Global Existence and Scattering for Einstein's equations and related equations satisfying the weak null condition

Hans Lindblad

Harvard

Overview

1. Stability of Minkowski Spacetime with matter fields
2. Asymoptotics and scattering
3. The weak null condition for systems of quasilinear wave equations
4. The weak null structure for Einstein's eq. and commutators
5. Asymptotics of the metric and the characteristic surfaces
6. Recasting Einstein in coord. adapted to Schwarzschild at null infinity
7. Einstein-Maxwell-Klein-Gordon
8. Einstein-Vlasov
9. The lifted vector fields to momentum space

Overview

General Relativity and the Einstein Equations

Mass and matter curve spacetime and gravity is a result of the curvature.

Einstein's equations determine a $4d$ manifold M with a Lorentzian metric g

$$Ric(g)_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R(g) = T_{\mu\nu},$$

where $Ric(g)_{\mu\nu}$ denotes the Ricci curvature, $R(g)$ the scalar curvature.

Here $T_{\mu\nu}$, the stress–energy–momentum tensor of matter, depends on additional fields that satisfy additional equations that imply $\nabla_\mu T^{\mu\nu} = 0$.
Minkowski $m = \text{diag}(-1, 1, 1, 1)$ is a solution to the vacuum case $T = 0$.

General Relativity and the Einstein Equations

Mass and matter curve spacetime and gravity is a result of the curvature.

Einstein's equations determine a $4d$ manifold M with a Lorentzian metric g

$$Ric(g)_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R(g) = T_{\mu\nu},$$

where $Ric(g)_{\mu\nu}$ denotes the Ricci curvature, $R(g)$ the scalar curvature.

Here $T_{\mu\nu}$, the stress–energy–momentum tensor of matter, depends on additional fields that satisfy additional equations that imply $\nabla_\mu T^{\mu\nu} = 0$. Minkowski $m = \mathrm{diag}(-1, 1, 1, 1)$ is a solution to the vacuum case $T = 0$.

The initial value problem: Given a 3d manifold Σ, with Riemannian metric $\overline{g}$, and a symmetric two-tensor k, we want to find a 4d manifold $(\mathcal{M}, g)$ satisfying Einstein equations, and an imbedding $\Sigma \subset \mathcal{M}$ such that $\overline{g}$ is the restriction of g to Σ and k is the second fundamental form of Σ.

The problem is over determined so data must satisfy *constraint equations*:

$$\overline{\mathrm{div}}\, k_j - (d\,\overline{\mathrm{tr}}\, k)_j = T_{0j}, \qquad \overline{R}(\overline{g}) + (\overline{\mathrm{tr}}\, k)^2 - |k|^2_{\overline{g}} = 2T_{00}.$$

Wave coordinates and local existence

EE are invariant under diffeomorphisms. Eliminate freedom by fixing *gauge*. *Harmonic gauge* or *wave coordinates* are solutions of the wave equations

$$\Box_g x^\mu = 0, \quad \text{where} \quad \Box_g = g^{\alpha\beta}\partial_\alpha\partial_\beta + g^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \nu}\partial_\nu$$

Here $g^{\alpha\beta}$ is the inverse of the metric. $\Gamma_{\alpha\beta}^{\ \ \nu}$ are the Christoffel symbols. The metric in wave coordinates satisfy

$$g_{\mu\nu}g^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \nu} = g^{\alpha\beta}\partial_\beta g_{\alpha\mu} - \tfrac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = 0. \tag{1}$$

In wave coord. Einstein's eq. are a system of nonlinear wave equations

$$\widetilde{\Box}_g\, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g) + \widehat{T}_{\mu\nu}, \quad \text{where} \quad \widetilde{\Box}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta, \tag{2}$$

and $\widehat{T}_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}\,\mathrm{tr}_g\, T/2$.

(1) is preserved under (2).

Wave coordinates and local existence

EE are invariant under diffeomorphisms. Eliminate freedom by fixing *gauge*. *Harmonic gauge* or *wave coordinates* are solutions of the wave equations

$$\Box_g x^\mu = 0, \quad \text{where} \quad \Box_g = g^{\alpha\beta}\partial_\alpha\partial_\beta + g^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \nu}\partial_\nu$$

Here $g^{\alpha\beta}$ is the inverse of the metric. $\Gamma_{\alpha\beta}^{\ \ \nu}$ are the Christoffel symbols. The metric in wave coordinates satisfy

$$g_{\mu\nu}g^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \nu} = g^{\alpha\beta}\partial_\beta g_{\alpha\mu} - \tfrac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = 0. \tag{1}$$

In wave coord. Einstein's eq. are a system of nonlinear wave equations

$$\widetilde{\Box}_g\, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g) + \widehat{T}_{\mu\nu}, \quad \text{where} \quad \widetilde{\Box}_g = g^{\alpha\beta}\partial_\alpha\partial_\beta, \tag{2}$$

and $\widehat{T}_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}\,\mathrm{tr}_g\, T/2$.

(1) is preserved under (2).

Chouquet-Bruhat (1952): Given smooth initial data on $\Sigma = \{t = 0\}$

$$g|_{t=0} = \overline{g}, \qquad \partial_t g|_{t=0} = k, \tag{3}$$

that satisfy constraint and coord. cond., Einstein's eq. (2)-(3) with vacuum or Vlasov matter has a local solution in $0 \le t \le t_0$ for some $t_0 > 0$

Global stability of Minkowski space for vacuum equations

Christodoulou-Klainerman (1989) [CK]
Constructing a global solution from initial data which is close to and asymptotically approaching the Minkowski metric $m = \mathrm{diag}(-1,1,1,1)$.

Smallness assumption on data $(\Sigma, \overline{g}, k)$: Σ is diffeomorphic to $\mathbf{R}^3$ and data are close to the data for Minkowski space $(\mathbf{R}^3, \delta, 0)$.

Initial data $(\mathbf{R}^3, \overline{g}, k)$ are *asymptotically flat*, i.e. for some $\gamma > 0$

$$\overline{g}_{ij} = (1 + 2M/r)\delta_{ij} + o(r^{-1-\gamma}), \quad k = o(r^{-2-\gamma}), \quad \text{as} \quad r = |x| \to \infty.$$

Here $M > 0$ by the positive mass theorem by Schoen-Yau (CK: $\gamma > 1/2$).

Global stability of Minkowski space for vacuum equations

Christodoulou-Klainerman (1989) [CK]
Constructing a global solution from initial data which is close to and asymptotically approaching the Minkowski metric $m = \mathrm{diag}(-1,1,1,1)$.

Smallness assumption on data $(\Sigma, \overline{g}, k)$: Σ is diffeomorphic to $\mathbf{R}^3$ and data are close to the data for Minkowski space $(\mathbf{R}^3, \delta, 0)$.

Initial data $(\mathbf{R}^3, \overline{g}, k)$ are *asymptotically flat*, i.e. for some $\gamma > 0$

$$\overline{g}_{ij} = (1 + 2M/r)\delta_{ij} + o(r^{-1-\gamma}), \quad k = o(r^{-2-\gamma}), \quad \text{as} \quad r = |x| \to \infty.$$

Here $M > 0$ by the positive mass theorem by Schoen-Yau (CK: $\gamma > 1/2$).

Proof > 500 pages uses

Equations for the *curvature* tensor $R_{\alpha\beta\gamma\mu\nu}$ since it was believed the metric in wave coordinates blows up.

Coordinates and *vector fields* adapted to the outgoing curved light cones or *characteristic surfaces*, $u = const$, where u solves the eikonal equation

$$g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0.$$

Global stability of Minkowski space for vacuum equations

Christodoulou-Klainerman (1989) [CK]
Constructing a global solution from initial data which is close to and asymptotically approaching the Minkowski metric $m = \mathrm{diag}(-1, 1, 1, 1)$.

Smallness assumption on data $(\Sigma, \overline{g}, k)$: Σ is diffeomorphic to $\mathbf{R}^3$ and data are close to the data for Minkowski space $(\mathbf{R}^3, \delta, 0)$.

Initial data $(\mathbf{R}^3, \overline{g}, k)$ are *asymptotically flat*, i.e. for some $\gamma > 0$

$$\overline{g}_{ij} = (1 + 2M/r)\delta_{ij} + o(r^{-1-\gamma}), \quad k = o(r^{-2-\gamma}), \quad \text{as} \quad r = |x| \to \infty.$$

Here $M > 0$ by the positive mass theorem by Schoen-Yau (CK: $\gamma > 1/2$).

Proof $>$ 500 pages uses

Equations for the *curvature* tensor $R_{\alpha\beta\gamma\mu\nu}$ since it was believed the metric in wave coordinates blows up.

Coordinates and *vector fields* adapted to the outgoing curved light cones or *characteristic surfaces*, $u = const$, where u solves the eikonal equation

$$g^{\alpha\beta}\partial_\alpha u\, \partial_\beta u = 0.$$

Global Existence for Einstein's eq. in Wave coordinates

L-Rodnianski (2010) [LR]
Global existence in wave coordinates
with matter field (scalar field)
$\gamma > 0$.

Global Existence for Einstein's eq. in Wave coordinates

L-Rodnianski (2010) [LR]
Global existence in wave coordinates
with matter field (scalar field)
$\gamma > 0$.

Proof $\sim$ 50 pages uses

Equations for the *metric* in wave coordinates

Identified a *weak null condition* that prevents blow up and showed that EE in wave coordinates satisfy this condition.

Vector fields adapted to the fixed background Minkowski light cones
$t - |x| = const$.

Work on Stability of Minkowski Space without symmetry

Using Bianchi system for curvature

- Christodoulou–Klainerman (1993): Vacuum;
- Zipser (2000): Maxwell;
- Klainerman–Nicolo (2003): Vacuum;
- Bieri (2007): Vacuum;
- Taylor (2015): Vlasov massless case

Using harmonic coordinates

- L–Rodnianski (2005,2010): Vacuum, Scalar Field;
- Loizelet (2008): Maxwell;
- Speck (2010): "Large class of nonlinear electromagnetic equations";
- Le Floch–Ma (2015): Klein–Gordon;
- Huneau (2015): Vacuum with translational symmetry.
- Lindblad-Taylor, Fajman–Joudioux–Smulevici (2017): Massive Vlasov.
- Kauffman-Lindblad(2019):Maxwell-Klein-Gordon(charged scalar field)

Using conformal method

- Friedrich (1986) For special data

Overview

Asymptotics

L (2017)

The outgoing light cones for the Schwarzschild metric in wave coord. satisfy $t - r^* = u^*$, where $r^* = r + M\ln r$. There is a solution to the eikonal equation that approaches the one for Schwarzschild with the same mass M.

$$g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0, \qquad u \to u^* = t - r^*, \quad \text{when} \qquad r > t/2 \to \infty.$$

Asymptotics

L (2017)

The outgoing light cones for the Schwarzschild metric in wave coord. satisfy $t - r^* = u^*$, where $r^* = r + M \ln r$. There is a solution to the eikonal equation that approaches the one for Schwarzschild with the same mass M.

$$g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0, \qquad u \to u^* = t - r^*, \quad \text{when} \qquad r > t/2 \to \infty.$$

The metric $g = m + h$ has the asymptotic expansion

$$h_{\mu\nu}(t, r\omega) \sim \frac{H_{\mu\nu}(r^* - t, \omega)}{r^*} + \ln\Big(\frac{r^*}{\langle r^* - t\rangle}\Big)\frac{N_{\mu\nu}(r^* - t, \omega)}{r^*}, \quad \text{as } \frac{r^*}{\langle r^* - t\rangle} \to \infty, \tag{4}$$

where

$$N_{\mu\nu}(q^*, \omega) = L_\mu L_\nu \int_{q^*}^{\infty} n(q, \omega)\,dq, \quad n(q, \omega) = P(\partial_q H, \partial_q H)(q, \omega),$$

$L_0 = 1$, $L_i = -\omega_i$, $P(H, K) = -\frac{1}{2}\delta^{AB}\delta^{CD}H_{AC}H_{BD}$. Here $H_{AB} = A^\mu B^\nu H_{\mu\nu}$, and the sums are over $A, B = S_1, S_2$, an orthonormal frame of S^2.

Scattering

Moreover $H^1_{\mu\nu} = H_{\mu\nu} - M\delta_{\mu\nu}$ satisfies

$$\delta^{AB} H^1_{AB} = H^1_{LT} = 0, \text{ for } T = L, S_1, S_2, \tag{5}$$

$$|(\langle q\rangle \partial_q)^k H_1(q,\omega)| \lesssim \langle q_+\rangle^{-\gamma} \tag{6}$$

$$\int\int n(q,\omega) dq d\omega = M \tag{7}$$

Here $q_+ = q$, when $q > 0$ and $q_+ = 0$, when $q < 0$, where $q = r - t$.

Scattering

Moreover $H^1_{\mu\nu} = H_{\mu\nu} - M\delta_{\mu\nu}$ satisfies

$$\delta^{AB} H^1_{AB} = H^1_{LT} = 0, \text{ for } T = L, S_1, S_2, \tag{5}$$

$$|(\langle q\rangle \partial_q)^k H_1(q,\omega)| \lesssim \langle q_+\rangle^{-\gamma} \tag{6}$$

$$\int\int n(q,\omega) dq d\omega = M \tag{7}$$

Here $q_+ = q$, when $q > 0$ and $q_+ = 0$, when $q < 0$, where $q = r - t$.

He-L-Schlue (Work in progress)

Given $H^1_{\mu\nu}$ satisfying (5)-(6), let $M > 0$ be defined by (7). Then there is a unique solution to EE with the asymptotics (4).

The Trautman-Bondi mass

L-Rodnianski

Change the metric to the $*$ coordinates (t, x^*), $x^* = r^*\omega$, $g^*_{\mu\nu}$.
Einstein-vacuum equations can be written in the form

$$|g^*|\pi^{\alpha\beta} = \partial_\mu \lambda^{\alpha\beta\mu},$$

where

$$\lambda^{\alpha\beta\mu} = \partial_\nu\big(|g^*|(g^{*\alpha\beta}g^{*\mu\nu} - g^{*\alpha\mu}g^{*\beta\nu})\big) = -\lambda^{\alpha\mu\beta}$$

and $\pi^{\alpha\beta} = \pi^{\alpha\beta}(\partial g^*, \partial g^*) = \pi^{\beta\alpha}$ is the Landau-Lifshitz pseudo tensor that is divergence free

$$\partial^*_\beta\big(|g^*|\pi^{\alpha\beta}\big) = 0.$$

We show that the limit exist $(q^* = r^* - t)$

$$m^\alpha_T(q^*, \omega) = \lim_{r^*\to\infty} (r^*)^2\big(\lambda^{\alpha\beta\gamma}\underline{L}_\gamma L_\beta\big)(r^*, q^*, \omega),$$

where $\underline{L}_0 = 1$, $\underline{L}_i = \omega_i$, and that the mass

$$M^\alpha_T(q^*) = \int_{\mathbf{S}^2} m^\alpha_T(q^*, \omega)\, dS(\omega),$$

satisfies the mass law

$$M^\alpha_T(q^*_2) = M^a_T(q^*_1) - \frac{1}{4}\int_{q^*_1}^{q^*_2}\int_{\mathbb{S}^2} L^\alpha\, n(q, \omega) d\omega\, dq$$

Overview

Einstein's equations in wave coordinates

$$\widetilde{\Box}_g \, h_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + \widehat{T}_{\mu\nu},$$

$$F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$
$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta} \, \partial_\nu h_{\alpha'\beta'} - \tfrac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'} \, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}.$$

$Q_{\mu\nu}(\partial h, \partial h)$ satisfy the standard null condition. $G_{\mu\nu}(h)(\partial h, \partial h) \lesssim |h||\partial h|^2$.

Einstein's equations in wave coordinates

$$\widetilde{\Box}_g \, h_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + \widehat{T}_{\mu\nu},$$

$$F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$
$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta} \, \partial_\nu h_{\alpha'\beta'} - \tfrac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'} \, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}.$$

$Q_{\mu\nu}(\partial h, \partial h)$ satisfy the standard null condition. $G_{\mu\nu}(h)(\partial h, \partial h) \lesssim |h||\partial h|^2$.

Einstein's equations in wave coordinates

$$\widetilde{\square}_g\, h_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + \widehat{T}_{\mu\nu},$$

$$F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$
$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta}\, \partial_\nu h_{\alpha'\beta'} - \tfrac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'}\, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}.$$

$Q_{\mu\nu}(\partial h, \partial h)$ satisfy the standard null condition. $G_{\mu\nu}(h)(\partial h, \partial h) \lesssim |h||\partial h|^2$.

With $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ the wave coordinate condition can be expressed

$$\partial_\mu \widehat{H}^{\mu\nu} = W^\nu(h, \partial h), \quad \text{where} \quad \widehat{H}^{\mu\nu} = H^{\mu\nu} - m^{\mu\nu}\mathrm{tr}_m H/2, \quad \mathrm{tr} H = m_{\alpha\beta} H^{\alpha\beta},$$

and $W^\nu(h, \partial h) = (g^{\mu\nu} g_{\alpha\beta} - m^{\mu\nu} m_{\alpha\beta})\partial_\mu g^{\alpha\beta}/2 = O(h\,\partial h)$.

Quasilinear wave equations and the null condition

Blow up for small data (John) e.g. $\Box\phi = (\partial_t \phi)^2$.

Quasilinear wave equations and the null condition

Blow up for small data (John) e.g. $\Box\phi = (\partial_t\phi)^2$.

Global existence if an algebraic condition called the null condition (Christodoulou,Klainerman) e.g. Nirenberg's example $\Box\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2$.

Need εt^{-1} decay for ϕ to handle general quadratic nonlinear terms.
Need better than $\varepsilon^2 t^{-2}$ decay of the quadratic terms to achieve this.

Quasilinear wave equations and the null condition

Blow up for small data (John) e.g. $\Box\phi = (\partial_t\phi)^2$.

Global existence if an algebraic condition called the null condition (Christodoulou,Klainerman) e.g. Nirenberg's example $\Box\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2$.

Need εt^{-1} decay for ϕ to handle general quadratic nonlinear terms.
Need better than $\varepsilon^2 t^{-2}$ decay of the quadratic terms to achieve this.

Problem: Einstein's eq. do not satisfy the null condition

Quasilinear wave equations and the null condition

Blow up for small data (John) e.g. $\Box\phi = (\partial_t \phi)^2$.

Global existence if an algebraic condition called the null condition (Christodoulou,Klainerman) e.g. Nirenberg's example $\Box\phi = (\partial_t\phi)^2 - |\nabla_x\phi|^2$

Need εt^{-1} decay for ϕ to handle general quadratic nonlinear terms.
Need better than $\varepsilon^2 t^{-2}$ decay of the quadratic terms to achieve this.

Problem: Einstein's eq. do not satisfy the null condition

Einstein's eq. satisfy a *weak null condition*(LR).
Essentially the system decouples in a *null-frame*

$$\Box\phi_2 = (\partial_t \phi_1)^2, \qquad \Box\phi_1 = 0$$

$\partial\phi_1 \sim \varepsilon t^{-1}$, $\partial\phi_2 \sim \varepsilon t^{-1} \ln|t|$

Global existence (L, Alinhac) $\Box\phi = \phi\triangle\phi$ but solutions decay like $\varepsilon t^{-1+c\varepsilon}$.

Decay and the role of vector fields

A solution to the linear wave eq. with compactly supported data decays

$$\Box\,\phi = m^{\alpha\beta}\partial_\alpha\partial_\beta\,\phi = 0 \implies |\,\phi| \leq \frac{C}{t}, \quad |\,\partial\phi| \leq \frac{C}{t}.$$

Decay and the role of vector fields

A solution to the linear wave eq. with compactly supported data decays

$$\square\,\phi = m^{\alpha\beta}\partial_\alpha\partial_\beta\,\phi = 0 \implies |\,\phi| \leq \frac{C}{t}, \quad |\,\partial\phi| \leq \frac{C}{t}.$$

A wave starting from the origin will after time t be supported close to sphere S_t of radius t with total conserved energy

$$E = \int |\partial\phi|^2 dx \sim |\partial\phi|^2 \mathrm{Area}(S_t) \sim |\partial\phi|^2 t^2 \sim const.$$

Decay and the role of vector fields

A solution to the linear wave eq. with compactly supported data decays

$$\Box\,\phi = m^{\alpha\beta}\partial_\alpha\partial_\beta\,\phi = 0 \implies |\,\phi| \le \frac{C}{t}, \quad |\,\partial\phi| \le \frac{C}{t}.$$

A wave starting from the origin will after time t be supported close to sphere S_t of radius t with total conserved energy

$$E = \int |\partial\phi|^2 dx \sim |\partial\phi|^2 \text{Area}(S_t) \sim |\partial\phi|^2 t^2 \sim const.$$

Derivatives tangential to the outgoing light cones $t = |x|$ decay faster

$\Box\phi = 0 \implies |\partial\phi| \le C/t$ and $|\bar{\partial}\phi| + |(\partial_t + \partial_r)\phi| \le C/t^2$, where $\bar{\partial}_i = \partial_i - \omega_i\partial_r$.

Decay and the role of vector fields

A solution to the linear wave eq. with compactly supported data decays
$\Box\,\phi = m^{\alpha\beta}\partial_\alpha\partial_\beta\,\phi = 0 \implies |\,\phi| \leq \frac{C}{t}, \quad |\,\partial\phi| \leq \frac{C}{t}.$

A wave starting from the origin will after time t be supported close to sphere S_t of radius t with total conserved energy

$$E = \int |\partial\phi|^2 dx \sim |\partial\phi|^2 \text{Area}(S_t) \sim |\partial\phi|^2 t^2 \sim const.$$

Derivatives tangential to the outgoing light cones $t = |x|$ decay faster
$\Box\phi=0 \implies |\partial\phi|\leq C/t$ and $|\bar{\partial}\phi|+|(\partial_t+\partial_r)\phi|\leq C/t^2$, where $\bar{\partial}_i = \partial_i - \omega_i\partial_r$.

From the invariance of the wave operator under Lorentz transformations.

Generator $Z = x_a\partial_b - x_b\partial_a$, $[\Box, Z] = 0$, $\Box\, Z\phi = 0 \Longrightarrow |Z\phi| \leq \dfrac{C}{t}$,

$Z = r\bar{\partial} \Longrightarrow |\bar{\partial}\phi| \leq \dfrac{C}{t^2}.$

The asymoptotic system for a system of nonlinear wave eq.

Consider a generic system of nonlinear wave equations

$$\Box\phi_I = \sum A^{JK}_{I,\alpha\beta}\, \partial^\alpha\phi_J\, \partial^\beta\phi_K + \text{cubic terms.} \tag{8}$$

The asymptotic system for $\Phi_I = r\phi_I$ is given by

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1} \sum A^{JK}_{I,nm}(\partial_t - \partial_r)^n \Phi_J\, (\partial_t - \partial_r)^m \Phi_K, \tag{9}$$

where

$$A^{JK}_{I,nm} = \sum\nolimits_{|\alpha|=n, |\beta|=m} \tfrac{1}{(-2)^{m+n}} A^{JK}_{I,\alpha\beta}\, \hat{\omega}^\alpha \hat{\omega}^\beta, \quad \hat{\omega} = (-1, \omega),\ \omega \in \mathbf{S}^2.$$

The asymoptotic system for a system of nonlinear wave eq.

Consider a generic system of nonlinear wave equations

$$\Box\phi_I = \sum A_{I,\alpha\beta}^{JK}\, \partial^\alpha \phi_J\, \partial^\beta \phi_K + \text{cubic terms.} \tag{8}$$

The asymptotic system for $\Phi_I = r\phi_I$ is given by

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1} \sum A_{I,nm}^{JK} (\partial_t - \partial_r)^n \Phi_J\, (\partial_t - \partial_r)^m \Phi_K, \tag{9}$$

where

$$A_{I,nm}^{JK} = \sum_{|\alpha|=n,|\beta|=m} \tfrac{1}{(-2)^{m+n}} A_{I,\alpha\beta}^{JK}\, \hat\omega^\alpha \hat\omega^\beta, \quad \hat\omega = (-1,\omega),\ \omega \in \mathbf{S}^2.$$

The asymptotic system (9) is obtained from the system (8) by neglecting derivatives tangential to the outgoing Minkowski light cones; $t = |x|$, and cubic terms, that are decaying faster

$$\Box\phi = r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)(r\phi) + \text{angular derivatives}$$

$$\partial_\mu = -\tfrac{1}{2}\hat\omega_\mu(\partial_t - \partial_r) + \text{tangential derivatives } \overline{\partial}_\mu$$

Blowup, global existence and the weak null condition

The asymptotic system for $\Phi_I = r\phi_I$:

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1} \sum A_{I,nm}^{JK} (\partial_t - \partial_r)^n \Phi_J \, (\partial_t - \partial_r)^m \Phi_K, \qquad (10)$$

was introduced by Hörmander to find blow-up.
Introducing variables $q = r - t$ and $s = \ln r$ he recast this lower order as

$$\partial_s \partial_q \Phi_I = \sum A_{I,nm}^{JK} \partial_q^n \Phi_J \, \partial_q^m \Phi_K, \qquad (11)$$

Blowup, global existence and the weak null condition

The asymptotic system for $\Phi_I = r\phi_I$:

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1}\sum A_{I,nm}^{JK}(\partial_t - \partial_r)^n\Phi_J \, (\partial_t - \partial_r)^m\Phi_K, \qquad (10)$$

was introduced by Hörmander to find blow-up.

Introducing variables $q = r - t$ and $s = \ln r$ he recast this lower order as

$$\partial_s\partial_q\Phi_I = \sum A_{I,nm}^{JK}\partial_q^n\Phi_J \, \partial_q^m\Phi_K, \qquad (11)$$

In particular $\Box\phi = (\partial_t\phi)^2$ has the asymptotic system $\partial_s\partial_q\Phi = (\partial_q\Phi)^2$.
The solution $\partial_q\Phi = \Psi$ is $\Psi(s, q, \omega) = \Psi_0(q, \omega)/(1 - \Phi_0(q, \omega)s)$ blows up.

Blowup, global existence and the weak null condition

The asymptotic system for $\Phi_I = r\phi_I$:

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1} \sum A_{I,nm}^{JK} (\partial_t - \partial_r)^n \Phi_J \; (\partial_t - \partial_r)^m \Phi_K, \qquad (10)$$

was introduced by Hörmander to find blow-up.
Introducing variables $q = r - t$ and $s = \ln r$ he recast this lower order as

$$\partial_s \partial_q \Phi_I = \sum A_{I,nm}^{JK} \partial_q^n \Phi_J \; \partial_q^m \Phi_K, \qquad (11)$$

In particular $\Box\phi = (\partial_t \phi)^2$ has the asymptotic system $\partial_s \partial_q \Phi = (\partial_q \Phi)^2$.
The solution $\partial_q \Phi = \Psi$ is $\Psi(s, q, \omega) = \Psi_0(q, \omega)/(1 - \Phi_0(q, \omega)s)$ blows up.

The usual null condition (Klainerman) is that $A_{I,nm}^{JK}(\omega) \equiv 0$.

Blowup, global existence and the weak null condition

The asymptotic system for $\Phi_I = r\phi_I$:

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1}\sum A^{JK}_{I,nm}(\partial_t - \partial_r)^n\Phi_J\,(\partial_t - \partial_r)^m\Phi_K, \qquad (10)$$

was introduced by Hörmander to find blow-up.
Introducing variables $q = r - t$ and $s = \ln r$ he recast this lower order as

$$\partial_s\partial_q\Phi_I = \sum A^{JK}_{I,nm}\partial_q^n\Phi_J\,\partial_q^m\Phi_K, \qquad (11)$$

In particular $\Box\phi = (\partial_t\phi)^2$ has the asymptotic system $\partial_s\partial_q\Phi = (\partial_q\Phi)^2$.
The solution $\partial_q\Phi = \Psi$ is $\Psi(s,q,\omega) = \Psi_0(q,\omega)/(1 - \Phi_0(q,\omega)s)$ blows up.

The usual null condition (Klainerman) is that $A^{JK}_{I,nm}(\omega) \equiv 0$.

The asymptotic system for $\Box\phi_2 = (\partial_t\phi_1)^2, \ \ \Box\phi_1 = 0.$ is
$\partial_s\partial_q\Phi_2 = (\partial_q\Phi_1)^2, \ \ \partial_s\partial_q\Phi_1 = 0$ which has the solution
$\Phi_1(s,q,\omega) = \Phi_{10}(q,\omega)$ and $\Phi_2(s,q,\omega) = \Phi_{20}(q,\omega) + s(\partial_q\Phi_{10}(q,\omega))^2$.

Blowup, global existence and the weak null condition

The asymptotic system for $\Phi_I = r\phi_I$:

$$(\partial_t + \partial_r)(\partial_t - \partial_r)\Phi_I \sim r^{-1}\sum A^{JK}_{I,nm}(\partial_t - \partial_r)^n\Phi_J\;(\partial_t - \partial_r)^m\Phi_K, \qquad (10)$$

was introduced by Hörmander to find blow-up.
Introducing variables $q = r - t$ and $s = \ln r$ he recast this lower order as

$$\partial_s\partial_q\Phi_I = \sum A^{JK}_{I,nm}\partial_q^n\Phi_J\;\partial_q^m\Phi_K, \qquad (11)$$

In particular $\Box\phi = (\partial_t\phi)^2$ has the asymptotic system $\partial_s\partial_q\Phi = (\partial_q\Phi)^2$.
The solution $\partial_q\Phi = \Psi$ is $\Psi(s,q,\omega) = \Psi_0(q,\omega)/(1 - \Phi_0(q,\omega)s)$ blows up.

The usual null condition (Klainerman) is that $A^{JK}_{I,nm}(\omega) \equiv 0$.

The asymptotic system for $\Box\phi_2 = (\partial_t\phi_1)^2$, $\;\Box\phi_1 = 0.$ is
$\partial_s\partial_q\Phi_2 = (\partial_q\Phi_1)^2$, $\;\partial_s\partial_q\Phi_1 = 0$ which has the solution
$\Phi_1(s,q,\omega) = \Phi_{10}(q,\omega)$ and $\Phi_2(s,q,\omega) = \Phi_{20}(q,\omega) + s(\partial_q\Phi_{10}(q,\omega))^2$.

The weak null condition (L-Rodnianski) is that the asymptotic system has global solutions decaying at a weaker rate $|\Phi| \le Ct^{\varepsilon}$. (e.g. $|\Phi| \le C\ln t$)

Overview

Einstein's equations in wave coordinates

$$\widetilde{\square}_g \, h_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + \widehat{T}_{\mu\nu},$$

$$F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$
$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta} \, \partial_\nu h_{\alpha'\beta'} - \tfrac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'} \, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}.$$

$Q_{\mu\nu}(\partial h, \partial h)$ satisfy the standard null condition. $G_{\mu\nu}(h)(\partial h, \partial h) \lesssim |h||\partial h|^2$.

Einstein's equations in wave coordinates

$$\widetilde{\square}_g\, h_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + \widehat{T}_{\mu\nu},$$

$$F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$
$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta}\, \partial_\nu h_{\alpha'\beta'} - \tfrac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'}\, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}.$$

$Q_{\mu\nu}(\partial h, \partial h)$ satisfy the standard null condition. $G_{\mu\nu}(h)(\partial h, \partial h) \lesssim |h||\partial h|^2$.

Einstein's equations in wave coordinates

$$\widetilde{\Box}_g \, h_{\mu\nu} = F_{\mu\nu}(g)(\partial h, \partial h) + \widehat{T}_{\mu\nu},$$

$$F_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h),$$
$$P(\partial_\mu h, \partial_\nu h) = \tfrac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \partial_\mu h_{\alpha\beta} \, \partial_\nu h_{\alpha'\beta'} - \tfrac{1}{4} m^{\alpha\alpha'} \partial_\mu h_{\alpha\alpha'} \, m^{\beta\beta'} \partial_\nu h_{\beta\beta'}.$$

$Q_{\mu\nu}(\partial h, \partial h)$ satisfy the standard null condition. $G_{\mu\nu}(h)(\partial h, \partial h) \lesssim |h||\partial h|^2$.

With $H^{\mu\nu} = g^{\mu\nu} - m^{\mu\nu}$ the wave coordinate condition can be expressed

$\partial_\mu \widehat{H}^{\mu\nu} = W^\nu(h, \partial h)$, where $\widehat{H}^{\mu\nu} = H^{\mu\nu} - m^{\mu\nu} \mathrm{tr}_m H/2$, $\mathrm{tr} H = m_{\alpha\beta} H^{\alpha\beta}$,

and $W^\nu(h, \partial h) = (g^{\mu\nu} g_{\alpha\beta} - m^{\mu\nu} m_{\alpha\beta}) \partial_\mu g^{\alpha\beta}/2 = O(h \, \partial h)$.

Nullframe

A null frame $\mathcal{N}$ is given by

$$\underline{L} = \partial_t - \partial_r, \quad L = \partial_t + \partial_r, \quad S_1, S_2 \in \mathbf{S}^2, \quad \langle S_i, S_j \rangle = \delta_{ij}$$

For solutions of wave equations derivatives tangential to the outgoing light cones $\overline{\partial} \in \mathcal{T} = \{L, S_1, S_2\}$ decay faster. Modulo tangential derivative $\overline{\partial} h$:

$$\partial_\mu h \sim L_\mu \partial_q h, \qquad \text{where} \quad \partial_q = (\partial_r - \partial_t)/2, \quad L_\mu = m_{\mu\nu} L^\nu, \quad L = (1, \omega).$$

Nullframe

A null frame $\mathcal{N}$ is given by

$$\underline{L} = \partial_t - \partial_r, \quad L = \partial_t + \partial_r, \quad S_1, S_2 \in \mathbf{S}^2, \quad \langle S_i, S_j \rangle = \delta_{ij}$$

For solutions of wave equations derivatives tangential to the outgoing light cones $\overline{\partial} \in \mathcal{T} = \{L, S_1, S_2\}$ decay faster. Modulo tangential derivative $\overline{\partial} h$:

$$\partial_\mu h \sim L_\mu \partial_q h, \qquad \text{where} \quad \partial_q = (\partial_r - \partial_t)/2, \quad L_\mu = m_{\mu\nu} L^\nu, \quad L = (1, \omega).$$

Applying this to Einstein equations we get

$$\widetilde{\Box}_g h_{\mu\nu} \sim L_\mu L_\nu P(\partial_q h, \partial_q h), \quad \text{where} \quad \widetilde{\Box}_g h_{\mu\nu} \sim \Box h_{\mu\nu} - h_{LL} \partial_q^2 h_{\mu\nu},$$

and $h_{LL} = h_{\alpha\beta} L^\alpha L^\beta$. In a null frame the semilinear terms become

$$(\widetilde{\Box}_g h)_{TU} \sim 0, \quad T \in \mathcal{T}, U \in \mathcal{N} \qquad (\widetilde{\Box}_g h)_{\underline{L}\underline{L}} \sim 4 P_{\mathcal{N}}(\partial_q h, \partial_q h),$$

The null structure

Here in a null frame

$$P_{\mathcal{N}}(D,E) = -\big(D_{LL}E_{\underline{LL}} + D_{\underline{LL}}E_{LL}\big)/8 - \big(2D_{AB}E^{AB} - D_A{}^A E_B{}^B\big)/4 \\ + \big(2D_{AL}E^A_{\underline{L}} + 2D_{A\underline{L}}E^A_{L} - D_A{}^A E_{L\underline{L}} - D_{L\underline{L}}E_A{}^A\big)/4.$$

Hence $P(\partial_q h, \partial_q h)$ only contain $\partial_q h_{\underline{LL}}$ through the term $\partial_q h_{LL}\partial_q h_{\underline{LL}}$, but the asymptotic wave coordinate condition in a null frame becomes

$$\partial_q h_{LT} \sim 0, \quad T \in \mathcal{T}, \qquad \delta^{AB}\partial_q h_{AB} \sim 0, \quad A, B \in \mathcal{S} = \{S_1, S_2\},$$

modulo tangential derivatives, so $P_{\mathcal{N}}(\partial_q h, \partial_q h) \sim P_{\mathcal{S}}(\partial_q h, \partial_q h)$, where

$$P_{\mathcal{S}}(D,E) = -\widehat{D}_{AB}\,\widehat{E}^{AB}/2, \quad A, B \in \mathcal{S}, \text{ where } \widehat{D}_{AB} = D_{AB} - \delta_{AB}\delta^{CD}D_{CD}.$$

Hence the right only depend on components we have better control on.

The null structure

Here in a null frame

$$P_{\mathcal{N}}(D,E) = -\big(D_{LL}E_{\underline{L}\underline{L}} + D_{\underline{L}\underline{L}}E_{LL}\big)/8 - \big(2D_{AB}E^{AB} - D_A{}^A E_B{}^B\big)/4$$
$$+ \big(2D_{AL}E^A_{\ \underline{L}} + 2D_{A\underline{L}}E^A_{\ L} - D_A{}^A E_{L\underline{L}} - D_{L\underline{L}}E_A{}^A\big)/4.$$

Hence $P(\partial_q h, \partial_q h)$ only contain $\partial_q h_{\underline{L}\underline{L}}$ through the term $\partial_q h_{LL}\partial_q h_{\underline{L}\underline{L}}$, but the asymptotic wave coordinate condition in a null frame becomes

$$\partial_q h_{LT} \sim 0, \quad T \in \mathcal{T}, \qquad \delta^{AB}\partial_q h_{AB} \sim 0, \quad A,B \in \mathcal{S} = \{S_1, S_2\},$$

modulo tangential derivatives, so $P_{\mathcal{N}}(\partial_q h, \partial_q h) \sim P_{\mathcal{S}}(\partial_q h, \partial_q h)$, where

$$P_{\mathcal{S}}(D,E) = -\widehat{D}_{AB}\,\widehat{E}^{AB}/2, \quad A,B \in \mathcal{S}, \ \text{ where } \ \widehat{D}_{AB} = D_{AB} - \delta_{AB}\delta^{CD}D_{CD}.$$

Hence the right only depend on components we have better control on.

Moreover, integrating the wave coordinate condition from data we get

$$h_{LL} \sim 2M/r.$$

Modified Lie derivatives and commutators

Let the modified Lie derivative (Lie density) be defined by

$$\widehat{\mathcal{L}}_Z K^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} = \mathcal{L}_Z K^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s} + \tfrac{r-s}{4}(\partial_\gamma Z^\gamma) K^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s}.$$

It has the following nice commutator properties

$$\mathcal{L}_Z \big(S_{\mu\nu}(\partial h, \partial k)\big) = S_{\mu\nu}(\partial \widehat{\mathcal{L}}_Z h, k) + S_{\mu\nu}(\partial h, \partial \widehat{\mathcal{L}}_Z k).$$

$$\mathcal{L}_Z \big(g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}\big) = \big(\widehat{\mathcal{L}}_Z g^{\alpha\beta}\big)\partial_\alpha\partial_\beta h_{\mu\nu} + g^{\alpha\beta}\partial_\alpha\partial_\beta \widehat{\mathcal{L}}_Z h_{\mu\nu}.$$

$$\partial_\mu \widehat{\mathcal{L}}_Z \widehat{H}^{\mu\nu} = \big(\widehat{\mathcal{L}}_Z + \tfrac{\partial_\gamma Z^\gamma}{2}\big)\partial_\mu \widehat{H}^{\mu\nu}$$

Overview

Asymptotics and Sources on light cones

A solution of a homogeneous wave equation $\Box\,\phi = 0$ has a radiation field

$$\phi(t,x) \sim \mathcal{F}(r-t,\omega)/r,$$

The same is true if only

$$|\Box\,\phi\,| + r^{-2}|\triangle_\omega \phi| \lesssim r^{-1}\langle t+r\rangle^{-1-\varepsilon}\langle t-r\rangle^{-1+\varepsilon}, \qquad \varepsilon > 0,$$

This is seen by expressing the wave operator in spherical coordinates:

$$\Box\,\phi = -r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)(r\phi) + r^{-2}\triangle_\omega \phi,$$

and integrating, in the $t+r$ direction and in the $t-r$ directions.

Asymptotics and Sources on light cones

A solution of a homogeneous wave equation $\Box\,\phi = 0$ has a radiation field

$$\phi(t,x) \sim \mathcal{F}(r-t,\omega)/r,$$

The same is true if only

$$|\Box\,\phi\,| + r^{-2}|\triangle_\omega \phi| \lesssim r^{-1}\langle t+r\rangle^{-1-\varepsilon}\langle t-r\rangle^{-1+\varepsilon}, \qquad \varepsilon > 0,$$

This is seen by expressing the wave operator in spherical coordinates:

$$\Box\,\phi = -r^{-1}(\partial_t+\partial_r)(\partial_t-\partial_r)(r\phi) + r^{-2}\triangle_\omega\phi,$$

and integrating, in the $t+r$ direction and in the $t-r$ directions.

However, general quadratic inhomogeneous terms do not decay enough:

$$\phi_t(t,x)^2 \sim \mathcal{F}_q(r-t,\omega)^2/r^2, \qquad q = r-t$$

The asymptotics for the wave equation with such sources along light cones

$$-\Box\,\psi = n(r-t,\omega)/r^2.$$

leads to a log correction of the asymptotic behavior:

$$\psi(t,r\omega) \sim \int_{r-t}^{\infty} \frac{1}{2r} \ln\Big|\frac{t+q+r}{t+q-r}\Big|\, n(q,\omega)\, dq \sim \ln|r|\, \frac{\mathcal{F}_1(r-t,\omega)}{r} + \frac{\mathcal{F}(r-t,\omega)}{r}$$

Asymptotics of the curved wave operator

L (2017) Vector fields adapted to the geometry of Schwarzschild-mass M. Due to the wave coordinate condition $h_{LL} \sim 2M/r$, from which it follows that the outgoing light cones of a solution with asymptotically flat data approach those of the Schwarzschild metric with the same mass M. The outgoing light cones for the Schwarzschild metric satisfy $t \sim r^* - u^*$, where $r^* = r + M \ln r$. Moreover, there is a solution to the eikonal equation that approaches the one for Schwarzschild

$$g^{\alpha\beta}\partial_\alpha u\,\partial_\beta u = 0, \qquad u \to u^* = t - r^*, \quad \text{when} \qquad r > t/2 \to \infty.$$

Moreover, the wave operator $\widetilde{\square}_g \sim \square - h_{LL}\partial_q^2 \sim \square - \frac{M}{r}\partial_q^2 \sim \square_S$ is asymptotic to the Schwarzschild wave opertor with the same mass. The change of variables $x = r\omega \to x^* = r^*\omega$, for $r >> 1$, reduces it to the constant coefficient wave operator $\square^*$ in the (t, x^*) coordinates.

Overview

We therefore make the change of variables

$$\widetilde{t} = t, \qquad \widetilde{x} = r^* x/r, \qquad \text{where} \quad r^* = r + \widetilde{\chi}(\tfrac{r}{t+1}) M \ln r,$$

Let

$$\widetilde{g}^{ab} = A^a_{\ \alpha} A^b_{\ \beta} g^{\alpha\beta}, \quad \text{where} \quad A^a_{\ \alpha} = \frac{\partial \widetilde{x}^{\,a}}{\partial x^\alpha}, \quad A^{\alpha}_{\ a} = \frac{\partial x^\alpha}{\partial \widetilde{x}^{\,a}}, \quad \widetilde{\partial}_a = A^\alpha_{\ a} \partial_\alpha,$$

Let ∇_a be covariant differentiation with respect to the Minkowski metric in the new coordinates $\widetilde{m}_{ab} = m_{\alpha\beta} A^\alpha_a A^\beta_b$ with Christoffel symbols $\widehat{\Gamma}^c_{ab} = \widetilde{m}^{cd}\big(\widetilde{\partial}_a \widetilde{m}_{bd} + \widetilde{\partial}_b \widetilde{m}_{ad} - \widetilde{\partial}_d \widetilde{m}_{ab}\big)/2 = O(Mr^{-2}\ln r)$. Then $\nabla_a h$ is equal to $\widetilde{\partial}_a h$ plus a correction of the form $\widehat{\Gamma} \cdot h$. Einstein's eq. become

$$\widetilde{g}^{ab} \nabla_a \nabla_b \widetilde{h}_{cd} = \widetilde{F}_{cd}(\widetilde{g})[\nabla \widetilde{h}, \nabla \widetilde{h}] + \widetilde{\widetilde{T}}_{cd},$$

and the wave coordinate condition become

$$\nabla_a\big(\widetilde{H}^{ac} - \tfrac{1}{2}\widetilde{m}^{ac}\widetilde{m}_{bd}\,\widetilde{H}^{bd}\big) = \widetilde{W}^c(\widetilde{g})[\widetilde{H}, \nabla \widetilde{H}].$$

The Modified Lorentz Fields

In addition to the modified radial coordinate and frame, we also define $x^{*i} = r^*\omega_i$ and consequently the Lorentz fields

$$\mathbb{L}^* = \{\partial_{x^{*\alpha}}, \Omega^*_{ij} = x^{*i}\partial_{x^{*j}} - x^{*j}\partial_{x^{*i}}, \Omega^*_{0i} = t\partial_{x^{*i}} + x^{*i}\partial_{t^*}, S^* = t\partial_{t^*} + r^*\partial_{r^*}\}$$

We also use the modified radial coordinate $r^* = r + M\chi \ln(1+r), t^* = t$ and weight $t - r^*$ as well as the modified null frame

$$L^* = \partial_{t^*} + \partial_{r^*}, \quad \underline{L}^* = \partial_{t^*} - \partial_{r^*}, \quad S^*_i = \frac{r}{r^*} S_i,$$

where S_i are piecewise defined orthonormal vectors tangent to the sphere.

Properties of the Metric

We recall the assumption that the metric is of the form

$$g_{\alpha\beta} = m_{\alpha\beta} + \frac{M\chi}{1+r}\delta_{\alpha\beta} + h^1_{\alpha\beta}.$$

Here m is the Minkowski metric, $M \le \epsilon$, and h satisfies the L^∞ estimates

$$|\mathcal{L}^I_{Z^*} h^1| \le \epsilon \tau_+^{-1+\delta},$$
$$|\mathcal{L}^I_{Z^*} h^1_{\mathcal{L}\mathcal{T}}| \le \epsilon \tau_+^{-1-\gamma'+\delta}\tau_-^{\gamma'},$$

for $\tau_+ = |t + r^*|, \tau_- = |t - r^*|, \mathcal{L} \in \{L^*\}$, $\mathcal{T} = \{L^*, S_1^*.S_2^*\}$. We take as well the L^2 estimates

$$\left\| (|\partial \mathcal{L}^I_{Z^*} h^1| + \tau_-^{-1}|\mathcal{L}^I_{Z^*} h^1|) w_1^{1/2} \right\|_{L^2(x)} \le \epsilon t^\delta,$$
$$\left\| \tau_-^{-1/2}\tau_+^{-\delta}|(\partial \mathcal{L}^I_{Z^*} h^1)_{\mathcal{L}\mathcal{L}}| + |\overline{\partial}\mathcal{L}^I_{Z^*} h^1| + \tau_-^{-1}|(\mathcal{L}^I_{Z^*} h^1)_{\mathcal{L}\mathcal{L}}|) w_1^{1/2} \right\|_{L^2(t,x)} \le \epsilon,$$

for the weight $w_1 = \langle (r^* - t)_+ \rangle$, and $Z \in \mathbb{L}^*$.

Overview

The Maxwell-Klein-Gordon system

We choose as our set of field equations the Maxwell-Klein-Gordon system,

$$\Box_g^{\mathbb{C}}\phi = D^\alpha D_\alpha \phi = 0, \tag{12a}$$

$$\nabla^\beta F_{\alpha\beta} = \Im(\phi\overline{D_\alpha\phi}), \tag{12b}$$

where, for a one-form A, we define

$$F = dA, \qquad D_\mu = \nabla_\mu + iA_\mu.$$

Additionally, we define the current vector

$$J_\alpha = \Im(\phi\overline{D_\alpha\phi}).$$

Einstein-Maxwell-Klein-Gordon

$$Ric(g)_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} = Q_{\mu\nu},$$

or in harmonic coordinates

$$\widetilde{\Box}_g \, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g) + \widehat{Q}_{\mu\nu}, \qquad \text{where} \qquad \widehat{Q}_{\mu\nu} := Q_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, \mathrm{tr}_g \, Q.$$

Maxwell-Klein-Gordon on a curved background

Theorem

(Kauffman) Given a background metric g satisfying bounds consistent with small-data solutions of the Einstein Vacuum Equations in harmonic gauge, the system (12) is globally well-posed for small initial data. Additionally, given some initial energy on k derivatives, $\mathcal{E}_k[F,\phi]$, with $k \geq 11$, the energy-momentum tensor,

$$Q[F,\phi]_{\alpha\beta} = \Re\left(D_\alpha\phi\overline{D_\beta\phi} - \frac{1}{2}g_{\alpha\beta}D_\gamma\phi\overline{D^\gamma\phi}\right) + F_{\alpha\gamma}F^\gamma_\beta - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}, \tag{13}$$

satisfies the following estimates:

$$\|Q[F,\phi](t,\cdot)_{TU}\|_{L^\infty} \lesssim t^{-3}\mathcal{E}_k[F,\phi] \tag{14}$$

$$\|Q[F,\phi](t,\cdot)\|_{L^2} \lesssim \mathcal{E}_k[F,\phi](1+t)^{-1} \tag{15}$$

Maxwell-Klein-Gordon Equations: Weak Null Condition

We recall the potential A_μ. We have some freedom in the choice of gauge; in particular, we can assume A satisfies the Lorenz gauge condition,

$$\nabla \cdot A = 0.$$

In Minkowski space, this reduces the Maxwell-Klein-Gordon system to a system of semilinear wave equations,

$$\Box A_\mu = \Im(\phi \overline{D_\mu \phi}), \tag{16}$$

$$\Box \phi = -2iA^\mu \partial_\mu \phi + A^\mu A_\mu \phi. \tag{17}$$

Intuitively, every time a bad derivative or component of A appears on the right of (17), it is paired with a good derivative or component. In (16), this holds everywhere except for the component $A_{\underline{L}}$, for which we can similarly expect decay like $(1+t)^{-1}\ln(1+t)$ along the light cone. A model system is $\Box\phi_2 = \phi_1 \partial_t \phi_1$ and $\Box\phi_1 = 0$. More tight than the model system for Einstein. If $\phi \sim t^{-1}$ then $\Box\phi \sim t^{-3}$ and $\phi\partial\phi \sim t^{-3}$ whereas $\partial\phi\partial\phi \sim t^{-4}$.

Overview

The Vlasov Equation

On a given spacetime (M, g), a particle at $\mathbf{x} = (t, x) \in M$ with momentum $\mathbf{p} = (p^0, p) \in T_{(t,x)}M$, will follow the unique geodesic γ such that

$$\gamma(0) = \mathbf{x}, \qquad \dot{\gamma}(0) = \mathbf{p}.$$

Vlasov equation says f conserved along trajectories of the geodesic flow:

$$f(\mathbf{x}, \mathbf{p}) = f(\gamma(s), \dot{\gamma}(s)).$$

In particular there is a paramter value s_0 such that $\gamma(s_0) = (0, y)$, $\dot{\gamma}(s_0) = \mathbf{q}$ so f is determined by its initial value f_0 when $t = 0$:

$$f(t, x, \mathbf{p}) = f_0(y, \mathbf{q}).$$

Equivalently,

$$\mathbb{X}(f) = 0, \qquad f\Big|_{t=0} = f_0$$

where

$$\mathbb{X} = p^\mu \partial_{x^\mu} - p^\alpha p^\beta \Gamma^\mu_{\alpha\beta} \partial_{p^\mu}.$$

Here $\Gamma = g \cdot \partial g$, are the Christoffel symbols of the metric g.

The Energy Momentum Tensor

Assume all particles have equal mass, i.e. f is supported on the mass shell,

$$\mathcal{P} = \{(t, x, \mathbf{p}) : g\big(\mathbf{p}, \mathbf{p}\big) = -m^2\}.$$

Distinguish two cases:

- Massive matter: $m > 0$;
- Massless matter: $m = 0$.

Energy momentum tensor takes the form,

$$T^{\alpha\beta}(t, x) = \int_{\mathcal{P}_{(t,x)}} f(t, x, p^0, p) p^\alpha p^\beta d\mu_{\mathcal{P}_x},$$

with p^0, p^1, p^2, p^3 components of p and $d\mu_{\mathcal{P}_x}$ volume form on $\mathcal{P}_x$.

Einstein-Vlassov

$$Ric(g)_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} = T_{\mu\nu},$$

or in harmonic coordinates

$$\widetilde{\Box}_g \, g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g) + \widehat{T}_{\mu\nu}, \qquad \text{where} \qquad \widehat{T}_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,\mathrm{tr}_g\, T.$$

Work on Einstein-Vlassov

Einstein–Vlasov in spherical symmetry:

- Rein–Rendall (1992): $m > 0$;
- Dafermos (2006): $m = 0$.

Einstein-Vlasov without symmetry

- Taylor (2016) m=0

Einstein–Vlasov without symmetry, with cosmological constant $\Lambda > 0$:

- Ringström (2013): Spatially homogeneous expanding spacetimes.

Related work on models

- Hwang-Rendall-Velázquez (2011) Vlasov-Poisson
- Smulevici (2016) Vlasov-Poisson
- Fajman–Joudioux–Smulevici (2017) Vlasov-Nordström

Global stability for massive Einstein-Vlasov

Theorem (L-Taylor, 2017)

Given asymptotically flat, smooth and compatible initial data $(\Sigma, \overline{g}, k, f_0)$ for the massive Einstein–Vlasov system suitably "close" to that of Minkowski space with f_0 compactly supported, the resulting spacetime "exists globally" and asymptotically approaches Minkowski space.

Fajman–Joudioux–Smulevici (2017) independent proof of a similar result.

Global stability for massive Einstein-Vlasov

Theorem (L-Taylor, 2017)

Given asymptotically flat, smooth and compatible initial data $(\Sigma, \overline{g}, k, f_0)$ for the massive Einstein–Vlasov system suitably "close" to that of Minkowski space with f_0 compactly supported, the resulting spacetime "exists globally" and asymptotically approaches Minkowski space.

Fajman–Joudioux–Smulevici (2017) independent proof of a similar result.

Proof uses harmonic coordinates as in [LR]. Main new idea is to construct a set of suitable vector fields on the mass hell $\mathcal{P} = \{(t, x, \mathbf{p}) : g\big(\mathbf{p}, \mathbf{p}\big) = 1\}$ that reduce to the Minkowski vector fields if acting on functions of (t, x).

Global stability for massive Einstein-Vlasov

Theorem (L-Taylor, 2017)

Given asymptotically flat, smooth and compatible initial data $(\Sigma, \overline{g}, k, f_0)$ for the massive Einstein–Vlasov system suitably "close" to that of Minkowski space with f_0 compactly supported, the resulting spacetime "exists globally" and asymptotically approaches Minkowski space.

Fajman–Joudioux–Smulevici (2017) independent proof of a similar result.

Proof uses harmonic coordinates as in [LR]. Main new idea is to construct a set of suitable vector fields on the mass hell $\mathcal{P} = \{(t, x, \mathbf{p}) : g\big(\mathbf{p}, \mathbf{p}\big) = 1\}$ that reduce to the Minkowski vector fields if acting on functions of (t, x).

This leads to an L^2 estimate for the energy momentum tensor T on a fixed metric background which is the main technical advance of the paper.

Precise stability statement in wave coordinates

Let $N \geq 11$, $0 < \gamma < 1$, $\mu > 0$ and set $\mathbb{D}_N = \sum_{k+\ell \leq N} \|\partial_x^k \partial_p^\ell f_0\|_{L^\infty}$ and write

$$g = m + h^0 + h^1, \quad \text{where } h^0_{\mu\nu}(t,x) = \chi\left(\frac{r}{t}\right)\chi(r)\frac{M}{r}\delta_{\mu\nu}, \ \chi(s) = \begin{cases} 1, & s > \frac{3}{4} \\ 0, & s < \frac{1}{4}. \end{cases}$$

$$E_N(t) = \sum_{|I| \leq N} \|w^{\frac{1}{2}} \partial Z^I h^1(t,\cdot)\|^2_{L^2}, \quad \text{where } w = \begin{cases} (1 + |r - t|)^{1+2\gamma}, & r > t \\ 1 + (1 + |r - t|)^{-2\mu}, & r \leq t. \end{cases}$$

Here I is a multi index and Z^I denotes $|I|$ of the vector fields,

$$\Omega_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i}, \quad B_i = x^i \partial_t + t\partial_{x^i}, \quad S = t\partial_t + x^k \partial_{x^k}, \quad \text{and} \quad \partial_\alpha,$$

Precise stability statement in wave coordinates

Let $N \geq 11$, $0 < \gamma < 1$, $\mu > 0$ and set $\mathbb{D}_N = \sum_{k+\ell \leq N} \|\partial_x^k \partial_p^\ell f_0\|_{L^\infty}$ and write

$$g = m + h^0 + h^1, \quad \text{where } h^0_{\mu\nu}(t,x) = \chi\left(\frac{r}{t}\right)\chi(r)\frac{M}{r}\delta_{\mu\nu}, \ \chi(s) = \begin{cases} 1, & s > \frac{3}{4} \\ 0, & s < \frac{1}{4}. \end{cases}$$

$$E_N(t) = \sum_{|I| \leq N} \|w^{\frac{1}{2}} \partial Z^I h^1(t,\cdot)\|^2_{L^2}, \quad \text{where } w = \begin{cases} (1+|r-t|)^{1+2\gamma}, & r > t \\ 1 + (1+|r-t|)^{-2\mu}, & r \leq t. \end{cases}$$

Here I is a multi index and Z^I denotes $|I|$ of the vector fields,

$$\Omega_{ij} = x^i \partial_{x^j} - x^j \partial_{x^i}, \quad B_i = x^i \partial_t + t\partial_{x^i}, \quad S = t\partial_t + x^k \partial_{x^k}, \quad \text{and} \quad \partial_\alpha,$$

If $E_N(0)^{1/2} + \mathbb{D}_N + M \leq \varepsilon$ then Einstein's eq. have a global sol. satisfying

$$(E_N(t))^{\frac{1}{2}} + \sum_{|I| \leq N} (1+t)\|Z^I T^{\mu\nu}(t,\cdot)\|_{L^2} \leq C_N \varepsilon (1+t)^{C'_N \varepsilon},$$

$$|Z^I h^1(t,x)| \leq \frac{C'_N \varepsilon (1+t)^{C'_N \varepsilon}}{(1+t+r)(1+q_+)^\gamma}, \quad |I| \leq N-3, \quad q_+ = \begin{cases} r-t, & r > t \\ 0, & r \leq t. \end{cases}$$

The support of the Energy momentum tensor T

Parameterise mass shell by $(t, x, \widehat{p})$, where $\widehat{p}^{\,i} = p^i/p^0$
Normalize so time is the parameter along geodesics and solve backwards

$$\frac{dX}{ds}(s,t,x,\widehat{p}) = \widehat{P}(s,t,x,\widehat{p}), \qquad X(t,t,x,\widehat{p}) = x,$$
$$\frac{d\widehat{P}}{ds}(s,t,x,\widehat{p}) = \widehat{\Gamma}\big(s, X(s,t,x,\widehat{p}), \widehat{P}(s,t,x,\widehat{p})\big), \qquad \widehat{P}(t,t,x,\widehat{p}) = \widehat{p}.$$

where $\widehat{\Gamma}^{\,\mu}(t,x,\widehat{p}) = \Gamma^0_{\alpha\beta}(t,x)\widehat{p}^{\,\alpha}\widehat{p}^{\,\beta}\widehat{p}^{\,\mu} - \Gamma^\mu_{\alpha\beta}(t,x)\widehat{p}^{\,\alpha}\widehat{p}^{\,\beta}$, where $\widehat{p}^{\,0} = 1$.
We have $f(t,x,\widehat{p}) = f_0(X(0,t,x,\widehat{p}), \widehat{P}(0,t,x,\widehat{p}))$ where we assume

$$\operatorname{supp} f_0 \subset \{(y,q);\, |y| \le K,\ |q| \le K'\}.$$

Since $g_{\alpha\beta}p^\alpha p^\beta = -1$ and $g_{\alpha\beta} \sim m_{\alpha\beta}$ we have ${p^0}^2 \sim 1 + {p^1}^2 + {p^2}^2 + {p^3}^2$
Under relatively mild smallness assumptions on the metric it follows that

$$\operatorname{supp} f \subset \{(t,x,p);\, |x| \le K + ct,\ |p| \le K' + 1,\ |\widehat{p}| \le c < 1\}. \tag{18}$$

Hence matter is supported away from the wave zone $r \sim t$ where the sol. of the wave equation for the metric propagate so the two issues decouple.

The L^2 estimate for the Energy momentum tensor T

Theorem (L-Taylor, 2017)

Suppose that f satisfies the support condition (18) *and that on the support* $\left|Z^J\Gamma(t,x)\right| \le C_N'(1+t)^{-1-a}$, *for some* $\frac{1}{2} < a < 1$, *for* $|J| \le \frac{|I|}{2} + 2$. *Then*

$$\|(Z^I T^{\mu\nu})(t,\cdot)\|_{L^2} \le \frac{D_k \mathbb{D}_k}{(1+t)^{3/2}}$$

$$+ D_k \mathbb{D}_{k'} \Big(\sum_{|J|\le|I|-1} \frac{\|(Z^J\Gamma)(t,\cdot)\|_{L^2}}{(1+t)^{1+a}} + \sum_{|J|\le|I|} \frac{1}{(1+t)^{3/2}} \int_0^t \frac{\|(Z^J\Gamma)(s,\cdot)\|_{L^2}}{(1+s)^{1/2}} ds \Big),$$

$$\|(Z^I T^{\mu\nu})(t,\cdot)\|_{L^1} \le D_k \mathbb{D}_k$$

$$+ D_k \mathbb{D}_{k'} \Big(\sum_{|J|\le|I|-1} \frac{\|(Z^J\Gamma)(t,\cdot)\|_{L^2}}{(1+t)^{a-1/2}} + \sum_{|J|\le|I|+1} \int_0^t \frac{\|(Z^J\Gamma)(s,\cdot)\|_{L^2}}{(1+s)^{1/2+a}} ds \Big).$$

Proof of the energy estimate using the L^2 estimate for T

We have $\widetilde{\Box}_g g = F + T$ so by the energy inequality

$$E_N(t)^{1/2} \lesssim E_N(0)^{1/2} + \sum_{|I|\leq N} \int_0^t \|Z^I F(s,\cdot)\|_{L^2} + \|[Z^I, \widetilde{\Box}_g] g(t,\cdot)\|_{L^2} + \|(Z^I T^{\mu\nu})(s,\cdot)\|_{L^2}\, ds$$

where by the previous theorem

$$\|(Z^I T^{\mu\nu})(t,\cdot)\|_{L^2} \lesssim \frac{C\varepsilon}{(1+t)^{3/2}} + \frac{C\varepsilon}{1+t} \sup_{0\leq s\leq t} E_N(s)^{1/2}, \quad |I| \leq N.$$

By Grönwall's inequality an estimate of the form

$$Q(t) \leq C\varepsilon + \int_0^t \frac{C\varepsilon}{1+s} Q(s)\, ds$$

gives a bound

$$Q(t) \leq C\varepsilon(1+t)^{C\varepsilon}$$

Overview

The vector fields

Let us for simplicity instead of T consider

$$\rho(t,x) = \int f(t,x,\widehat{p})d\widehat{p}$$

For the vector fields Z, $|Zf|$ is unbounded. Instead given $Z = Z^\alpha \partial_{x^\alpha}$ we want to construct vector fields $\overline{Z} = Z + \tilde{Z}$, where $\tilde{Z} = \tilde{Z}^\alpha \partial_{\widehat{p}^\alpha}$. Then

$$Z\rho(t,x) = \int \overline{Z} f(t,x,\widehat{p})d\widehat{p} - \int (\partial_{\widehat{p}^\alpha} \tilde{Z}^\alpha) f(t,x,\widehat{p})d\widehat{p}$$

The vector fields

Let us for simplicity instead of T consider

$$\rho(t,x) = \int f(t,x,\widehat{p})d\widehat{p}$$

For the vector fields Z, $|Zf|$ is unbounded. Instead given $Z = Z^\alpha \partial_{x^\alpha}$ we want to construct vector fields $\overline{Z} = Z + \tilde{Z}$, where $\tilde{Z} = \tilde{Z}^\alpha \partial_{\widehat{p}^\alpha}$. Then

$$Z\rho(t,x) = \int \overline{Z} f(t,x,\widehat{p})d\widehat{p} - \int (\partial_{\widehat{p}^\alpha} \tilde{Z}^\alpha) f(t,x,\widehat{p})d\widehat{p}$$

Since $f(t,x,\widehat{p}) = f\big(t_0, X(t_0,t,x,\widehat{p}), \widehat{P}(t_0,t,x,\widehat{p})\big)$ we have

$$\overline{Z} f(t,x,\widehat{p}) = \overline{Z}\left(X(t_0,t,x,\widehat{p})^i\right)(\partial_{x^i} f)(t_0, X(t_0), \widehat{P}(t_0)) \\ + \overline{Z}\big(\widehat{P}(t_0,t,x,\widehat{p})^i\big)(\partial_{\widehat{p}^i} f)(t_0, X(t_0), \widehat{P}(t_0)),$$

It follows that $|\overline{Z} f(t,x,\widehat{p})| \leq C$ independently of t if $|\overline{Z}\left(X(t_0,t,x,\widehat{p})^i\right)| \leq C$ and $|\overline{Z}\big(\widehat{P}(t_0,t,x,\widehat{p})^i\big)| \leq C$.

The geometric construction

For a given vector field Z on $\mathcal{M}$, let $\Phi_\lambda^Z : \mathcal{M} \to \mathcal{M}$ denote the associated one parameter family of diffeomorphisms, so that

$$\frac{d\Phi_\lambda^Z(t,x)}{d\lambda}\Big|_{\lambda=0} = Z|_{(t,x)}.$$

For fixed τ, any point $(t,x,\widehat{p}\,) \in \mathcal{P}$ can be given uniquely by a pair of points $\{(t,x),(\tau,y)\}$, where $y = X(\tau,t,x,\widehat{p}\,)$, is the point where the geodesic emanating from (t,x) with velocity $\widehat{p}$ intersects the hypersurface $\{t=\tau\}$, i.e. $(t,x,\widehat{p}\,) \in \mathcal{P}$ can be parameterised by $\{(t,x),(\tau,y)\}$,

$$(t,x,\widehat{p}\,) = (t,x,\widehat{p}_X(t,x,\tau,y)).$$

The action of Φ_λ^Z on (t,x) and (τ,y) induces an action on $\mathcal{P}$ at time t:

$$\overline{\Phi}_{\lambda,\tau}^{Z,X}(t,x,\widehat{p}\,) := \left(\Phi_\lambda^Z(t,x),\widehat{p}_X\left(\Phi_\lambda^Z(t,x),\Phi_\lambda^Z(\tau,y)\right)\right)$$

For fixed t_0 we define the vector field $\overline{Z}$ by

$$\overline{Z}|_{(t,x,\widehat{p}\,)} = \frac{d\overline{\Phi}_{\lambda,\tau}^{Z,X}(t,x,\widehat{p}\,)}{d\lambda}\Big|_{\lambda=0,\,\tau=t_0}.$$

The initial bounds

The good bound for $|\overline{Z}(X(t_0)^i)|$ follows from the indentity

$$\overline{Z}|_{(t,x,\widehat{p}\,)}\left(X(t_0,t,x,\widehat{p}\,)^i\right) = Z^i|_{(t_0,X(t_0))} - Z^0|_{(t_0,X(t_0))}\widehat{P}\,(t_0,t,x,\widehat{p}\,)^i, \quad (19)$$

In particular, if $X^i(t_0,t,x,\widehat{p}\,)$ and $\widehat{P}\,^i(t_0,t,x,\widehat{p}\,)$ are bounded in the support of $f(t_0,X(t_0),\widehat{P}\,(t_0))$, (19) guarantees that $|\overline{Z}(X(t_0)^i)|$ is bounded.

The initial bounds

The good bound for $|\overline{Z}(X(t_0)^i)|$ follows from the indentity

$$\overline{Z}|_{(t,x,\widehat{p}\,)}\left(X(t_0,t,x,\widehat{p}\,)^i\right) = Z^i|_{(t_0,X(t_0))} - Z^0|_{(t_0,X(t_0))}\widehat{P}\,(t_0,t,x,\widehat{p}\,)^i, \quad (19)$$

In particular, if $X^i(t_0,t,x,\widehat{p}\,)$ and $\widehat{P}^{\,i}(t_0,t,x,\widehat{p}\,)$ are bounded in the support of $f(t_0,X(t_0),\widehat{P}\,(t_0))$, (19) guarantees that $|\overline{Z}(X(t_0)^i)|$ is bounded.

To see that (19) holds, first note that the left is the derivative of $X\big(t_0,\overline{\Phi}^{Z,X}_{\lambda,t_0}(t,x,\widehat{p}\,)\big)^i$ with respect to λ at $\lambda=0$. The first term on the right is the derivative of $\Phi^Z_\lambda(t_0,y)^i$ at $\lambda=0$. The equality (19) follows from taking the derivative, with respect to λ at $\lambda=0$ of both sides of

$$X\big(\Phi^Z_\lambda(t_0,y)^0,\overline{\Phi}^{Z,X}_{\lambda,t_0}(t,x,\widehat{p}\,)\big)^i = \Phi^Z_\lambda(t_0,y)^i. \quad (20)$$

The initial bounds

The good bound for $|\overline{Z}(X(t_0)^i)|$ follows from the indentity

$$\overline{Z}|_{(t,x,\widehat{p}\,)}\left(X(t_0,t,x,\widehat{p}\,)^i\right)=Z^i|_{(t_0,X(t_0))}-Z^0|_{(t_0,X(t_0))}\widehat{P}\,(t_0,t,x,\widehat{p}\,)^i,\quad(19)$$

In particular, if $X^i(t_0,t,x,\widehat{p}\,)$ and $\widehat{P}^{\,i}(t_0,t,x,\widehat{p}\,)$ are bounded in the support of $f(t_0,X(t_0),\widehat{P}\,(t_0))$, (19) guarantees that $|\overline{Z}(X(t_0)^i)|$ is bounded.

To see that (19) holds, first note that the left is the derivative of $X\big(t_0,\overline{\Phi}^{Z,X}_{\lambda,t_0}(t,x,\widehat{p}\,)\big)^i$ with respect to λ at $\lambda=0$. The first term on the right is the derivative of $\Phi^Z_\lambda(t_0,y)^i$ at $\lambda=0$. The equality (19) follows from taking the derivative, with respect to λ at $\lambda=0$ of both sides of

$$X\big(\Phi^Z_\lambda(t_0,y)^0,\overline{\Phi}^{Z,X}_{\lambda,t_0}(t,x,\widehat{p}\,)\big)^i=\Phi^Z_\lambda(t_0,y)^i.\qquad(20)$$

Similarly a bound for $|\overline{Z}(\widehat{P}\,(t_0)^i)|$ follows from an indentity

$$\overline{Z}|_{(t,x,\widehat{p})}(\widehat{P}\,(t_0,t,x,\widehat{p}\,)^i)=\frac{Z^i|_{(t,x)}-Z^i|_{(t_0,X(t_0))}}{t-t_0}-\frac{Z^0|_{(t,x)}-Z^0|_{(t_0,X(t_0))}}{t-t_0}\widehat{p}^{\,i}+\dots$$

The Minkowski case

On Minkowski background, $y = x - (t-\tau)\widehat{p}$ and, when Z is chosen to be Ω_{ij}, B_i, S, a computation, shows that the resulting vectors $\overline{Z}^M$ are

$$\overline{\Omega}_{ij}^M = x^i\partial_{x^j} - x^j\partial_{x^i} + \widehat{p}^{\,i}\partial_{\widehat{p}^{\,j}} - \widehat{p}^{\,j}\partial_{\widehat{p}^{\,i}},$$
$$\overline{B}_i^M = x^i\partial_t + t\partial_{x^i} + \left(\delta_i^j - \widehat{p}^{\,i}\widehat{p}^{\,j}\right)\partial_{\widehat{p}^{\,j}},$$
$$\overline{S}^M = t\partial_t + x^k\partial_{x^k}.$$

These 'Minkowski vector fields' were used by Fajman–Joudioux–Smulevici (2016) for Vlasov–Nordstrom system in higher dimensions, who notice $\overline{\Omega}_{ij}$ and $\overline{B}_i$ generate symmetries of the tangent bundle of Minkowski space. Related vector fields were used by Taylor (2016) for massless Einstein–Vlasov. Also Fajman–Joudioux–Smulevici (2017) use a different modification of the 'Minkowksi vector fields' in case of curved spacetime. The geometric construction we present here is new and completely general. It will lead to a regularity issue which can be avoided by using the same construction for approximate geodesics as follows.