Since the sum of coefficients equal to 2019, $P(1) = 2019$, $P(-1) = -2019$, $P(0) = 0$. Set $P(x) = Q(x)(x^3 - x) + ax^2 + bx + c$. Then $P(1) = a + b + c = 2019$, $P(-1) = a - b + c = -2019$, $P(0) = c = 0$. Hence $a = 0, b = 2019$, and so the remainder is $2019x$. Thus, the answer is $\boxed{2019}$.

5. The relation $(X+1)P(X) = (X-10)P(X+1)$ shows that P(x) is divisible by (x-10). Shifting the variable, we get $xP(x-1) = (x-11)P(x)$, which shows that $P(x)$ is also divisible by x. Hence $P(x) = x(x-10)P_1(x)$ for some polynomial $P_1(x)$. Substituting in the original equation and canceling common factors, we find that $P_1(x)$ satisfies

$$xP_1(x) = (x-9)P_1(x+1)$$

. Arguing as before, we find that $P_1(x) = (x-1)(x-9)P_2(x)$. Repeating the argument, we eventually see that $P(x) = x(x-1)(x-1)...(x-10)Q(x)$, where $Q(x)$ satisfies $Q(x) = Q(x+1)$, which means that $Q(x)$ is constant. Thus, $P(x) = ax(x-1)(x-1)...(x-10)$, but since the leading coefficient=1, $P(x) = x(x-1)(x-1)...(x-10)$. $P(5) = \boxed{0}$.

6. Using a complex number approach: note that $1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)$. From de Moivre's formula, we have

$$(1+i)^{1000} = (\sqrt{2})^{1000}(\cos(1000 * \pi/4) + i\sin(1000 * \pi/4)$$
$$= 2^{500}(\cos(250\pi) + i\sin(250\pi) = 2^{500}$$

Using binomial expansion, $(1+i)^{1000} = \binom{1000}{0} + \binom{1000}{1}i + \binom{1000}{2}i^2 + ... + \binom{1000}{1000}i^{1000}$ So by looking at the real and imaginary parts. $\binom{1000}{0} - \binom{1000}{2} + \binom{1000}{4} - ... + \binom{1000}{1000} = 2^{500}$. Thus $A = \boxed{500}$.

7. Using Viete's relations we obtain the system

$$a + b = \frac{b}{a}$$
$$ab = \frac{c}{a}$$
$$b^2 - 4ac = c$$

write this as

$$a^2 + ab = b$$
$$a^2 b = c$$
$$b^2 - 4ac = c$$

Eliminating c we get

$$a^2 + ab = b$$
$$b^2 - 4a^3 b = a^2 b$$

The first equation shows that $b \neq 0$, so the second can be divided by b, yielding $b - 4a^3 = a^2$ or $b = 4a^3 + a^2$. Canceling a^2 we $4a = 3$, so $a = \frac{3}{4}$. Then $b = \frac{9}{4}$ and $c = \frac{81}{64}$. Thus $\frac{4c}{ab} = \boxed{3}$.

8. Using the identity $a^3+b^3=(a+b)(a^2-ab+b^2)$ and pairing x^3 with $(x+3)^3$ and $(x+1)^3$ with $(x+2)^2$, we get

$$(2x+3)(x^2-x(x+3)+(x+3)^2)+(2x+3)((x+1)^2-(x+1)(x+2)+(x+2)^2)$$

which reduces to

$$2(2x+3)(x^2+3x+6)=0$$

The quadratic $x^2+3x+6=(x+3/2)^2+\frac{15}{4}$, so it is strictly positive. The only way the equation is satisfied is when $2x+3=0$, or $x=\frac{3}{2}$. Thus, $\boxed{5}$.

9. Transform $P(x)=(1+x+^2+x^3+...+x^{17})^2-x^{17}$ using geometric series formula, yielding

$$P(x)=(\frac{x^{18}-1}{x-1})^2-x^{17}=\frac{x^{36}-2x^{18}+1}{x^2-2x+1}-x^{17}$$

$$=\frac{x^{36}-x^{19}-x^{17}+1}{(x-1)^2}=\frac{(x^{19}-1)(x^{17}-1)}{(x-1)^2}$$

This equation has as roots all of 17th and 19th roots of unity, excluding 1. Hence we want to find the smallest fraction of the form $\frac{m}{19}$ or $\frac{n}{17}$ for $m,n>0$ We find: $\frac{1}{19},\frac{2}{19},\frac{3}{19},\frac{1}{17},\frac{2}{17}$, yielding $\frac{1}{19}+\frac{2}{19}+\frac{3}{19}+\frac{1}{17}+\frac{2}{17}=\frac{6}{19}+\frac{3}{17}=\frac{159}{323}$ Thus, $\alpha+\beta=159+323=\boxed{482}$.

10. Recall Heron's formula for area of a triangle with sides a,b,c:

$$A=\frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

From Vieta's, we know that $a+b+c=4$, so $a+b-c=4-2c$ and analogue for a and b. Therefore, the area is

$$A=\frac{1}{4}\sqrt{4(4-2a)(4-2b)(4-2c)}$$

Furthermore,

$$(4-2a)(4-2b)(4-2c)=8(-4(a+b+c)+2(ab+bc+ac)-abc+8)$$

By Vieta's,

$$ab+bc+ac=5, abc=19/10$$

Thus the area is

$$A=\frac{1}{4}\sqrt{4*8(-16+10-\frac{19}{10}+8)}=\frac{1}{\sqrt{5}}$$

In simplest radical form, we have the fraction $\sqrt{5}/5$, thus the answer is 10.