The motion of the free surface of a slightly compressible liquid and the incompressible limit

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Motion of a liquid body in vacuum
(water drop, the ocean, a star or a galaxy)
Incompressible or compressible perfect fluid
Without surface tension and gravitation
v-velocity, p-pressure, ρ-density, t-time

$\mathcal{D}_t$

$\mathcal{D}$

$\mathcal{D}_0$

Free boundary problem:
The velocity tells the boundary where to move. The boundary is the zero set of the pressure and the pressure determines the acceleration. (Regularity of the boundary is intimately connected to the regularity of the velocity.)

Euler's equations

$$\rho\,(\partial_t + V^k\partial_k)v_i = -\partial_i p \quad \text{in } \mathcal{D} \quad i{=}1,...,n$$
$$(\partial_t + V^k\partial_k)\rho + \rho\,\text{div}\,V = 0, \quad \text{in } \mathcal{D}$$

$\partial_k{=}\frac{\partial}{\partial x^k}$, $V^k{=}v_k$, $V^k\partial_k{=}\sum_{k=1}^n V^k\partial_k$, $\text{div}V{=}\partial_k V^k$

Equation of state

Incompressible case: $\rho = \text{constant}$

Compressible case: $p = p(\rho)$, $p'(\rho) > 0$

$p(\bar\rho_0) = 0$, Liquid: $\bar\rho_0 > 0$, Gas: $\bar\rho_0 = 0$

Boundary conditions

$$(\partial_t + V^k\partial_k)|_{\partial\mathcal{D}} \ \in T(\partial\mathcal{D})$$
$$p = 0, \qquad \text{on } \partial\mathcal{D}$$

$T(\partial\mathcal{D})$ is the tangent space of the boundary.

Initial conditions

$$\{x; (0,x) \in \mathcal{D}\} = \mathcal{D}_0$$
$$V(0,x) = V_0(x), \ \rho(0,x) = \rho_0(x), \ \text{in } \mathcal{D}_0$$

Compatibility cond. If $\bar\rho_0 > 0$ formal series solution $(\tilde V, \tilde\rho)$ in time should satisfy bound. cond.

$$(\partial_t + \tilde V^k\partial_k)^j(\tilde\rho - \bar\rho_0)\Big|_{\{0\}\times\partial\mathcal{D}_0} = 0, \ j = 0,...$$

Local Existence?:
Given a domain $\mathcal{D}_0 \subset \mathbf{R^n}$, a vector field V_0 and a function ρ_0 in $\mathcal{D}_0$ satisfying the compatibility conditions, find a domain $\mathcal{D} = \cup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$, $\mathcal{D}_t \subset \mathbf{R^n}$, a vector field V and a function ρ in $\mathcal{D}$, such that Euler's eq. hold.

Local existence for analytic data
Baouendi-Goulaouic, Nishida
(incompressible irrotational case)

Instability in Sobolev norms?
Rayleigh-Taylor Instability
(heavier fluid above lighter)
Ebin's counterexample (when $p < 0$, $\nabla_N p > 0$).

Physical condition

$$\nabla_N p \le -c_0 < 0, \quad \text{on } \partial\mathcal{D}_0,$$

where $\nabla_N = N^k \partial_k$ and N is the exterior normal
Since the pressure of a fluid has to be positive
Needed for local existence in Sobolev Spaces.

Vorticity: $\operatorname{curl} v_{ij} = \partial_i v_j - \partial_j v_i$
Incompressible fluid: $\operatorname{div} V = 0$
Irrotational fluid: $\operatorname{curl} v = 0$.

Local existence in Sobolev spaces:

I) Incompressible Irrotational case:
Local existence for Water wave problem:
Yosihara, Nalimov: close to still water in $\mathbf{R^2}$
Wu: in general in $\mathbf{R^2}$ and $\mathbf{R^3}$
(no instability when water wave turns over, physical cond. hold in the irrotational case)

II) General Incompressible case:
Christodoulou-L: Local *a priori* bounds for Sobolev norms. Sobolev norms remain bounded as long as the physical cond. hold, first order derivatives of the velocity and the second fundamental form of the free surface are bounded.
L: Local existence assuming physical cond.
Coutand-Shkoller: Local exist-surface tension.
Nordgren: Local exist-Newtonian self gravity.
Shatah-Zeng: Local wellposed-Fluid interface.

III) General Compressible case for a liquid
L: Local existence assuming physical cond.
Oliynyk:Local apriori bounds relativistic case

IV) General Compressible case for a gas
Makino-Local existence in degenerate case
Jang-Masmoudi: Local exist in one space dim.
Coutand-L-Shkoller: Local apriori bounds.
Coutand-Shkoller: Local exist.
Hadzic-Shkoller-Speck:Bounds relativistic case

Global existence for water wave problem
for a small localized disturbance of still water.
Wu, Germain-Masmoudi-Shatah: in $\mathbf{R^3}$
Ionescu-Putsateri, Ifrim-Tataru: in $\mathbf{R^2}$.

Irrotational Incompressible case

$$(\partial_t + V^k\partial_k)v_i = -\partial_i p$$
$$\text{div}\, V = 0, \qquad \text{curl}\, v = 0$$

Taking the divergence of Euler's eq's:

$$\triangle p = -(\partial_i V^j)(\partial_j V^i) < 0, \qquad p\Big|_{\partial\mathcal{D}} = 0$$

By strong maximum principle $\nabla_N p\Big|_{\partial\mathcal{D}} < 0$. Water wave problem, uniform gravitational field g

Incompressibility cond, $p > 0$ holds it together

If div$v$ = curl$v$ = 0 then $\triangle v_i = 0$ so V is determined by its boundary values and hence one can reduce to equations on the boundary only.

If the boundary was smooth, then inverting $\triangle p$ would give that $\partial p = O(V)$ and Euler's eq's would be an O.D.E. $(\partial_t + V^k\partial_k)V = O(V)$.

In general improved eq. for div V and curl v.

Lagrangian coordinates: $f_t : y \to x = \eta(t,y)$:

$$d\eta/dt = V(t,\eta), \qquad \eta(0,y) = f_0(y),\ y \in \Omega$$

Boundary becomes fixed in the (t,y) coord.

$\mathcal{D}_t$

Lagrangian (t,y)	Eulerian (t,x)
$[0,T] \times \Omega$	$\mathcal{D} = \cup_{0\le t\le T}\{t\} \times \mathcal{D}_t$
$D_t = \partial_t$	$D_t = \partial_t + V^k \partial_k$
$\partial_k = \frac{\partial y^a}{\partial x^k}\frac{\partial}{\partial y^a}$	$\partial_k = \frac{\partial}{\partial x^k}$

Euler's eq:

$$\rho\, D_t v_i = -\partial_i p, \qquad D_t \rho + \rho \operatorname{div} V = 0$$

Coordinates: $D_t \eta^i = V^i$

$$D_t J - J \operatorname{div} V = 0, \qquad J = \det(\partial\eta/\partial y)$$

$(D_t \det(M) = \det(M)\operatorname{tr}(M^{-1} D_t M).)$ **so:**

$$\rho\, D_t^2 \eta^i = -\partial_i p, \qquad \rho = \rho_0 J_0 / J, \qquad p = p(\rho)$$

Energy Conservation $E_0(t) = E_0(0)$ where

$$E_0(t) = \int_{\mathcal{D}_t} \Big(|V|^2 + Q(\rho)\Big)\rho dx, \quad Q(\rho) = 2\int \frac{p(\rho)}{\rho^2} d\rho$$

Proof of Energy conservation: We have

$$\int_{\mathcal{D}_t} h\,\rho dx = \int_\Omega h\,\rho J dy, \quad J = \det(\partial x/\partial y), \quad D_t(\rho J) = 0$$

so by the above and the divergence theorem

$$\frac{d}{dt} E_0 = \int_{\mathcal{D}_t} (D_t(|V|^2 + Q(\rho)))\rho dx$$

$$= \int_{\mathcal{D}_t} (-2V^i \partial_i p + 2p\rho^{-1} D_t \rho)\, dx$$

$$= -\int_{\partial\mathcal{D}_t} 2N_i V^i p\, dS + \int_{\mathcal{D}_t} 2(\partial_i V^i)p + 2p\rho^{-1} D_t \rho dx = 0$$

by the boundary cond. and Euler's eq.

Higher order Energies Incompressible Liquid (with Christodoulou)

$$E_r(t) = \|v\|_{H^r(\mathcal{D}_t)} + \|\theta\|_{H^{r-2}(\partial\mathcal{D}_t)},$$

$\theta_{ij} = \bar{\partial}_i N_j$ is the second fundamental form of $\partial\mathcal{D}_t$

Energy bound: If $\nabla_N p \leq -c_0 < 0$ then

$$E_r(t) \leq C_r(t, c_0^{-1}, ...)E_r(0).$$

Higher order Energies-Compressible Gas (with Coutand-Shkoller), For simplicity take $p = \rho^\gamma$, $\gamma = 2$. For $1 \leq \gamma \leq 2$ can reduce to this.

$$
\begin{aligned}
E_r(t) = \sum_{k=0}^{r} \|D_t^{2k}\eta\|_{H^{r-k}(\mathcal{D}_t)} \\
+ \sum_{k=0}^{r} \|\rho_0 D_t^{2k}\overline{D}^{r-k} D\eta\|_{L^2(\mathcal{D}_t)} \\
+ \sum_{k=0}^{r} \|\rho_0^{1/2} D_t^{2k}\overline{D}^{r-k} v\|_{L^2(\mathcal{D}_t)} \\
+ \sum_{k=0}^{r} \|\rho_0 D_t^{2k} J\|_{H^{r-k}(\mathcal{D}_t)} \\
+ \|\mathrm{curl}\, v\|_{H^{r-1}(\mathcal{D}_t)} + \|\rho_0 \overline{D}^r \mathrm{curl}\, v\|_{L^2(\mathcal{D}_t)}
\end{aligned}
$$

where $D = D_y$ and $\overline{D}$ are derivatives in y that are tangential at the boundary. Here $J = \det(D\eta)$

Energy bound: If $\nabla_N p/\rho \leq -c_0 < 0$ and $r \geq 4$

$$
E_r(t) \leq C_r(t, c_0^{-1}, ..., E_{r-1}(t)) E_r(0).
$$

Higher order Energies Compressible Liquid (with Chenyun Luo)

$$\rho\,(\partial_t + V^k\partial_k)v_i = -\partial_i p \quad \text{in } \mathcal{D} \quad i{=}1,...,n$$
$$(\partial_t + V^k\partial_k)\rho + \rho\,\text{div}\,V = 0, \quad \text{in } \mathcal{D}$$

$$p = p(\rho),\;\; p'(\rho) > 0,\;\; p(\bar\rho_0) = 0,\; \bar\rho_0 > 0$$

Higher order Energies roughly:

$$E_r(t) = \|v\|_{H^r(\mathcal{D}_t)} + \sum_{k=0,1} \|D_t^k\rho\|_{H^r(\mathcal{D}_t)} + \|\theta\|_{H^{r-2}(\partial\mathcal{D}_t)},$$

$\theta_{ij} = \bar\partial_i N_j$ is the second fundamental form of $\partial\mathcal{D}_t$

Energy bound roughly: If $\nabla_N p \leq -c_0 < 0$ then

$$E_r(t) \leq C_r(t, c_0^{-1}, ..., E_{r-1}(t))E_r(0).$$

Incompressible limit exist: $\rho{=}\rho_\kappa(p)$.

If $\rho_\kappa(p) \to \bar\rho_0$, $\kappa \to \infty$, then the sol. v_κ to the free boundary problem for the compressible Euler with density $\rho = \rho_\kappa(p)$ converges to a sol. to the incompressible case with the 'same divergence free data'.

Long time existence for the free boundary problem for compressible Euler? Blowup for compressible Euler in all space (Sideris)

Euler's eq. If $h(\rho)$ the enthalphy ($h' = p'/\rho$):

$$D_t v^i = -\partial_i h,$$
$$D_t e(h) = -\text{div}\, v,$$
$$D_t \eta^i = v^i$$

where $x = \eta(t,y)$, $D_t = \partial_t\big|_{y=const}$, $\partial_i = (\partial y^a/\partial x^i)\partial/\partial y^a$
Here $e(h) = \log\rho(h)$ is a given smooth function:

$$e'(h) > 0,\ \ e(0) = e_0 > 0,\ |e^{(k)}(h)| \le C\sqrt{e'(h)} \le C'$$

We assume that initial data (v_0, h_0) satisfies

$$h_0\Big|_{\partial\mathcal{D}_0} = 0, \quad \text{and} \quad h_0 > 0, \text{ in } \mathcal{D}_0$$

and Physical condition

$$\nabla_N h_0\Big|_{\partial\mathcal{D}_0} \le -c_0 < 0$$

Commuting divergence with Euler's eq. using

$$[D_t, \partial_i] = -(\partial_i v^k)\partial_k.$$

gives a wave equation for the enthalpy

$$D_t^2 e(h) - \triangle h = (\partial_i v^j)(\partial_j v^i), \text{ in } [0,T]\times\Omega,$$

with boundary conditions $h|_{\partial\Omega} = 0$.

Energies for Euler's eq.

$$E_r(t) = \sum_{k+s=r} \frac{1}{2}\int_{\mathcal{D}_t} \delta^{ij} Q(\partial^s D_t^k v_i, \partial^s D_t^k v_j)\, \rho dx$$
$$+ \sum_{k+s=r} \frac{1}{2}\int_{\mathcal{D}_t} e'(h) Q(\partial^s D_t^k h, \partial^s D_t^k h)\, \rho dx$$
$$+ \sum_{k+s=r} \frac{1}{2}\int_{\partial\mathcal{D}_t} Q(\partial^s D_t^k h, \partial^s D_t^k h)\nu\, \rho dS,$$

$\nu = (-\nabla_N h)^{-1}$. Q is a quad. form restricting to norm of the tangential components at $\partial\mathcal{D}_t$.

$$Q(\alpha,\beta) = q^{i_1 j_1}\cdots q^{i_s j_s}\alpha_{i_1\cdots i_s}\beta_{j_1\cdots j_s},$$

$$q^{ij} = \delta^{ij} - \eta(d)^2 \mathcal{N}^i\mathcal{N}^j,$$
$$d(x) = \text{dist}(x, \partial\mathcal{D}_t), \quad \mathcal{N}^i = -\delta^{ij}\partial_j d.$$

In addition one also has to add

$$F_r(t) = \int_{\mathcal{D}_t} |\partial^{r-1}\text{curl} v|^2\, \rho dx$$

and the energy for the wave equation

$$W_{r+1}(t) = \frac{1}{2}\int_{\mathcal{D}_t} e'(h)(D_t^{r+1}h)^2 + |\nabla D_t^r h|^2\, \rho dx$$

Energy bound $\tilde{E}_r = E_r + F_r + W_{r+1}$.

$$|\frac{d\tilde{E}_r(t)}{dt}| \leq C_r\Big(K, \frac{1}{\epsilon}, M, \text{vol}\mathcal{D}_t, \tilde{E}_{r-1}(t)\Big)(\tilde{E}_r(t),$$

provided that

$$|\theta| + \frac{1}{l_0} \leq K, \;\; \text{on}\, \partial\mathcal{D}_t,$$

$$-\nabla_N h \geq \epsilon > 0, \;\; \text{on}\, \partial\mathcal{D}_t$$

$$\bar{\rho}_0 \leq |\rho| \leq M, \;\; \text{in}\, \mathcal{D}_t$$

$$|\partial v| + |\partial h| + |\partial^2 h| + |\partial D_t h| \leq M, \;\; \text{in}\, \mathcal{D}_t.$$

Elliptic bounds Let

$$\langle\langle h \rangle\rangle_r^2 = \sum_{s+k=r} \int_{\partial\mathcal{D}_t} |\partial^s D_t^k h|^2 \, dS,$$

$$\|h\|_r^2 = \int_{\mathcal{D}_t} \sum_{s+k=r, s<r} |\partial^s D_t^k h|^2 + e'(h)(D_t^r h)^2 dx$$

$$\langle\langle h \rangle\rangle_r^2 + \|v\|_r^2 + \|h\|_r^2 + \|D_t h\|_r^2 \leq C_r(...)\tilde{E}_r$$

Energy estimate:

$$\frac{dE_r(t)}{dt} = \sum_{k+s=r} \int_{\mathcal{D}_t} \delta^{ij} Q(\partial^s D_t^k v_i, D_t \partial^s D_t^k v_j)\, \rho dx$$
$$+ \sum_{k+s=r} \int_{\mathcal{D}_t} e'(h) Q(\partial^s D_t^k h, D_t \partial^s D_t^k h)\, \rho dx$$
$$+ \sum_{k+s=r} \int_{\partial\mathcal{D}_t} Q(\partial^s D_t^k h, D_t \partial^s D_t^k h) \nu\, \rho dS + \text{L.O.},$$

Using the commutator $[D_t, \partial_i] = -\partial_i v^k \partial_k$ and the equation $D_t v_i = -\partial_i h$ we get

$$D_t \partial^s D_t^k v_j = -\partial^s D_t^k \partial_j h + \text{L.O.}$$
$$D_t \partial^r h + (\partial_j h) \partial^r v^j = \partial^r D_t h + \text{L.O.},$$
$$D_t \partial^s D_t^k h = \partial^s D_t^{k+1} h + \text{L.O.}$$

The first term become after integration by parts

$$-\int_{\mathcal{D}_t} \delta^{ij} Q(\partial^s D_t^k \partial^i v_i, D_t \partial^s D_t^k h)\, \rho dx$$
$$+ \int_{\partial\mathcal{D}_t} Q(\partial^s D_t^k v_i, D_t \partial^s D_t^k h) \nu\, \rho N^i dS + \text{L.O.}$$

which cancel the main contribution of the other.

The term $Q(\cdot, \partial^r D_t h)$ is lower order because Q projects to the boundary where $D_t h = 0$.

Elliptic Estimate We have

$$\int_{\partial\mathcal{D}_t} Q(\partial^r q, \partial^r q)\nu\, \rho dS = \int_{\partial\mathcal{D}_t} |\Pi\partial^r q|^2 \nu\, \rho dS$$

where Π is the projection to the tangent space of the boundary. We have

$$\Pi\nabla^2 q = \overline{\nabla}^2 q + \theta\nabla_N q, \tag{1}$$

where the tangential component $\overline{\nabla}^2 q = 0$ on the boundary. In general, the higher order projection formula is of the form

$$\Pi\nabla^r q = (\overline{\nabla}^{r-2}\theta)\nabla_N q + O(\nabla^{r-1} q) + ...,$$