Minimal regularity solutions of nonlinear wave equations

Hans Lindblad
University of California San Diego
e-mail lindbladmath.ucsd.edu

Main goal: Prove global existence for systems of nonlinear wave equations from physics, for example:

Einstein's equations, Yang-Mills equations,
Equations of Elasticity

and for the basic model equations.

Two approaches to prove global existence.

Method 1: For small initial data:

Method 2: If there is a conserved energy norm.

Method 1: For small initial data:
The equation can be considered to be a small perturbation of the linear case. To do this one needs global estimates catching the right the decay of the solution as time tends to infinity. The classical approach, developed by

John, Klainerman, Christodoulou et. al.

is to use the energy method together with the differential operators coming from the invariances of the wave equation.

In this lecture we present joint work of Georgiev-L-Sogge proving global existence for a certain class of semi-linear wave equations, with nonlinearities depending only on the solution.

This result was a conjecture of Strauss, following an initial result of John. The energy method does not work. To prove the conjecture we develop weighted Strichartz estimates using techniques from Harmonic analysis which take into account the symmetries of the wave equation.

Global existence for small initial data

$$\Box u = |u|^p, \quad (t,x) \in \mathbf{R}^{1+n}, \quad \Box = \partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2$$
$$u(0,x) = \epsilon f(x) \in C_0^\infty, \quad \partial_t u(0,x) = \epsilon g(x) \in C_0^\infty$$

Question: *When is there global existence for small data, i.e. for small ε ? (For which p?)*

Asymptotics for the linear solution: $\Box v = 0$, data as above:

$$|v(t,x)| \leq \frac{C\epsilon}{(1+t)^{(n-1)/2}(1+|t-|x||)^{(n-1)/2}}$$

A first guess If u satisfied same bounds as v, then by the **energy inequality:**

$$\sup_{t<T} \|u'(t,\cdot)\|_2 \le \|u'(0,\cdot)\|_2 + \int_0^T \| \, |u|^p(t,\cdot)\|_2 dt$$
$$\le C\epsilon + (C'\epsilon)^{p-1} \int_0^T (1+t)^{-\frac{n-1}{2}(p-1)} \|u(t,\,\cdot\,)\|_2 dt$$

(Here $\|u\|_2 = \|u\|_{L^2}$)

If (f,g) vanishes outside a ball of radius R the by (the weak) Huygens principle $u(t,x)$ vanishes when $|x| \ge R + t$ so $\|u(t,\,\cdot\,)\|_2 \le (R+t)\|u'(t,\,\cdot\,)\|_2$. Hence we would expect global existence if

$$(C'\epsilon)^{p-1} \int_0^\infty (1+t)^{-\frac{n-1}{2}(p-1)} (R+t)\, dt < \frac{1}{2}$$

i.e. if $p-1 > 4/(n-1)$. (If $n=3$ this means $p>3$.)

On the other hand if the strong Huygens principle was valid, i.e. $u(t,x)$ would be vanishing when $||x|-t| \ge R$ then $\|u(t,\,\cdot\,)\|_2 \le 2R\|u'(t,\,\cdot\,)\|_2$ and we would guess $p > 2/(n-1)$. (If $n=3$ then $p>2$.)

Theorem (John) *There is blow-up for all small data if $p < 1+\sqrt{2}$, and global existence for all small data if $p > 1+\sqrt{2}$.*

Conjecture (Strauss): *For $n \geq 2$ blow-up for all small data if $p < p_n$ and global existence for all small data if $p > p_n$, where $p_n > 1$ solves*

$$(n-1)p_n^2 - (n+1)p_n - 2 = 0, \qquad p_n \sim 1 + 4/n$$

Verified:

$n = 2$ Glassey

Blowup $n \geq 2$ Sideris

Existence $n = 4$, Yi Zhou

Existence $n \leq 8$ and spherically symmetric case for all n, L-Sogge

Spherically symmetric odd n, H. Kubo

Existence for all n, Georgiev-L-Sogge

John's Estimate:
$1+\sqrt{2}<p\leq 3$, $\Box w = F$ zero data, supp $F \subset \{|x|< t-1\}$:

$$\|(1+t)(1+|t-r|)^{p-2}w\|_\infty \leq C\|(1+t)^p(1+|t-r|)^{p(p-2)}F\|_\infty$$

Gives existence for $\Box u = |u|^p$ by a standard iteration argument: Let $\|w\|$ be the norm in the LHS. John's estimate applied to $\Box(u-v) = |u|^p$ gives

$$\|u\| \leq \|v\| + C\|u\|^p$$

It follows that $\|u\| \leq 2\|v\|$ if $C(2\|v\|)^{p-1} \leq 1/2$.

Problem in higher dimensions: No point wise estimate can hold since the fundamental solution of $\Box$ no longer is a measure if $n \geq 4$.

Space-time estimates from harmonic analysis: Classical Strichartz estimate;

$$\|w\|_{L^q(\mathbf{R}_+\times\mathbf{R}^n)} \leq C\|F\|_{L^{q/(q-1)}\mathbf{R}_+\times\mathbf{R}^n)}, \qquad q = \frac{2(n+1)}{(n-1)}$$

Does not catch the right decay as $t\to\infty$: If $\Box v = 0$ with compactly supported data then

$$\|v\|_{L^q}^q \sim \int_{|t-r|\leq R} \frac{r^{n-1}drdt}{(1+t)^{q(n-1)/2}} < \infty, \qquad \text{if } q > \tfrac{2n}{n-1}.$$

Theorem 1(GLS) $n \geq 2$, $2 \leq q \leq \frac{2(n+1)}{n-1}$, $\Box w = F$, *zero data*, supp $F \subset \{|x|<t-1\}$:

$$\|(t^2-|x|^2)^{\gamma_1} w\|_{L^q(} \leq C\|(t^2-|x|^2)^{\gamma_2} F\|_{L^{q/(q-1)}},$$

provided that

$$\gamma_1 < n(1/2 - 1/q) - 1/2, \qquad \gamma_2 > 1/q$$

Here the integrals are over $\{(t,x);\, |x|<t-1\}$.

One should think of this estimate as a weighted version of Strichartz estimate:

$$\|w\|_{L^q} \leq C\|F\|_{L^{q/(q-1)}}, \quad q = \frac{2(n+1)}{(n-1)}$$

Proof of estimate and generalizations

The proof of Theorem 3 uses a decomposition into regions, where the weights $(t^2 - |x|^2)$ are essentially constant, together with the invariance of the norms and the equation under Lorentz transformations. In each case we get the desired estimate by using analytic interpolation, between an $L^1 \to L^\infty$ and an $L^2 \to L^2$ estimate with weights, for the Fourier integral operators associated with the wave equation.

We also prove a stronger scale invariant weighted Strichartz estimate under the assumption of radial symmetry. This assumption was later removed by Tataru:

Theorem 2 Suppose that F is supported in the forward light cone $\{(t,x) \in \mathbf{R}^{1+n} : |x| \le t\}$. Then if $\Box w = F$, $w = w_t = 0$ when $t = 0$ we have

$$\|(t^2 - |x|^2)^{-\alpha} w\|_{L^q} \le C_\gamma \|(t^2 - |x|^2)^{\beta} F\|_{L^{q/(q-1)}},$$

$$\beta < 1/q, \quad \alpha + \beta + \gamma = 2/q,$$
$$\gamma = (n-1)(1/2 - 1/q),$$
$$2 < q \le 2(n+1)/(n-1)$$

We use the usual Strichartz estimates: If $\Box u = 0$, $u|_{t=0} = 0$, $u_t|_{t=0} = g$ then

$$\|u(t,\cdot)\|_{H^{1-s,p}} \leq C t^{-r(1/p'-1/p)} \|g\|_{H^{s,p'}}$$

where $1/p + 1/p' = 1$ and

$$2s = (r+1)(\frac{1}{p'} - \frac{1}{p}), \quad 2 \leq p < \infty$$

Tataru improved the proof by using a similar estimate between hyperboloids; $e^\tau = \sqrt{t^2 - |x|^2} = constant$: If $\Box u = 0$, $u|_{\tau=0} = 0$, $u|_{\tau=0} = g$ then

$$\|u(\tau,\cdot)\|_{H^{1-s,p}} \leq \frac{C(1+\tau)}{(\sin \tau)^{r(1/p'-1/p)}} \|g\|_{H^{s,p'}}$$

Open problems:

To prove a scale invariant weighted Strichartz estimate when the restriction on the support of $F = \Box w$ is removed.

To prove a scale invariant weighted Strichartz estimate for the homogeneous initial value problem $\Box v = 0$, with minimal regularity and decay assumptions on initial data.

Does decay require more regularity than what is needed for local existence?

Method 2: If there is a conserved energy norm. If one can prove local existence and uniqueness assuming only that the energy norm of initial data is bounded, then global existence and uniqueness follow from the conserved energy. This leads to the:

Question: What is the minimal amount of regularity of the initial data needed to ensure local existence?

Recently there have been improvements of the classical local existence results using space-time estimates known as Strichartz estimates and generalizations of these. This can be found in work of

Klainerman-Machedon, Ponce-Sideris, L-Sogge,
Grillakis, Tataru, Bahouri-Chemin.

Related results for KdV and Nonlinear Schrödinger

Bourgain, Kenig-Ponce-Vega

In this lecture we present counterexamples to local existence for the typical nonlinear wave equations.

Example Consider the Cauchy problem

$$\Box u \pm |u|^{p-1} = 0, \quad u|_{t=0} = f, \ u_t|_{t=0} = g,$$

for which we have a conserved energy $E(t) = E(0)$;

$$E(t) = \int \frac{u_t(t,x)^2 + |\nabla_x u(t,x)|^2}{2}\, dx \pm \int \frac{|u(t,x)|^{p+1}}{p+1}\, dx$$

Global existence for the good sign +: Jörgens if $p < 5$, Struwe, Grillakis if $p = 5$, Open if $p > 5$?

Notation: $\|f\|_{H^\gamma} = \sqrt{\int (|D_x|^\gamma f(x))^2\, dx}$.

Local existence: L-Sogge: If $f \in H^\gamma$ and $g \in H^{\gamma-1}$ then we have a local solution $(u, u_t) \in H^\gamma \times H^{\gamma-1}$ for $0 \leq t < T$ when

$$\gamma \geq \gamma(p) = \begin{cases} 1 - 1/(p-1), & p \leq 3 \\ 3/2 - 2/(p-1) & p \geq 3 \end{cases}$$

and when $\gamma > \gamma(p)$ then T is a function of the norm $\|f\|_{H^\gamma} + \|g\|_{H^{\gamma-1}}$ only. Note that $\gamma(5) = 1$.

Hence, if we have the **good sign +** and $p \leq 5$ we have local existence for $t \leq T$ for data with bounded energy $E(0)$ and if $p < 5$ then $T = T(E)$. Since $E(T) = E(0)$ we can then iterate the argument to conclude that we have **global existence** for all t.

Counterexamples to local existence for the bad sign$-$: If $\gamma < \gamma(p)$ then we have counterexamples to local existence in H^γ, i.e. there are data $(f, g) \in H^\gamma \times H^{\gamma-1}$ such that there is no solution $(u(t,\cdot), u_t(t,\cdot)) \in H^\gamma \times H^{\gamma-1}$ for $t > 0$.

The scaling argument:
If $u(t,x)$ is a solution which blows up when $t = T$:

Then $u_\varepsilon(t,x) = \varepsilon^{-2/(p-1)} u(t/\varepsilon, x/\varepsilon)$ is a solution which blows up when $t = \varepsilon T$. Norms when $t = 0$;

$$\begin{aligned} &\|u_\varepsilon(0,\cdot)\|_{H^\gamma} + \|\partial_t u_\varepsilon(0,\cdot)\|_{H^{\gamma-1}} \\ &= \varepsilon^{3/2-2/(p-1)}(\|u(0,\cdot)\|_{H^\gamma} + \|\partial_t u(0,\cdot)\|_{H^{\gamma-1}} \to 0 \end{aligned}$$

if $\gamma < \gamma(p) = 3/2 - 2/(p-1)$. Hence the initial value problem is ill posed in H^γ if $\gamma < \gamma(p)$.

Counterexamples to local existence for the good sign$+$?: For the good sign one can construct weak solutions even if $p > 5$ by using the energy bound (Segal-Strauss). These might however not be unique and probably do not depend continuously of data in the energy norm. The question of global existence of smooth solutions might be related to the question if we have local existence in H^γ for $\gamma > 1$ if $p > 5$.

Counterexamples to local existence

Linear Wave Operator: $\Box = \partial_t^2 - \sum_{i=1}^3 \partial_{x_i}^2$.
Coefficient matrix: $(m^{ij}) = \text{diag}\,(1, -1, -1, -1)$.

Cauchy problem for a quasi linear wave equation:

$$(t, x) \in [0, T) \times \mathbf{R^3},$$

$$\sum_{i,j=0}^{3} g^{ij}(u, u')\partial_{x_i}\partial_{x_j} u \;= F(u, u'), \quad x_0 = t, \qquad (1)$$

$$u(0, x) = f(x), \qquad u_t(0, x) = g(x), \qquad (2)$$

g^{ij} and F are smooth functions of u and $u' = (u_t, u_x)$. (g^{ij}) is a symmetric matrix close to (m^{ij}) so that (1) is hyperbolic.

Question: What is the smallest possible γ such that

$$(f, g) \;\in\; \dot{H}^\gamma(\mathbf{R}^3) \times \dot{H}^{\gamma-1}(\mathbf{R}^3), \qquad (3)$$

$$\text{supp } f \cup \text{ supp } g \;\subset\; \{x; |x| \le 2\}$$

implies that for some $T > 0$ there is a unique distributional solution u of (1) which satisfies

$$(u, \partial_t u) \;\in\; C(\,[0, T]\,;\, \dot{H}^\gamma(\mathbf{R^3}) \times \dot{H}^{\gamma-1}(\mathbf{R^3})\,)? \quad (4)$$

Here, $\dot{H}^\gamma$ denotes the **homogeneous Sobolev space** with norm $\|f\|_{\dot{H}^\gamma} = \|\,|D_x|^\gamma f\|_{L^2}$ where $|D_x| = \sqrt{-\Delta_x}$.

Abbreviation: We say the solution u is in H^γ if

$$(u, \partial_t u) \in L^\infty(\,[0,T]\,;\, \dot{H}^\gamma(\mathbf{R^3}) \times \dot{H}^{\gamma-1}(\mathbf{R^3})\,)$$

i.e. if the norm

$$||u(t,\cdot)||_\gamma^2 = \int |\,|D_x|^{\gamma-1} u_t(t,x)|^2 + |\,|D_x|^\gamma u(t,x)|^2\, dx$$

is uniformly bounded.
Similarly, we will say that data (f,g) is in H^k if $f \in H^k$ and $g \in H^{k-1}$, i.e. if the above norm is bounded when $t = 0$.

Note that the norm $||u(t,\cdot)||_\gamma$ is invariant for the linear wave equation $\Box u = 0$ and is more or less the only norm that is preserved.

Smooth solutions are unique! If data (f, g) are smooth then there is a unique smooth solution to (1).

Distributional solutions may not be unique! In fact, for smooth data (f, g) there can be a smooth solution to (1) and a non-smooth distributional solution in H^γ.

Example: $u(t, x) = 2H(t - |x|)/t$ satisfies $\Box u = u^3$, and $||u(t, \cdot)||_\gamma \to 0$ when $t \to 0$ if $\gamma < 1/2$, by homogeneity. But $u = 0$ is also a solution!

Definition: u is a **proper solution** of (1) if it is a distributional solution and it is the weak limit of a sequence of smooth solutions u_ε to (1) with data $(\phi_\varepsilon * f, \phi_\varepsilon * g)$, where

$$\phi_\varepsilon(x) = \phi(x/\varepsilon)\varepsilon^{-n}, \qquad \phi \in C_0^\infty, \qquad \int \phi \, dx = 1.$$

Theorem 3. Consider the Cauchy problem in $\mathbf{R}^{1+3}$;

$$\Box u = (\,D_-^{\ell}\, u) D_-^{k-\ell} u, \qquad D_- = \partial_{x_1} - \partial_t, \tag{5}$$

$$u(0,x) = f(x), \quad u_t(0,x) = g(x), \tag{6}$$

where

$$0 \le \ell \le k - \ell \le 2, \qquad \ell = 0, 1.$$

There are data $(f, g) \in \dot{H}^k \times \dot{H}^{k-1}$ with compact support such that (5)-(6) does not have any proper H^k solution in $[0, T) \times \mathbf{R}^3$ for any $T > 0$. In fact $\|f\|_{\dot{H}^k} + \|g\|_{\dot{H}^{k-1}}$ can be arbitrarily small.

Remark: It also follows from the proof that either there is no distributional solution in H^k or the solution is non-unique.

In some cases stronger statements hold. E.g. if $\ell = k - \ell = 0$, then there is no distributional solution in $L^2([0, T] \times \mathbf{R}^3)$.

Remark: Theorem 1 is sharp in the semi-linear case; $k - \ell \le 1$.

First guess: A Scaling argument gives counterexamples to well posedness in H^γ for $\gamma < k-1/2$, half a derivative less regular than the counterexample in Theorem 1.

The Scaling Argument: If u is a solution of (5) which blows up when $t = T$, then

$$u_\varepsilon(t,x) = \varepsilon^{k-2} u(t/\varepsilon, x/\varepsilon)$$

is a solution of (5) with lifespan $T_\varepsilon = \varepsilon T$ and $\|u_\varepsilon(0,\cdot)\|_\gamma = \varepsilon^{k-2+3/2-\gamma}\|u(0,\cdot)\|_\gamma \to 0$ if $\gamma < k-1/2$.

The counterexamples in Theorem 1, by contrast, are designed to concentrate in one direction close to a characteristic.

Special equations can behave better! Klainerman-Machedon proved that for semi-linear wave equations satisfying the "null condition" (e.g. $\Box u = u_t^2 - |\nabla_x|^2$) there is local existence for initial data with the regularity predicted by the scaling argument. Although our Theorem is formulated for a special nonlinearity, the same result should hold for any nonlinearity that doesn't satisfy the "null condition".

Writing the equation of Theorem 1 in general form:

$$\Box u = (D_-^\ell \, u) D_-^{k-\ell} u, \qquad D_- = \partial_{x_1} - \partial_t, \tag{7}$$

with $0 \le \ell \le k - \ell$, $\ell = 0, 1$ can be written in the form

$$\sum_{i,j=0}^{3} g^{ij}(u, u') \partial_{x_i} \partial_{x_j} u \;= F(u, u'), \quad x_0 = t. \tag{8}$$

Semi-linear case: $k - \ell \le 1$ the equation is already in this form with $g^{ij} = m^{ij}$.

Quasi-linear case $k - \ell = 2$**:**

$$\sum_{i,j=0}^{3} g^{ij} \xi_i \xi_j = \xi_0^2 - \sum_{i,j=0}^{3} \xi_i^2 - D_-^\ell u_1(t, x_1)(\xi_1 - \xi_0)^2,$$

so with $v = D^\ell u_1(t, x_1)$

$$(g^{ij}) \;=\; \begin{pmatrix} 1-v & v & 0 & 0 \\ v & -1-v & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

A calculation gives

$$(g_{ij}) \;=\; \begin{pmatrix} 1+v & v & 0 & 0 \\ v & -1+v & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Domain of Dependence

Let $\Omega \subset \mathbf{R}_+ \times \mathbf{R}^3$ be an open set equipped with a Lorentzian metric $g_{jk} \in C(\Omega)$ such that inverse g^{jk} satisfies.

$$\sum_{j,k=0}^{3} |g^{jk} - m^{jk}| \leq 1/2, \qquad (m^{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

For $(t', x') \in \Omega$, the **Causal Past** $\Lambda_{t',x'}$ is defined to be all points in Ω that can be joined to (t', x') by a Lipschitz continuous curve $(t, x(t)) \in \Omega$, $t \leq t'$, satisfying

$$g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} \geq 0, \qquad x_0(t) = t$$

almost everywhere.

Ω is said to be a **domain of dependence** for the metric g_{ij} if the closure of the causal past $\Lambda_{t',x'}$ of each point $(t', x') \in \Omega$ is contained in Ω.

Since a solution u to (5) gives rise to a unique metric g_{jk} we say that Ω is a *domain of dependence for the solution* u if it is a domain of dependence for g_{jk}.

Hyugen's principle: Changing the initial data outside the domain of dependence intersected with the initial plane $t = 0$ should not change the solution in the domain of dependence.

Uniqueness and continuity of solutions: Using the energy inequality, one can prove that smooth solutions depend uniquely and continuously on the initial data within a domain of dependence.
Hence if there is a smooth solution in a domain of dependence it is unique in the class of *proper* solutions with the same data.

Theorem 3 follows from:

Proposition 4. There is an open set $\Omega \subset \mathbf{R}_+ \times \mathbf{R}^3$ and a solution $u \in C^\infty(\Omega)$ of

$$\Box u = (\, D_-^\ell \, u) D_-^{k-\ell} u, \qquad D_- = \partial_{x_1} - \partial_t,$$

$0 \le \ell \le k - \ell$, $\ell = 0, 1$, such that writing

$$\Omega_t = \{x; (t,x) \in \Omega\},$$

the following are true:
Ω is a domain of dependence for u.
$\partial\Omega_0$ is smooth.

$$\sum_{|\beta| \le k,\ |\beta_0| \le 1} \int_{\Omega_t} (\partial^\beta u(t,x))^2 \, dx \quad \begin{matrix} < \infty, & t = 0, \\ = \infty, & t > 0 \end{matrix} \tag{9}$$

Furthermore in the quasi linear case, $k - l = 2$, the norms $\|D^l u\|_{L^\infty(\Omega)}$ can be chosen to be arbitrarily small.

Proof of Proposition 4.

Step 1: Find a solution $u(t,x) = u_1(t,x_1)$, depending only on one space variable $x_1 \in \mathbf{R}$, that develops a certain singularity for $t > 0$ along a **non time like** curve $x_1 = \mu(t)$. i.e. $\Omega^1 = \{(t,x_1); x_1 > \mu(t)\}$ is a domain of dependence.

Step 2: Find a domain of dependence $\Omega \subset \Omega^1 \times \mathbf{R}^2$, for u such that the solution is in H^k in Ω_0 but not in Ω_t for any $t > 0$.

The equation for u_1:
$\Box u \;=\; D_-^{\ell}\, u\, D_-^{k-\ell}\, u$ becomes

$$D_+\, D_-\, u_1 \;+\; D_-^{\ell}\, u_1\, D_-^{k-\ell}\, u_1 \;=\; 0, \qquad D_\pm = \partial_{x_1} \pm \partial_t$$

This equation can be solved by integrating along characteristics.

Choice of initial data:

$$D_- u_1(0, x_1) = \chi^{(3-k)}(x_1), \quad D_+ u_1(0, x_1) = 0$$

$$\chi(x_1) = \int_0^{x_1} -\varepsilon |\log|s/4||^{\alpha}\, ds, \quad 0 < \alpha < \frac{1}{2},\ \varepsilon > 0$$

Notice that data has a singularity at $x_1 = 0$. However, this singularity is not too strong. This will later ensure that $u(0, x)$ is in $H^k(\Omega_0)$.

We get a solution $u_1 \in C^\infty(\Omega^1)$ where Ω^1 is a domain of dependence for u_1 given by

$$\Omega^1 = \{(t, x_1); \mu(t) < x_1 < 2 - t\} \subset \mathbf{R}_+ \times \mathbf{R}^1$$

for some function $\mu(t)$ with $\mu(0) = 0$.
$u_1(t, x_1)$ has a singularity along $x_1 = \mu(t)$.

The non-linear effect: For the linear equation $D_+ D_- u_1 = 0$, the singularity in the data at $x_1 = 0$ would just propagate along a characteristic. However, nonlinearity causes the singularity to strengthen for $t > 0$. This is the same phenomena that causes blow-up for smooth initial data. Because $\chi'(0+) = -\infty$, the blow up happens directly close to $x_1 = 0$.

Defining Ω: Define $\Omega \subset \Omega^1 \times \mathbf{R}^2$ to be the largest domain of dependence for the metric obtained from the solution $u_1(t, x_1)$ such that $\Omega_0 = B_0$

$$\Omega_0 = \{x; (0, x) \in \Omega\}, \qquad B_0 = \{x; |x-(1,0,0)| < 1\}.$$

Initial data was chosen to just be in $H^k(\Omega_0)$. We must now show that the solution is not in $H^k(\Omega_t)$ for $t > 0$. **Integrate out x_2 and x_3:** Set

$$\Omega_t = \{x; (t, x) \in \Omega\}, \quad a_t(x_1) = \int_{(x_1,x_2,x_3)\in\Omega_t} dx_2\, dx_3,$$

so

$$\int_{\Omega_t} (D_-^{k-\ell} u_1(t, x_1))^2\, dx = \int_{\mu(t)}^{2-t} a_t(x_1)(D_-^{k-\ell} u_1(t, x_1))^2\, dx_1$$

To show that this equals infinity, prove lower bounds on the integrand close to $x_1 = \mu(t)$. In the quasi-linear case, estimating $a_t(x_1)$ from below involves controlling the geometry of the causal past $\Lambda_{t'x'}$.

Explicit formulas in the semi-linear case:
$g^{jk} = m^{jk}$, so Ω_1 is a domain of dependence if $\mu'(t) \geq 1$ and it follows that $\Omega = \Omega^1 \times \mathbf{R}^2 \cap \Lambda$, where $\Lambda = \{(t,x);\ |x-(1,0,0)| < 1-t\}$. Hence for $x > \mu(t)$:

$$a_t(x_1) = \int_{x_2^2+x_3^2<(1-t)^2-(1-x_1)^2} dx_2 dx_3$$
$$= \pi(2-t-x_1)(x_1-t)$$

Also the solution formulas are relatively simple, for example if $k-l=l=1$ then

$$D_- u_1(t,x_1) = \frac{\chi'(x_1-t)}{1+t\chi'(x_1-t)}, \quad u_1(0,x)=0$$

satisfies $D_+ D_- u_1 = (D_- u_1)^2$ when $1+t\chi'(x_1-t) > 0$

Since $\chi'(0+) = -\infty$ and $\chi'' > 0$ it follows that there is a function $\mu(t)$, with $\mu'(t) > 1$ and $\mu(0)=0$, such that $1+t\chi'(x_1-t)=0$, when $x_1 = \mu(t)$.

Hence $1+t\chi'(x_1-t) \leq C(t)(x_1-\mu(t))$ so

$$\int_{\mu(t)}^{1/2} a_t(x_1)(D_- u_1(t,x_1))^2\, dx_1 \geq$$
$$\int_{\mu(t)}^{1/2} \frac{(x_1-t)\, dx_1}{C(t)^2(x_1-\mu(t))^2} = \infty.$$

Open Problems

1) What is the minimal amount of regularity which ensures that a local solution u in H^γ exists for

$$\sum_{i,j=0}^{3} g^{ij}(u)\partial_{x_i}\partial_{x_j}u = F(u,u'), \quad x_0 = t,$$
$$u(0,x) = f(x) \in H^\gamma, \qquad u_t(0,x) = g(x) \in H^{\gamma-1}?$$

Note: the counterexamples give $\gamma > 2$.

A recent in provement by Tataru respectively Bahouri-Chemin shows that we have existence if $\gamma > 2+1/4$.

2) Einstein's equations can be written as a system of equations of the above form, where the non-linear term has some special structure. What is the minimal amount of regularity needed to ensure local existence for Einstein's equations? The answer to this question might depend on choice of coordinates. Einstein's equations do satisfy a "null-condition" but not in all coordinate systems. Ultimately, one would like to have an existence theorem in terms of the curvature so its invariant under coordinate changes.

3) What is the minimal amount of regularity γ needed for local existence in H^γ of

$$\Box u = -|u|^{p-1}u$$

if $p > 5$? Using the energy-bound one can prove that we always have weak solutions in H^1. These solutions might however not be unique and they probably don't depend continuously of initial data. We also have local existence in $H^{3/2}$. So the question is if we have local existence in H^γ for $1 < \gamma < 3/2$. Counterexample?