Lectures on black holes and linear waves

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Abstract

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These lecture notes, based on a course given at the Zürich Clay Summer School (June 23-July 18 2008), review our current mathematical understanding of the global behaviour of waves on black hole exterior backgrounds. Interest in this problem stems from its relationship to the non-linear stability of the black hole spacetimes themselves as solutions to the Einstein equations, one of the central open problems of general relativity. After an introductory discussion of the Schwarzschild geometry and the black hole concept, the classical theorem of Kay and Wald on the boundedness of scalar waves on the exterior region of Schwarzschild is reviewed. The original proof is presented, followed by a new more robust proof of a stronger boundedness statement. The problem of decay of scalar waves on Schwarzschild is then addressed, and a theorem proving quantitative decay is stated and its proof sketched. This decay statement is carefully contrasted with the type of statements derived heuristically in the physics literature for the asymptotic tails of individual spherical harmonics. Following this, our recent proof of the boundedness of solutions to the wave equation on axisymmetric stationary backgrounds (including slowly-rotating Kerr and Kerr-Newman) is reviewed and a new decay result for slowly-rotating Kerr spacetimes is stated and proved. This last result was announced at the summer school and appears in print here for the first time. A discussion of the analogue of these problems for spacetimes with a positive cosmological constant $\Lambda > 0$ follows. Finally, a general framework is given for capturing the red-shift effect for nonextremal black holes. This unifies and extends some of the analysis of the previous sections. The notes end with a collection of open problems.

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1 Introduction: General relativity and evolution

Black holes are one of the fundamental predictions of general relativity. At the same time, they are one of its least understood (and most often misunderstood) aspects. These lectures intend to introduce the black hole concept and the analysis of waves on black hole backgrounds (\mathcal{M},g) by means of the example of the scalar wave equation

$$\Box_g \psi = 0. \tag{1}$$

We do not assume the reader is familiar with general relativity, only basic analysis and differential geometry. In this introductory section, we briefly describe general relativity in outline form, taking from the beginning the evolutionary point of view which puts the Cauchy problem for the Einstein equations—the system of nonlinear partial differential equations (see (2) below) governing the theory—at the centre. The problem (1) can be viewed as a poor man's linearisation for the Einstein equations. Study of (1) is then intimately related to the problem of the dynamic stability of the black hole spacetimes (\mathcal{M}, g) themselves. Thus, one should view the subject of these lectures as intimately connected to the very tenability of the black hole concept in the theory.

1.1 General relativity and the Einstein equations

General relativity postulates a 4-dimensional Lorentzian manifold (\mathcal{M}, g) -space-time-which is to satisfy the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}.$$
 (2)

Here, $R_{\mu\nu}$, R denote the Ricci and scalar curvature of g, respectively, and $T_{\mu\nu}$ denotes a symmetric 2-tensor on \mathcal{M} termed the stress-energy-momentum tensor of matter. (Necessary background on Lorentzian geometry to understand the above notation is given in Appendix A.) The equations (2) in of themselves do not close, but must be coupled to "matter equations" satisfied by a collection $\{\Psi_i\}$ of matter fields defined on \mathcal{M} , together with a constitutive relation determining $T_{\mu\nu}$ from $\{g, \Psi_i\}$. These equations and relations are stipulated by the relevant continuum field theory (electromagnetism, fluid dynamics, etc.) describing the matter. The formulation of general relativity represents the culmination of the classical field-theoretic world-view where physics is governed by a closed system of partial differential equations.

Einstein was led to the system (2) in 1915, after a 7-year struggle to incorporate gravity into his earlier principle of relativity. In the field-theoretic formulation of the "Newtonian" theory, gravity was described by the Newtonian potential ϕ satisfying the Poisson equation

$$\Delta \phi = 4\pi \mu,\tag{3}$$

where μ denotes the mass-density of matter. It is truly remarkable that the constraints of consistency were so rigid that encorporating gravitation required finally a complete reworking of the principle of relativity, leading to a theory where Newtonian gravity, special relativity and Euclidean geometry each emerge as limiting aspects of one dynamic geometrical structure—the Lorentzian metric—naturally living on a 4-dimensional spacetime continuum. A second remarkable aspect of general relativity is that, in contrast to its Newtonian predecessor, the theory is non-trivial even in the absence of matter. In that case, we set $T_{\mu\nu} = 0$ and the system (2) takes the form

$$R_{\mu\nu} = 0. (4)$$

The equations (4) are known as the *Einstein vacuum equations*. Whereas (3) is a linear elliptic equation, (4) can be seen to form a closed system of non-linear (but *quasilinear*) wave equations. Essentially all of the characteristic features of the dynamics of the Einstein equations are already present in the study of the vacuum equations (4).

1.2 Special solutions: Minkowski, Schwarzschild, Kerr

To understand a theory like general relativity where the fundamental equations (4) are nonlinear, the first goal often is to identify and study important explicit solutions, i.e., solutions which can be written in closed form. Much of the early history of general relativity centred around the discovery and interpretation of such solutions. The simplest explicit solution to the Einstein vacuum equations (4) is Minkowski space \mathbb{R}^{3+1} . The next simplest solution of (4) is the so-called Schwarzschild solution, written down [139] already in 1916. This is in fact a one-parameter family of solutions (\mathcal{M}, g_M) , the parameter M identified with mass. See (5) below for the metric form. The Schwarzschild family lives as a subfamily in a larger two-parameter family of explicit solutions $(\mathcal{M}, g_{M,a})$ known as the Kerr solutions, discussed in Section 5.1. These were discovered only much later [99] (1963).

When the Schwarzschild solution was first written down in local coordinates, the necessary concepts to understand its geometry had not yet been developed. It took nearly 50 years from the time when Schwarzschild was first discovered for its global geometry to be sufficiently well understood so as to be given a suitable name: Schwarzschild and Kerr were examples of what came to be known as black hole spacetimes². The Schwarzschild solution also illustrates another feature of the Einstein equations, namely, the presence of singularities.

We will spend Section 2 telling the story of the emergence of the black hole notion and sorting out what the distinct notions of "black hole" and "singularity" mean. For the purpose of the present introductory section, let us take the notion of "black hole" as a "black box" and make some general remarks on the role of explicit solutions, whatever might be their properties. These remarks are relevant for any physical theory governed by an evolution equation.

1.3 Dynamics and the stability problem

Explicit solutions are indeed suggestive as to how general solutions behave, but only if they are appropriately "stable". In general relativity, this notion can in turn only be understood after the problem of dynamics for (4) has been formulated, that is to say, the *Cauchy problem*.

In contrast to other non-linear field theories arising in physics, in the case of general relativity, even formulating the Cauchy problem requires addressing several conceptual issues (e.g. in what sense is (4) hyperbolic?), and these took a long time to be correctly sorted out. Important advances in this process include

¹The traditional terminology in general relativity for such solutions is *exact solutions*.

²This name is due to John Wheeler.

the identification of the harmonic gauge by de Donder [70], the existence and uniqueness theorems for general quasilinear wave equations in the 1930's based on work of Friedrichs, Schauder, Sobolev, Petrovsky, Leray and others, and Leray's notion of global hyperbolicity [112]. The well-posedness of the appropriate Cauchy problem for the vacuum equations (4) was finally formulated and proven in celebrated work of Choquet-Bruhat [33] (1952) and Choquet-Bruhat—Geroch [35] (1969). See Appendix B for a concise survey of these developments and the precise statement of the existence and uniqueness theorems and some comments on their proof.

In retrospect, much of the confusion in early discussions of the Schwarzschild solution can be traced to the lack of a dynamic framework to understand the theory. It is only in the context of the language provided by [35] that one can then formulate the dynamical stability problem and examine the relevance of various explicit solutions.

The stability of Minkowski space was first proven in the monumental work of Christodoulou and Klainerman [51]. See Appendix B.5 for a formulation of this result. The dynamical stability of the Kerr family as a family of solutions to the Cauchy problem for the Einstein equations, even restricted to parameter values near Schwarzschild, i.e. $|a| \ll M$, 3 is yet to be understood and poses an important challenge for the mathematical study of general relativity in the coming years. See Section 5.6 for a formulation of this problem. In fact, even the most basic linear properties of waves (e.g. solutions of (1)) on Kerr spacetime backgrounds (or more generally, backgrounds near Kerr) have only recently been understood. In view of the wave-like features of the Einstein equations (4) (see in particular Appendix B.4), this latter problem should be thought of as a prerequisite for understanding the non-linear stability problem.

1.4 Outline of the lectures

The above linear problem will be the main topic of these lectures: We shall here develop from the beginning the study of the linear homogeneous wave equation (1) on fixed black hole spacetime backgrounds (\mathcal{M}, g) . We have already referred in passing to the content of some of the later sections. Let us give here a complete outline: Section 2 will introduce the black hole concept and the Schwarzschild geometry in the wider context of open problems in general relativity. Section 3 will concern the basic boundedness properties for solutions ψ of (1) on Schwarzschild exterior backgrounds. Section 4 will concern quantitative decay properties for ψ . Section 5 will move on to spacetimes (\mathcal{M}, g) "near" Schwarzschild, including slowly rotating Kerr, discussing boundedness and decay properties for solutions to (1) on such (\mathcal{M}, g) , and ending in Section 5.6 with a formulation of the non-linear stability problem for Kerr, the open problem which in some sense provides the central motivation for these notes. Section 6 will consider the analogues of these problems in spacetimes with a

³Note that without symmetry assumptions one cannot study the stability problem for Schwarzschild per se. Only the larger Kerr family can be stable.

positive cosmological constant Λ , Section 7 will give a multiplier-type estimate valid for general non-degenerate Killing horizons which quantifies the classical red-shift effect. The importance of the red-shift effect as a stabilising mechanism for the analysis of waves on black hole backgrounds will be a common theme throughout these lectures. The notes end with a collection of open problems in Section 8.

The proof of Theorem 5.2 of Section 5 as well as all results of Section 7 appear in print in these notes for the first time. The discussion of Section 3.3 as well as the proof of Theorem 4.1 have also been streamlined in comparision with previous presentations. We have given a guide to background literature in Sections 3.4, 4.4, 5.5 and 6.3.

We have tried to strike a balance in these notes between making the discussion self-containted and providing the necessary background to appreciate the place of the problem (1) in the context of the current state of the art of the Cauchy problem for the Einstein equations (2) or (4) and the main open problems and conjectures which will guide this subject in the future. Our solution has been to use the history of the Schwarzschild solution as a starting point in Section 2 for a number of digressions into the study of gravitational collapse, singularities, and the weak and strong cosmic censorship conjectures, deferring, however, formal development of various important notions relating to Lorentzian geometry and the well-posedness of the Einstein equations to a series of Appendices. We have already referred to these appendices in the text. The informal nature of Section 2 should make it clear that the discussion is not intended as a proper survey, but merely to expose the reader to important open problems in the field and point to some references for further study. The impatient reader is encouraged to move quickly through Section 2 at a first reading. The problem (1) is itself rather self-contained, requiring only basic analysis and differential geometry, together with a good understanding of the black hole spacetimes, in particular, their so-called causal geometry. The discussion of Section 2 should be more than enough for the latter, although the reader may want to supplement this with a more general discussion, for instance [55].

These notes accompanied a series of lectures at a summer school on "Evolution Equations" organized by the Clay Mathematics Institute, June–July 2008. The centrality of the evolutionary point of view in general relativity is often absent from textbook discussions. (See however the recent [133].) We hope that these notes contribute to the point of view that puts general relativity at the centre of modern developments in partial differential equations of evolution.

2 The Schwarzschild metric and black holes

Practically all concepts in the development of general relativity and much of its history can be told from the point of view of the Schwarzschild solution. We now readily associate this solution with the black hole concept. It is important to remember, however, that the Schwarzschild solution was first discovered in a thoroughly classical astrophysical setting: it was to represent the vacuum region

outside a star. The black hole interpretation—though in some sense inevitable—historically only emerged much later.

The most efficient way to present the Schwarzschild solution is to begin at the onset with Kruskal's maximal extension as a point of departure. Instead, we shall take advantage of the informal nature of the present notes to attempt a more conversational and "historical" presentation of the Schwarzschild metric and its interpretation. Although certainly not the quickest route, this approach has the advantage of highlighting the themes which have become so important in the subject—in particular, singularities, black holes and their event horizons—with the excitement of their step-by-step unravelling from their origin in a model for the simplest of general relativistic stars. The Schwarzschild solution will naturally lead to discussions of the Oppenheimer-Snyder collapse model, the cosmic censorship conjectures, trapped surfaces and Penrose's incompleteness theorems, and recent work of Christodoulou on trapped surface formation in vacuum collapse, and we elaborate on these topics in Sections 2.6–2.8. (The discussion in these three last sections was not included in the lectures, however, and is not necessary for understanding the rest of the notes.)

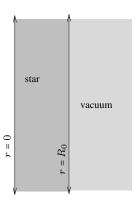
2.1 Schwarzschild's stars

The most basic self-gravitating objects are stars. In the most primitive stellar models, dating from the 19th century, stars are modeled by a self-gravitating fluid surrounded by vacuum. Moreover, to a first approximation, classically stars are *spherically symmetric* and *static*.

It should not be surprising then that early research on the Einstein equations (2) would address the question of the existence and structure of general relativistic stars in the new theory. In view of our above discussion, the most basic problem is to understand spherically symmetric, static metrics, represented in coordinates (t, r, θ, ϕ) , such that the spacetime has two regions: In the region $r \leq R_0$ —the interior of the star—the metric should solve a suitable Einstein-matter system (2) with appropriate matter, and in the region $r \geq R_0$ —the exterior of

⁴This in no way should be considered as a true attempt at the history of the solution, simply a pedagogical approach to its study. See for example [76].

the star-the spacetime should be vacuum, i.e. the metric should solve (4).



This is the problem first addressed by Schwarzschild [139, 140], already in 1916. Schwarzschild considered the vacuum region first [139] and arrived⁵ at the one-parameter family of solutions:

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \tag{5}$$

Every student of this subject should explicitly check that this solves (4) (Exercise).

In [140], Schwarzschild found interior metrics for the darker shaded region $r \leq R_0$ above. In this region, matter is described by a perfect fluid. We shall not write down explicitly such metrics here, as this would require a long digression into fluids, their equations of state, etc. See [44]. Suffice it to say here that the existence of such solutions required that one take the constant M positive, and the value R_0 marking the boundary of the star always satisfied $R_0 > 2M$. The constant M could then be identified with the total mass of the star as measured by considering the orbits of far-away test particles.⁶ In fact, for most reasonable matter models, static solutions of the type described above only exist under a stronger restriction on R_0 (namely $R_0 \geq 9M/4$) now known as the Buchdahl inequality. See [14, 2, 97].

The restriction on R_0 necessary for the existence of Schwarzschild's stars appears quite fortuitous: It is manifest from the form (5) that the components of g are singular if the (t, r, θ, ϕ) coordinate system for the vacuum region is extended to r = 2M. But a natural (if perhaps seemingly of only academic interest) question arises, namely, what happens if one does away completely

 $^{^5}$ As is often the case, the actual history is more complicated. Schwarzschild based his work on an earlier version of Einstein's theory which, while obtaining the correct vacuum equations, imposed a condition on admissible coordinate systems which would in fact exclude the coordinates of (5). Thus he had to use a rescaled r as a coordinate. Once this condition was removed from the theory, there is no reason not to take r itself as the coordinate. It is in this sense that these coordinates can reasonably be called "Schwarzschild coordinates".

⁶Test particles in general relativity follow timelike geodesics of the spacetime metric. **Exercise**: Explain the statement claimed about far-away test particles. See also Appendix B.2.3.

with the star and tries simply to consider the expression (5) for all values of r? This at first glance would appear to be the problem of understanding the gravitational field of a "point particle" with the particle removed.⁷

For much of the history of general relativity, the degeneration of the metric functions at r = 2M, when written in these coordinates, was understood as meaning that the gravitational field should be considered singular there. This was the famous Schwarzschild "singularity". Since "singularities" were considered "bad" by most pioneers of the theory, various arguments were concoted to show that the behaviour of g where r = 2M is to be thought of as "pathological", "unstable", "unphysical" and thus, the solution should not be considered there. The constraint on R_0 related to the Buchdahl inequality seemed to give support to this point of view. See also [75].

With the benefit of hindsight, we now know that the interpretation of the previous paragraph is incorrect, on essentially every level: neither is r=2M a singularity, nor are singularities—which do in fact occur!—necessarily to be discarded! Nor is it true that non-existence of static stars renders the behaviour at r=2M—whatever it is—"unstable" or "unphysical"; on the contrary, it was an early hint of gravitational collapse! Let us put aside this hindsight for now and try to discover for ourselves the geometry and "true" singularities hidden in (5), as well as the correct framework for identifying "physical" solutions. In so doing, we are retracing in part the steps of early pioneers who studied these issues without the benefit of the global geometric framework we now have at our disposal. All the notions referred to above will reveal themselves in the next subsections.

2.2 Extensions beyond the horizon

The fact that the behaviour of the metric at r=2M is not singular, but simply akin to the well-known breakdown of the coordinates (5) at $\theta=0,\pi$ (this latter breakdown having never confused anyone...), is actually quite easy to see, and there is no better way to appreciate this than by doing the actual calculations. Let us see how to proceed.

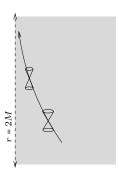
First of all, before even attempting a change of coordinates, the following is already suggestive: Consider say a future-directed⁹ ingoing radial null geodesic.

⁷Hence the title of [139].

⁸Let the reader keep in mind that there is a good reason for the quotation marks here and for those that follow.

⁹We time-orient the metric by ∂_t . See Appendix A.

The image of such a null ray is in fact depicted below:



One can compute that this has finite affine length to the future, i.e. these null geodesics are *future-incomplete*, while scalar curvature invariants remain bounded as $s \to \infty$. It is an amusing **exercise** to put oneself in this point of view and carry out the above computations in these coordinates.

Of course, as such the above doesn't show anything. ¹⁰ But it turns out that indeed the metric can be extended to be defined on a "bigger" manifold. One defines a new coordinate

$$t^* = t + 2M\log(r - 2M).$$

This metric then takes the form

$$g = -\left(1 - \frac{2M}{r}\right)(dt^*)^2 + \frac{4M}{r}dt^*dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2d\sigma_{\mathbb{S}^2}$$
 (6)

on r > 2M. Note that $\frac{\partial}{\partial t^*} = \frac{\partial}{\partial t}$, each interpreted in its respective coordinate system. But now (6) can clearly be defined in the region r > 0, $-\infty < t^* < \infty$, and, by explicit computation or better, by analytic continuation, the metric (6) must satisfy (4) for all r > 0.

Transformations similar to the above were already known to Eddington and Lemaitre [111] in the early 1930's. Nonetheless, from the point of view of that time, it was difficult to interpret their significance. The formalisation of the manifold concept and associated language had not yet become common knowledge to physicists (or most mathematicians for that matter), and in any case, there was no selection principle as to what should the underlying manifold \mathcal{M} be on which a solution g to (4) should live, or, to put it another way, the domain of g in (4) is not specified a priori by the theory. So, even if the solutions (6) exist, how do we know that they are "physical"?

This problem can in fact only be clarifed in the context of the Cauchy problem for (2) coupled to appropriate matter. Once the Cauchy problem for (4) is formulated correctly, then one can assign a unique spacetime to an appropriate notion of initial data set. This is the maximal development of Appendix B. It is only the initial data set, and the matter model, which can be judged for

 $^{^{10}}$ Consider for instance a cone with the vertex removed...

"physicality". One cannot throw away the resulting maximal development just because one does not like its properties!

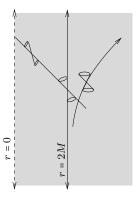
From this point of view, the question of whether the extension (6) was "physical" was resolved in 1939 by Oppenheimer and Snyder [125]. Specifically, they showed that the extension (6) for $t \ge 0$ arose as a subset of the solution to the Einstein equations coupled to a reasonable (to a first approximation at least) matter model, evolving from physically plausible initial data. With hindsight, the notion of black hole was born in that paper.

Had history proceeded differently, we could base our futher discussion on [125]. Unfortunately, the model [125] was ahead of its time. As mentioned in the introduction, the proper language to formulate the Cauchy problem in general only came in 1969 [35]. The interpretation of explicit solutions remained the main route to understanding the theory. We will follow thus this route to the black hole concept—via the geometric study of so-called maximally extended Schwarzschild—even though this spacetime is not to be regarded as "physical". It was through the study of this spacetime that the relevant notions were first understood and the important Penrose diagrammatic notation was developed. We shall return to [125] only in Section 2.5.3.

2.3 The maximal extension of Synge and Kruskal

Let us for now avoid the question of what the underlying manifold "should" be, a question whose answer requires physical input (see paragraphs above), and simply ask the purely mathematical question of how big the underlying manifold "can" be. This leads to the notion of a "maximally extended" solution. In the case of Schwarzschild, this will be a spacetime which, although not to be taken as a model for anything *per se*, can serve as a reference for the formulation of all important concepts in the subject.

To motivate this notion of "maximally extended" solution, let us examine our first extension a little more closely. The light cones can be drawn as follows:



Let us look say at null geodesics. One can see (**Exercise**) that future directed null geodesics either approach r = 0 or are future-complete. In the former case, scalar invariants of the curvature blow up in the limit as the affine

parameter approaches its supremum (**Exercise**). The spacetime is thus "singular" in this sense. It thus follows from the above properties that the above spacetime is future null geodesically incomplete, but also future null geodesically inextendible as a C^2 Lorentzian metric, i.e. there does not exist a larger 4-dimensional Lorentzian manifold with C^2 metric such that the spacetime above embeds isometrically into the larger one such that a future null geodesic passes into the extension.

On the other hand, one can see that past-directed null geodesics are not all complete, yet no curvature quantity blows up along them (**Exercise**). Again, this suggests that something may still be missing!

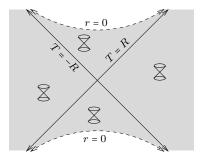
Synge was the first to consider these issues systematically and construct "maximal extensions" of the original Schwarzschild metric in a paper [146] of 1950. A more concise approach to such a construction was given in a celebrated 1960 paper [107] of Kruskal. Indeed, let \mathcal{M} be the manifold with differentiable structure given by $\mathcal{U} \times \mathbb{S}^2$ where \mathcal{U} is the open subset $T^2 - R^2 < 1$ of the (T, R)-plane. Consider the metric g

$$g = \frac{32M^3}{r}e^{-r/2M}(-dT^2 + dR^2) + r^2d\sigma_{S}^2$$

where r is defined implicitly by

$$T^2 - R^2 = \left(1 - \frac{r}{2M}\right)e^{r/2M}.$$

The region \mathcal{U} is depicted below:



This is a spherically symmetric 4-dimensional Lorentzian manifold satisfying (4) such that the original Schwarzschild metric is isometric to the region R > |T| (where t is given by $\tanh\left(\frac{t}{4M}\right) = T/R$), and our previous partial extension is isometric to the region T > -R (**Exercise**). It can be shown now (**Exercise**) that (\mathcal{M}, g) is inextendible as a C^2 (in fact C^0) Lorentzian manifold, that is to say, if

$$i: (\mathcal{M}, g) \to (\widetilde{\mathcal{M}}, \widetilde{g})$$

is an isometric embedding, where $(\widetilde{M}, \widetilde{g})$ is a C^2 (in fact C^0) 4-dimensional Lorentzian manifold, then necessarily $i(\mathcal{M}) = \widetilde{\mathcal{M}}$.

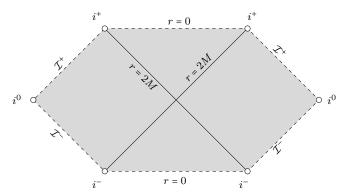
The above property defines the sense in which our spacetime is "maximally" extended, and thus, (\mathcal{M}, g) is called sometimes maximally-extended Schwarz-schild. In later sections, we will often just call it "the Schwarzschild solution".

Note that the form of the metric is such that the light cones are as depicted. Thus, one can read off much of the causal structure by sight.

It may come as a surprise that in maximally-extended Schwarzschild, there are two regions which are isometric to the original r > 2M Schwarzschild region. Alternatively, a Cauchy surface¹¹ will have topology $\mathbb{S}^2 \times \mathbb{R}$ with two asymptotically flat ends. This suggests that this spacetime is not to be taken as a physical model. We will discuss this later on. For now, let us simply try to understand better the global geometry of the metric.

2.4 The Penrose diagram of Schwarzschild

There is an even more useful way to represent the above spacetime. First, let us define null coordinates U = T - R, V = T + R. These coordinates have infinite range. We may rescale them by u = u(U), v = v(V) to have finite range. (Note the freedom in the choice of u and v!) The domain of (u, v) coordinates, when represented in the plane where the axes are at 45 and 135 degrees with the horizontal, is known as a *Penrose diagram* of Schwarzschild. Such a Penrose diagram is depicted below¹²:



In more geometric language, one says that a Penrose diagram corresponds to the image of a bounded conformal map

$$\mathcal{M}/SO(3) = \mathcal{Q} \to \mathbb{R}^{1+1},$$

where one makes the identification v = t + x, u = t - x where (t, x) are now the standard coordinates \mathbb{R}^{1+1} represented in the standard way on the plane. We further assume that the map preserves the time orientation, where Minkowski space is oriented by ∂_t . (In our application, this is a fancy way of saying that u'(U), v'(V) > 0). It follows that the map preserves the causal structure of

¹¹See Appendix A.

¹²How can (u, v) be chosen so that the r = 0 boundaries are horizontal lines? (Exercise)

Q. In particular, we can "read off" the radial null geodesics of \mathcal{M} from the depiction.

Now we may turn to the boundary induced by the causal embedding. We define \mathcal{I}^{\pm} to be the boundary components as depicted.¹³ These are characterized geometrically as follows: \mathcal{I}^{+} are limit points of future-directed null rays in \mathcal{Q} along which $r \to \infty$. Similarly, \mathcal{I}^{-} are limit points of past-directed null rays for which $r \to \infty$. We call \mathcal{I}^{+} future null infinity and \mathcal{I}^{-} past null infinity. The remaining boundary components i^{0} and i^{\pm} depicted are often given the names spacelike infinity and future (past) timelike infinity, respectively.

In the physical application, it is important to remember that asymptotically flat^{14} spacetimes like our (\mathcal{M},g) are not meant to represent the whole universe¹⁵, but rather, the gravitational field in the vicinity of an isolated self-gravitating system. \mathcal{I}^+ is an idealization of far away observers who can receive radiation from the system. In this sense, "we"-as astrophysical observers of stellar collapse, say-are located at \mathcal{I}^+ . The ambient causal structure of \mathbb{R}^{1+1} allows us to talk about $J^-(p) \cap \mathcal{Q}$ for $p \in \mathcal{I}^{+16}$ and this will lead us to the black hole concept. Therein lies the use of the Penrose diagram representation.

The systematic use of the conformal point of view to represent the global geometry of spacetimes is one of the many great contributions of Penrose to general relativity. These representations can be traced back to the well-known "spacetime diagrams" of special relativity, promoted especially by Synge [147]. The "formal" use of Penrose diagrams in the sense above goes back to Carter [28], in whose hands these diagrams became a powerful tool for determining the global structure of all classical black hole spacetimes. It is hard to overemphasize how important it is for the student of this subject to become comfortable with these representations.

2.5 The black hole concept

With Penrose diagram notation, we may now explain the black hole concept.

2.5.1 The definitions for Schwarzschild

First an important remark: In Schwarzschild, the boundary component \mathcal{I}^+ enjoys a limiting affine completeness. More specifically, normalising a sequence of ingoing radial null vectors by parallel transport along an outgoing geodesic meeting \mathcal{I}^+ , the affine length of the null geodesics generated by these vectors,

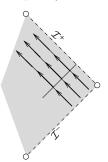
¹³Our convention is that open endpoint circles are not contained in the intervals they bound, and dotted lines are not contained in the regions they bound, whereas solid lines are.

¹⁴See Appendix B.2.3 for a definition.

¹⁵The study of that problem is what is known as "cosmology". See Section 6.

¹⁶Refer to Appendix A for J^{\pm} .

parametrized by their parallel transport (restricted to $J^{-}(\mathcal{I}^{+})$), tends to infinity:



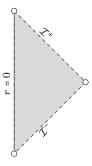
This has the interpretation that far-away observers in the radiation zone can observe for all time. (This is in some sense related to the presence of timelike geodesics near infinity of infinite length, but the completeness is best formulated with respect to \mathcal{I}^+ .) A similar statement clearly holds for \mathcal{I}^- .

Given this completeness property, we define now the black hole region to be $Q \setminus J^-(\mathcal{I}^+)$, and the white hole region to be $Q \setminus J^+(\mathcal{I}^-)$. Thus, the black hole corresponds to those points of spacetime which cannot "send signals" to future null infinity, or, in the physical interpetation, to far-away observers who (in view of the completeness property!) nonetheless can observe radiation for infinite time.

The future boundary of $J^-(\mathcal{I}^+)$ in \mathcal{Q} (alternatively characterized as the past boundary of the black hole region) is a null hypersurface known as the future event horizon, and is denoted by \mathcal{H}^+ . Exchanging past and future, we obtain the past event horizon \mathcal{H}^- . In maximal Schwarzschild, $\{r = 2M\} = \mathcal{H}^+ \cup \mathcal{H}^-$. The subset $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$ is known as the domain of outer communications.

2.5.2 Minkowski space

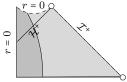
Note that in the case of Minkowski space, $Q = \mathbb{R}^{3+1}/SO(3)$ is a manifold with boundary since the SO(3) action has a locus of fixed points, the centre of symmetry. A Penrose diagram of Minkowski space is easily seen to be:



Here \mathcal{I}^+ and \mathcal{I}^- are characterized as before, and enjoy the same completeness property as in Schwarzschild. One reads off immediately that $J^-(\mathcal{I}^+) \cap \mathcal{Q} = \mathcal{Q}$, i.e. \mathbb{R}^{3+1} does not contain a black hole under the above definitions.

2.5.3 Oppenheimer-Snyder

Having now the notation of Penrose diagrams, we can concisely describe the geometry of the Oppenheimer-Snyder solutions referred to earlier, without giving explicit forms of the metric. Like Schwarzschild's original picture of the gravitational field of a spherically symmetric star, these solutions involve a region $r \leq R_0$ solving (2) and $r \geq R_0$ satisfying (4). The matter is described now by a pressureless fluid which is initially assumed homogeneous in addition to being spherically symmetric. The assumption of staticity is however dropped, and for appropriate initial conditions, it follows that $R_0(t^*) \to 0$ with respect to a suitable time coordinate t^* . (In fact, the Einstein equations can be reduced to an o.d.e. for $R_0(t^*)$.) We say that the star "collapses".¹⁷ A Penrose diagram of such a solution (to the future of a Cauchy hypersurface) can be seen to be of the form:



The lighter shaded region is isometric to a subset of maximal Schwarzschild, in fact a subset of the original extension of Section 2.2. In particular, the completeness property of \mathcal{I}^+ holds, and as before, we identify the black hole region to be $\mathcal{Q} \setminus J^-(\mathcal{I}^+)$.

In contrast to maximal Schwarzschild, where the initial configuration is unphysical (the Cauchy surface has two ends and topology $\mathbb{R} \times \mathbb{S}^2$), here the initial configuration is entirely plausible: the Cauchy surface is topologically \mathbb{R}^3 , and its geometry is not far from Euclidean space. The Oppenheimer-Snyder model [125] should be viewed as the most basic black hole solution arising from physically plausible regular initial data.¹⁸

It is traditional in general relativity to "think" Oppenheimer-Snyder but "write" maximally-extended Schwarzschild. In particular, one often imports terminology like "collapse" in discussing Schwarzschild, and one often reformulates our definitions replacing \mathcal{I}^+ with one of its connected components, that is to say, we will often write $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^+)$ meaning $J^-(\mathcal{I}^+_A) \cap J^+(\mathcal{I}^-_A)$, etc. In any case, the precise relation between the two solutions should be clear from the above discussion. In view of Cauchy stability results [91], sufficiently general theorems about the Cauchy problem on maximal Schwarzschild lead immediately to such results on Oppenheimer-Snyder. (See for instance the exercise in Section 3.2.6.) One should always keep this relation in mind.

¹⁷Note that $R_0(t^*) \to 0$ does not mean that the star collapses to "a point", merely that the spheres which foliate the interior of the star shrink to 0 area. The limiting singular boundary is a spacelike hypersurface as depicted.

¹⁸ Note however the end of Section 2.6.2.

2.5.4 General definitions?

The above definition of black hole for the Schwarzschild metric should be thought of as a blueprint for how to define the notion of black hole region in general. That is to say, to define the black hole region, one needs

- 1. some notion of future null infinity \mathcal{I}^+ ,
- 2. a way of identifying $J^{-}(\mathcal{I}^{+})$, and
- 3. some characterization of the "completeness" of \mathcal{I}^{+} . 19

If \mathcal{I}^+ is indeed complete, we can define the black hole region as

"the complement in \mathcal{M} of $J^-(\mathcal{I}^+)$ ".

For spherically symmetric spacetimes arising as solutions of the Cauchy problem for (2), one can show that there always exists a Penrose diagram, and thus, a definition can be formalised along precisely these lines (see [60]). For spacetimes without symmetry, however, even defining the relevant asymptotic structure so that this structure is compatible with the theorems one is to prove is a main part of the problem. This has been accomplished definitively only in the case of perturbations of Minkowski space. In particular, Christodoulou and Klainerman [51] have shown that spacetimes arising from perturbations of Minkowski initial data have a complete \mathcal{I}^+ in a well defined sense, whose past can be identified and is indeed the whole spacetime. See Appendix B.5. That is to say, small perturbations of Minkowski space cannot form black holes.

2.6 Birkhoff's theorem

Formal Penrose diagrams are a powerful tool for understanding the global causal structure of spherically symmetric spacetimes. Unfortunately, however, it turns out that the study of spherically symmetric vacuum spacetimes is not that rich. In fact, the Schwarzschild family parametrizes all spherically symmetric vacuum spacetimes in a sense to be explained in this section.

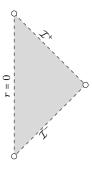
2.6.1 Schwarzschild for M < 0

Before stating the theorem, recall that in discussing Schwarzschild we have previously restricted to parameter value M>0. For the uniqueness statement, we must enlarge the family to include all parameter values.

¹⁹The characterization of completeness can be formulated for general asymptotically flat vacuum space times using the results of [51]. This formulation is due to Christodoulou [47]. Previous attempts to formalise these notions rested on "asymptotic simplicity" and "weak asymptotic simplicity". See [91]. Although the qualitative picture suggested by these notions appears plausible, the detailed asymptotic behaviour of solutions to the Einstein equations turns out to be much more subtle, and Christodoulou has proven [48] that these notions cannot capture even the simplest generic physically interesting systems.

If we set M = 0 in (5), we of course obtain Minkowski space in spherical polar coordinates. A suitable maximal extension is Minkowski space as we know it, represented by the Penrose diagram of Section 2.5.2.

On the other hand, we may also take M < 0 in (5). This is so-called negative mass Schwarzschild. The metric element (5) for such M is now regular for all r > 0. The limiting singular behaviour of the metric at r = 0 is in fact essential, i.e. one can show that along inextendible incomplete geodesics the curvature blows up. Thus, one immediately arrives at a maximally extended solution which can be seen to have Penrose diagram:



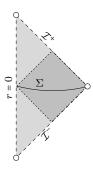
Note that in contrast to the case of \mathbb{R}^{3+1} , the boundary r = 0 is here depicted by a dotted line denoting (according to our conventions) that it is not part of \mathcal{Q} !

2.6.2 Naked singularities and weak cosmic censorship

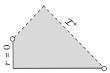
The above spacetime is interpreted as having a "naked singularity". The traditional way of describing this in the physics literature is to remark that the "singularity" $\mathcal{B} = \{r = 0\}$ is "visible" to \mathcal{I}^+ , i.e., $J^-(\mathcal{I}^+) \cap \mathcal{B} \neq \emptyset$. From the point of view of the Cauchy problem, however, this characterization is meaningless because the above maximal extension is not globally hyperbolic, i.e. it is not uniquely characterized by an appropriate notion of initial data.²⁰ From the point of view of the Cauchy problem, one must not consider maximal extensions but the maximal Cauchy development of initial data, which by definition is globally hyperbolic (see Theorem B.4 of Appendix B). Considering an inextendible spacelike hypersurface Σ as a Cauchy surface, the maximal Cauchy

²⁰See Appendix A for the definition of global hyperbolicity.

development of Σ would be the darker shaded region depicted below:



The proper characterization of "having a naked singularity", from the point of view of the darker shaded spacetime, is that its \mathcal{I}^+ is incomplete. Of course, this example does not say anything about the dynamic formation of naked singularities, because the inital data hypersurface Σ is already in some sense "singular", for instance, it is geodesically incomplete, and the curvature blows up along incomplete geodesics. The dynamic formation of a naked singularity from regular, complete initial data would be pictured by:



where we are to understand also in the above that \mathcal{I}^+ is incomplete. The conjecture that for generic asymptoically flat²¹ initial data for "reasonable" Einsteinmatter systems, the maximal Cauchy development "possesses a complete \mathcal{I}^+ " is known as weak cosmic censorship.²²

In light of the above conjecture, the story of the Oppenheimer-Snyder solution and its role in the emergence of the black hole concept does have an interesting epilogue. Recall that in the Oppenheimer-Snyder solutions, the region $r \leq R_0$, in addition to being spherically symmetric, is homogeneous. It turns out that by considering spherically symmetric initial data for which the "star" is no longer homogeneous, Christodoulou has proven that one can arrive at spacetimes for which "naked singularities" form [39] with Penrose diagram as above and with \mathcal{I}^+ incomplete. Moreover, it is shown in [39] that this occurs for an open subset of initial data within spherical symmetry, with respect to a suitable topology on the set of spherically symmetric initial data. Thus, weak cosmic censorship is violated in this model, at least if the conjecture is restricted to spherically symmetric data.

 $^{^{21}\}mathrm{See}$ Appendix B.2.3 for a formulation of this notion. Note that asymptotically flat data are in particular complete.

²²This conjecture is originally due to Penrose [127]. The present formulation is taken from Christodoulou [47].

The fact that in the Oppenheimer-Snyder solutions black holes formed appears thus to be a rather fortuitous accident! Nonetheless, we should note that the failure of weak cosmic censorship in this context is believed to be due to the inappropriateness of the pressureless model, not as indicative of actual phenomena. Hence, the restriction on the matter model to be "reasonable" in the formulation of the conjecture. In a remarkable series of papers, Christodoulou [45, 47] has shown weak cosmic censorship to be true for the Einstein-scalar field system under spherical symmetry. On the other hand, he has also shown [43] that the assumption of genericity is still necessary by explicitly constructing solutions of this system with incomplete \mathcal{I}^+ and Penrose diagram as depicted above.²³

2.6.3 Birkhoff's theorem

Let us understand now by "Schwarzschild solution with parameter M" (where $M \in \mathbb{R}$) the maximally extended Schwarzschild metrics described above.

We have the so-called Birkhoff's theorem:

Theorem 2.1. Let (\mathcal{M}, g) be a spherically symmetric solution to the vacuum equations (4). Then it is locally isometric to a Schwarzschild solution with parameter M, for some $M \in \mathbb{R}$.

In particular, spherically symmetric solutions to (4) possess an additional Killing field not in the Lie algebra so(3). (**Exercise**: Prove Theorem 2.1. Formulate and prove a global version of the result.)

2.6.4 Higher dimensions

In 3+1 dimensions, spherical symmetry is the only symmetry assumption compatible with asymptotic flatness (see Appendix B.2.3), such that moreover the symmetry group acts transitively on 2-dimensional orbits. Thus, Birkhoff's theorem means that vacuum gravitational collapse cannot be studied in a 1+1 dimensional setting by imposing symmetry. The simplest models for dynamic gravitational collapse thus necessarily involve matter, as in the Oppenheimer-Snyder model [125] or the Einstein-scalar field system studied by Christodoulou [41, 45]

Moving, however, to 4+1 dimensions, asymptotically flat manifolds can admit a more general SU(2) symmetry acting transitively on 3-dimensional group orbits. The Einstein vacuum equations (4) under this symmetry admit 2 dynamical degrees of freedom and can be written as a nonlinear system on a 1+1-dimensional Lorentzian quotient $\mathcal{Q}=\mathcal{M}/SU(2)$, where the dynamical degrees of freedom of the metric are reflected by two nonlinear scalar fields on \mathcal{Q} . This symmetry–known as "Triaxial Bianchi IX"–was first identified by Bizon, Chmaj and Schmidt [16, 17] who derived the equations on \mathcal{Q} and studied the resulting system numerically. The symmetry includes spherical symmetry as a

²³The discovery [43] of these naked singularities led to the discovery of so-called critical collapse phenomena [37] which has since become a popular topic of investigation [87].

special case, and thus, is admitted in particular by 4+1-dimensional Schwarzschild²⁴. The nonlinear stability of the Schwarzschild family as solutions of the vacuum equations (4) can then be studied—within the class of Triaxial Bianchi IX initial data—as a 1+1 dimensional problem. Asymptotic stability for the Schwarzschild spacetime in this setting has been recently shown in the thesis of Holzegel [93, 62, 94], adapting vector field multiplier estimates similar to Section 4 to a situation where the metric is not known a priori. The construction of the relevant mutipliers is then quite subtle, as they must be normalised "from the future" in a bootstrap setting. The thesis [93] is a good reference for understanding the relation of the linear theory to the non-linear black hole stability problem. See also Open problem 13 in Section 8.6.

2.7 Geodesic incompleteness and "singularities"

Is the picture of gravitational collapse as exhibited by Schwarzschild (or better, Oppenheimer-Snyder) stable? This question is behind the later chapters in the notes, where essentially the considerations hope to be part of a future understanding of the stability of the exterior region up to the event horizon, i.e. the closure of the past of null infinity to the future of a Cauchy surface. (See Section 5.6 for a formulation of this open problem.) What is remarkable, however, is that there is a feature of Schwarzschild which can easily be shown to be "stable", without understanding the p.d.e. aspects of (2): its geodesic incompleteness.

2.7.1 Trapped surfaces

First a definition: Let (\mathcal{M}, g) be a time-oriented Lorentzian manifold, and S a closed spacelike 2-surface. For any point $p \in S$, we may define two null mean curvatures $\operatorname{tr}\chi$ and $\operatorname{tr}\bar{\chi}$, corresponding to the two future-directed null vectors n(x), $\bar{n}(x)$, where n, \bar{n} are normal to S at x. We say that S is $\operatorname{trapped}$ if $\operatorname{tr}\chi < 0$, $\operatorname{tr}\bar{\chi} < 0$.

Exercise: Show that points $p \in \mathcal{Q} \setminus \operatorname{clos}(J^-(\mathcal{I}^+))$ correspond to trapped surfaces of \mathcal{M} . Can there be other trapped surfaces? (Refer also for instance to [12].)

2.7.2 Penrose's incompleteness theorem

Theorem 2.2. (Penrose 1965 [126]) Let (\mathcal{M}, g) be globally hyperbolic²⁵ with non-compact Cauchy surface Σ , where g is a C^2 metric, and let

$$R_{\mu\nu}V^{\mu}V^{\nu} \ge 0 \tag{7}$$

for all null vectors V. Then if \mathcal{M} contains a closed trapped two-surface S, it follows that (\mathcal{M}, g) is future causally geodesically incomplete.

 $^{^{24}}$ Exercise: Work out explicitly the higher dimensional analogue of the Schwarzschild solution for all dimensions.

 $^{^{25}\}mathrm{See}$ Appendix A.

This is the celebrated *Penrose incompleteness theorem*.

Note that solutions of the Einstein vacuum equations (4) satisfy (7). (Inequality (7), known as the null convergence condition, is also satisfied for solutions to the Einstein equations (2) coupled to most plausible matter models, specifically, if the energy momentum tensor $T_{\mu\nu}$ satisfies $T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$ for all null V^{μ} .) On the other hand, by definition, the unique solution to the Cauchy problem (the so-called maximal Cauchy development of initial data) is globally hyperbolic (see Appendix B.3). Thus, the theorem applies to the maximal development of (say) asymptotically flat (see Appendix B.2.3) vacuum initial data containing a trapped surface. Note finally that by Cauchy stability [91], the presence of a trapped surface in \mathcal{M} is clearly "stable" to perturbation of initial data.

From the point of view of gravitational collapse, it is more appropriate to define a slightly different notion of trapped. We restrict to $S \subset \Sigma$ a Cauchy surface such that S bounds a disc in Σ . We then can define a unique outward null vector field n along S, and we say that S is trapped if $\operatorname{tr}\chi < 0$ and antitrapped if $\operatorname{tr}\bar{\chi} < 0$, where $\operatorname{tr}\bar{\chi}$ denotes the mean curvature with respect to a conjugate "inward" null vector field. The analogue of Penrose's incompleteness theorem holds under this definition. One may also prove the interesting result that antitrapped surface cannot not form if they are not present initially. See [49].

Note finally that there are related incompleteness statements due to Penrose and Hawking [91] relevant in cosmological (see Section 6) settings.

2.7.3 "Singularities" and strong cosmic censorship

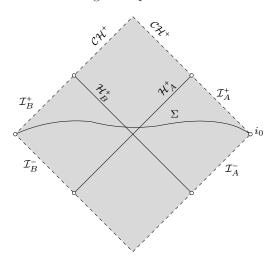
Following [49], we have called Theorem 2.2 an "incompleteness theorem" and not a "singularity theorem". This is of course an issue of semantics, but let us further discuss this point briefly as it may serve to clarify various issues. The term "singularity" has had a tortuous history in the context of general relativity. As we have seen, its first appearance was to describe something that turned out not to be a singularity at all—the "Schwarzschild singularity". It was later realised that behaviour which could indeed reasonably be described by the word "singularity" did in fact occur in solutions, as exemplified by the r=0 singular "boundary" of Schwarzschild towards which curvature scalars blow up. The presense of this singular behaviour "coincides" in Schwarzschild with the fact that the spacetime is future causally geodesically incomplete—in fact, the curvature blows up along all incomplete causal geodesics. In view of the fact that it is the incompleteness property which can be inferred from Theorem 2.2, it was tempting to redefine "singularity" as geodesic incompleteness (see [91]) and to call Theorem 2.2 a "singularity theorem".

This is of course a perfectly valid point of view. But is it correct then to associate the incompleteness of Theorem 2.2 to "singularity" in the sense of "breakdown" of the metric? Breakdown of the metric is most easily understood with curvature blowup as above, but more generally, it is captured by the no-

²⁶Note that there exist other conventions in the literature for this terminology. See [12].

tion of "inextendibility" of the Lorentzian manifold in some regularity class. We have already remarked that maximally-extended Schwarzschild is inextendible in the strongest of senses, i.e. as a C^0 Lorentzian metric. It turns out, however, that the statement of Theorem 2.2, even when applied to the maximal development of complete initial data for (4), is compatible with the solution being extendible as a C^{∞} Lorentzian metric such that every incomplete causal geodesic of the original spacetime enter the extension! This is in fact what happens in the case of Kerr initial data. (See Section 5.1 for a discussion of the Kerr metric.) The reason that the existence of such extensions does not contradict the "maximality" of the "maximal development" is that these extensions fail to be globally hyperbolic, while the "maximal development" is "maximal" in the class of globally hyperbolic spacetimes (see Theorem B.4 of Appendix B). In the context of Kerr initial data, Theorem 2.2 is thus not saying that breakdown of the metric occurs, merely that globally hyperbolicity breaks down, and thus further extensions cease to be predictable from initial data.²⁷

A similar phenomenon is exhibited by the Reissner-Nordström solution of the Einstein-Maxwell equations [91], which, unlike Kerr, is spherically symmetric and thus admits a Penrose diagram representation:



What is drawn above is the maximal development of Σ . The spacetime is future causally geodesically incomplete, but can be extended smoothly to a $(\tilde{\mathcal{M}}, \tilde{g})$ such that all inextendible geodesics leave the original spacetime. The boundary of (\mathcal{M}, g) in the extension corresponds to \mathcal{CH}^+ above. Such boundaries are known as *Cauchy horizons*.

²⁷Further confusion can arise from the fact that "maximal extensions" of Kerr constructed with the help of analyticity are still geodesically incomplete and inextendible, in particular, with the curvature blowing up along all incomplete causal geodesics. Thus, one often talks of the "singularities" of Kerr, referring to the ideal singular boundaries one can attach to such extensions. One must remember, however, that these extensions are of no relevance from the point of view of the Cauchy problem, and in any case, their singular behaviour in principle has nothing to do with Theorem 2.2.

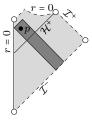
The strong cosmic censorship conjecture says that the maximal development of generic asymptotically flat initial data for the vacuum Einstein equations is inextendible as a suitably regular Lorentzian metric.²⁸ One can view this conjecture as saying that whenever one has geodesic incompleteness, it is due to breakdown of the metric in the sense discussed above. (In view of the above comments, for this conjecture to be true, the behaviour of the Kerr metric described above would have to be unstable to perturbation.²⁹) Thus, if by the term "singularity" one wants to suggest "breakdown of the metric", it is only a positive resolution of the strong cosmic censorship conjecture that would in particular (generically) make Theorem 2.2 into a true "singularity theorem".

2.8 Christodoulou's work on trapped surface formation in vacuum

These notes would not be complete without a brief discussion of the recent breakthrough by Christodoulou [53] on the understanding of trapped surface formation for the vacuum.

The story begins with Christodoulou's earlier [41], where a condition is given ensuring that trapped surfaces form for spherically symmetric solutions of the Einstein-scalar field system. The condition is that the difference in so-called Hawking mass m of two concentric spheres on an outgoing null hupersurface be sufficiently large with respect to the difference in area radius r of the spheres. This is a surprising result as it shows that trapped surface formation can arise from initial conditions which are as close to dispersed as possible, in the sense that the supremum of the quantity 2m/r can be taken arbitrarily small initially.

The results of [41] lead immediately (see for instance [61]) to the existence of smooth spherically symmetric solutions of the Einstein-scalar field system with Penrose diagram



 $^{^{28}}$ As with weak cosmic censorship, the original formulation of this conjecture is due to Penrose [128]. The formulation given here is from [47]. Related formulations are given in [54, 118]. One can also pose the conjecture for compact initial data, and for various Einsteinmatter systems. It should be emphasized that "strong cosmic censorship" does not imply "weak cosmic censorship". For instance, one can imagine a spacetime with Penrose diagram as in the last diagram of Section 2.6.2, with incomplete \mathcal{I}^+ , but still inextendible across the null "boundary" emerging from the centre.

²⁹Note that the instability concerns a region "far inside" the black hole interior. The black hole exterior is expected to be stable (as in the formulation of Section 5.6), hence these notes. See [58, 59] for the resolution of a spherically symmetric version of this problem, where the role of the Kerr metric is played by Reissner-Nordström metrics.

where the point p depicted corresponds to a trapped surface, and the spacetime is past geodesically complete with a complete past null infinity, whose future is the entire spacetime, i.e., the spacetime contains no white holes.³⁰ Thus, black hole formation can arise from spacetimes with a complete regular past.³¹

In [53], Christodoulou constructs vacuum solutions by prescribing a characteristic initial value problem with data on (what will be) \mathcal{I}^- . This \mathcal{I}^- is taken to be past complete, and in fact, the data is taken to be trivial to the past of a sphere on \mathcal{I}^- . Thus, the development will include a region where the metric is Minkowski, corresponding precisely to the lower lighter shaded triangle above. It is shown that—as long as the incoming energy per unit solid angle in all directions³² is sufficiently large in a strip of \mathcal{I}^- right after the trivial part, where sufficiently large is taken in comparison with the affine length of the generators of \mathcal{I}^- —a trapped surface arises in the domain of development of the data restricted to the past of this strip. Comparing with the spherically symmetric picture above, this trapped surface would arise precisely as before in the analogue of the darker shaded region depicted.

In contrast to the spherically symmetric case, where given the lower triangle, existence of the solution in the darker shaded region (at least as far as trapped surface formation) follows immediately, for vacuum collapse, showing the existence of a sufficiently "big" spacetime is a major difficulty. For this, the results of [53] exploit a hierarchy in the Einstein equations (4) in the context of what is there called the "short pulse method". This method may have many other applications for nonlinear problems.

One could in principle hope to extend [53] to show the formation of black hole spacetimes in the sense described previously. For this, one must first extend the initial data suitably, for instance so that \mathcal{I}^- is complete. If the resulting spacetime can be shown to possess a complete future null infinity \mathcal{I}^+ , then, since the trapped surface shown to form can be proven (using the methods of the proof of Theorem 2.2) not to be in the past of null infinity, the spacetime will indeed contain a black hole region.³³ Of course, resolution of this problem would appear comparable in difficulty to the stability problem for the Kerr family (see the formulation of Section 5.6).

 $^{^{30}}$ The triangle "under" the darker shaded region can in fact be taken to be Minkowski.

 $^{^{31}}$ The singular boundary in general consists of a possibly empty null component emanating from the regular centre, and a spacelike component where r=0 in the limit and across which the spacetime is inextendible as a C^0 Lorentzian metric. (This boundary could "bite off" the top corner of the darker shaded rectangle.) The null component arising from the centre can be shown to be empty generically after passing to a slightly less regular class of solutions, for which well-posedness still holds. See Christodoulou's proof of the cosmic censorship conjectures [45] for the Einstein-scalar field system.

 $^{^{32}}$ This is defined in terms of the shear of \mathcal{I}^- .

³³In spherical symmetry, the completeness of null infinity follows immediately once a single trapped surface has formed, for the Einstein equations coupled to a wide class of matter models. See for instance [60]. For vacuum collapse, Christodoulou has formulated a statement on trapped surface formation that would imply weak cosmic censorship. See [47].

3 The wave equation on Schwarzschild I: uniform boundedness

In the remainder of these lectures, we will concern ourselves solely with linear wave equations on black hole backgrounds, specifically, the scalar linear homogeneous wave equation (1). As explained in the introduction, the study of the solutions to such equations is motivated by the stability problem for the black hole spacetimes themselves as solutions to (4). The equation (1) can be viewed as a poor man's linearisation of (4), neglecting tensorial structure. Other linear problems with a much closer relationship to the study of the Einstein equations will be discussed in Section 8.

3.1 Preliminaries

Let (\mathcal{M}, g) denote (maximally-extended) Schwarzschild with parameter M > 0. Let Σ be an arbitrary *Cauchy surface*, that is to say, a hypersurface with the property that every inextendible causal geodesic in \mathcal{M} intersects Σ precisely once. (See Appendix A.)

Proposition 3.1.1. If $\psi \in H^2_{loc}(\Sigma)$, $\psi' \in H^1_{loc}(\Sigma)$, then there is a unique ψ with $\psi|_{\mathcal{S}} \in H^2_{loc}(\mathcal{S})$, $n_{\mathcal{S}}\psi|_{\mathcal{S}} \in H^1_{loc}(\mathcal{S})$, for all spacelike $\mathcal{S} \subset \mathcal{M}$, satisfying

$$\Box_g \psi = 0, \qquad \psi|_{\Sigma} = \psi, \qquad n_{\Sigma} \psi|_{\Sigma} = \psi',$$

where n_{Σ} denotes the future unit normal of Σ . For $m \geq 1$, if $\psi \in H^{m+1}_{loc}$, $\psi' \in H^m_{loc}$, then $\psi|_{\mathcal{S}} \in H^{m+1}_{loc}(\mathcal{S})$, $n_{\mathcal{S}}\psi|_{\mathcal{S}} \in H^m_{loc}(\mathcal{S})$. Moreover, if ψ_1, ψ'_1 , and ψ_2, ψ'_2 are as above and $\psi_1 = \psi_2$, $\psi'_1 = \psi'_2$ in an open set $\mathcal{U} \subset \Sigma$, then $\psi_1 = \psi_2$ in $\mathcal{M} \setminus J^{\pm}(\Sigma \setminus clos(\mathcal{U}))$.

We will be interested in understanding the behaviour of ψ in the exterior of the black hole and white hole regions, up to and including the horizons. It is enough of course to understand the behaviour in the region

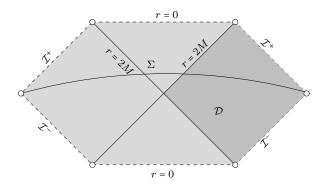
$$\mathcal{D} \doteq \operatorname{clos} \left(J^{-}(\mathcal{I}_{A}^{+}) \cap J^{+}(\mathcal{I}_{A}^{-}) \right) \cap \mathcal{Q}$$

where \mathcal{I}_A^{\pm} denote a pair of connected components of \mathcal{I}^{\pm} , respectively, with a common limit point.³⁴

Moreover, it suffices (**Exercise**: Why?) to assume that $\Sigma \cap \mathcal{H}^- = \emptyset$, and that we are interested in the behaviour in $J^-(\mathcal{I}^+) \cap J^+(\Sigma)$. Note that in this case, by the domain of dependence property of the above proposition, we have that the solution in this region is determined by $\psi|_{\mathcal{D}\cap\Sigma}$, $\psi'|_{\mathcal{D}\cap\Sigma}$. In the case

³⁴We will sometimes be sloppy with distinguishing between $\pi^{-1}(p)$ and p, where $\pi: \mathcal{M} \to \mathcal{Q}$ denotes the natural projection, distinguishing $J^-(p)$ and $J^-(p) \cap \mathcal{Q}$, etc. The context should make clear what is meant.

where Σ itself is spherically symmetric, then its projection to \mathcal{Q} will look like:



If Σ is not itself spherically symmetric, then its projection to \mathcal{Q} will in general have open interior. Nonetheless, we shall always depict Σ as above.

3.2 The Kay–Wald boundedness theorem

The most basic problem is to obtain uniform boundedness for ψ . This is resolved in the celebrated:

Theorem 3.1. Let ψ , ψ , ψ' be as in Proposition 3.1.1, with $\psi \in H^{m+1}_{loc}(\Sigma)$, $\psi' \in H^m_{loc}(\Sigma)$ for a sufficiently high m, and such that ψ , ψ' decay suitably at i^0 . Then there is a constant D depending on ψ , ψ' such that

$$|\psi| \le D$$

in \mathcal{D} .

The proof of this theorem is due to Wald [151] and Kay-Wald [98]. The "easy part" of the proof (Section 3.2.3) is a classic application of vector field commutators and multipliers, together with elliptic estimates and the Sobolev inequality. The main difficulties arise at the horizon, and these are overcome by what is essentially a clever trick. In this section, we will go through the original argument, as it is a nice introduction to vector field multiplier and commutator techniques, as well as to the geometry of Schwarzschild. We will then point out (Section 3.2.7) various disadvantages of the method of proof. Afterwards, we give a new proof that in fact achieves a stronger result (Theorem 3.2). As we shall see, the techniques of this proof will be essential for future applications.

3.2.1 The Killing fields of Schwarzschild

Recall the symmetries of (\mathcal{M}, g) : (\mathcal{M}, g) is spherically symmetric, i.e. there is a basis of Killing vectors $\{\Omega_i\}_{i=1}^3$ spanning the Lie algebra so(3). These are sometimes known as angular momentum operators. In addition, there is another Killing field T (equal to ∂_t in the coordinates (5)) which is hypersurface orthogonal and future directed timelike near i_0 . This Killing field is in fact timelike

everywhere in $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$, becoming null and tangent to the horizon, vanishing at $\mathcal{H}^+ \cap \mathcal{H}^-$. We say that the Schwarzschild metric in $J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)$ is static. T is spacelike in the black hole and white hole regions.

Note that whereas in Minkowski space \mathbb{R}^{3+1} , the Killing fields at any point span the tangent space, this is no longer the case for Schwarzschild. We shall return to this point later.

3.2.2 The current J^T and its energy estimate

Let φ_t denote the 1-parameter group of diffeomorphisms generated by the Killing field T. Define $\Sigma_{\tau} = \varphi_t(\Sigma \cap \mathcal{D})$. We have that $\{\Sigma_{\tau}\}_{\tau \geq 0}$ defines a spacelike foliation of

$$\mathcal{R} \doteq \cup_{\tau > 0} \Sigma_{\tau}$$
.

Define

$$\mathcal{H}^+(0,\tau) \doteq \mathcal{H}^+ \cap J^+(\Sigma_0) \cap J^-(\Sigma_\tau),$$

and

$$\mathcal{R}(0,\tau) \doteq \cup_{0 < \bar{\tau} < \tau} \Sigma_{\bar{\tau}}.$$

Let n_{Σ}^{μ} denote the future directed unit normal of Σ , and let $n_{\mathcal{H}}^{\mu}$ define a null generator of \mathcal{H}^+ , and give \mathcal{H}^+ the associated volume form.³⁵

Let $J_{\mu}^{T}(\psi)$ denote the energy current defined by applying the vector field T as a multiplier, i.e.

$$J_{\mu}^{T}(\psi) = T_{\mu\nu}(\psi)T^{\nu} = (\partial_{\mu}\psi\partial_{\nu}\psi - \frac{1}{2}g_{\mu\nu}\partial^{\alpha}\psi\partial_{\alpha}\psi)T^{\nu}$$

with its associated current $K^{T}(\psi)$,

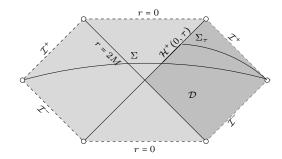
$$K^{T}(\psi) = {}^{T}\pi^{\mu\nu}T_{\mu\nu}(\psi) = \nabla^{\mu}J_{\mu}^{T}(\psi),$$

where $T_{\mu\nu}$ denotes the standard energy momentum tensor of ψ (see Appendix D). Since T is Killing, and $\nabla^{\mu}T_{\mu\nu} = 0$, it follows that $K^{T}(\psi) = 0$, and the divergence theorem (See Appendix C) applied to J_{μ}^{T} in the region $\mathcal{R}(0,\tau)$ yields

$$\int_{\Sigma_{\tau}} J_{\mu}^{T}(\psi) n_{\Sigma_{\tau}}^{\mu} + \int_{\mathcal{H}^{+}(0,\tau)} J_{\mu}^{T}(\psi) n_{\mathcal{H}}^{\mu} = \int_{\Sigma_{0}} J_{\mu}^{T}(\psi) n_{\Sigma_{0}}^{\mu}.$$
 (8)

³⁵Recall that for null surfaces, the definition of a volume form relies on the choice of a normal. All integrals in what follow will always be with respect to the natural volume form, and in the case of a null hypersurface, with respect to the volume form related to the given choice of normal. See Appendix C.

See



Since T is future-directed causal in \mathcal{D} , we have

$$J_{\mu}^{T}(\psi)n_{\Sigma}^{\mu} \ge 0, \qquad J_{\mu}^{T}(\psi)n_{\mathcal{H}}^{\mu} \ge 0. \tag{9}$$

Let us fix an $r_0 > 2M$. It follows from (8), (9) that

$$\int_{\Sigma_{\tau} \cap \{r \geq r_0\}} J_{\mu}^T(\psi) n_{\Sigma_{\tau}}^{\mu} \leq \int_{\Sigma_0} J_{\mu}^T(\psi) n_{\Sigma_0}^{\mu}.$$

As long as $-g(T, n_{\Sigma_0}) \leq B$ for some constant B^{36} , we have

$$B(r_0, \Sigma)((\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2) \ge J_u^T(\psi)n^\mu \ge b(r_0, \Sigma)((\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2).$$

Here, $|\nabla \psi|^2$ denotes the induced norm on the group orbits of the SO(3) action, with ∇ the gradient of the induced metric on the group orbits. We thus have

$$\int_{\Sigma_{\tau} \cap \{r \geq r_0\}} (\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \leq B(r_0, \Sigma) \int_{\Sigma_0} J_{\mu}^T(\psi) n_{\Sigma_0}^{\mu}.$$

3.2.3 *T* as a commutator and pointwise estimates away from the horizon

We may now commute the equation with T (See Appendix E), i.e., since $[\Box, T] = 0$, if $\Box_g \psi = 0$ then $\Box_g (T\psi) = 0$. We thus obtain an estimate

$$\int_{\Sigma_{\tau} \cap \{r \ge r_0\}} (\partial_t^2 \psi)^2 + (\partial_r \partial_t \psi)^2 + |\nabla \partial_t \psi|^2 \le B(r_0, \Sigma) \int_{\Sigma_0} J_{\mu}^T (T\psi) n_{\Sigma_0}^{\mu}. \tag{10}$$

Exercise: By elliptic estimates and a Sobolev estimate show that if $\psi(x) \to 0$ as $x \to i_0$, then (10) implies that for $r \ge r_0$,

$$|\psi|^2 \le B(r_0, \Sigma) \left(\int_{\Sigma_0} J_\mu^T(\psi) n_{\Sigma_0}^\mu + \int_{\Sigma_0} J_\mu^T(T\psi) n_{\Sigma_0}^\mu \right),$$
 (11)

for solutions ψ of $\Box_q \psi = 0$.

The right hand side of (11) is finite under the assumptions of Theorem 3.1, for m = 1. Thus, proving the estimate of Theorem 3.1 away from the horizon

 $^{^{36}}$ For definiteness, one could choose Σ to be a surface of constant t^* defined in Section 2.2, or alternatively, require that it be of constant t for large r.

poses no difficulty. The difficulty of Theorem 3.1 is obtaining estimates which hold up to the horizon.

Remark: The above argument via elliptic estimates clearly also holds for Minkowski space. But in that case, there is an alternative "easier" argument, namely, to commute with all translations.³⁷ We see thus already that the lack of Killing fields in Schwarzschild makes things more difficult. We shall again return to this point later.

3.2.4 Degeneration at the horizon

As one takes $r_0 \to 2M$, the constant $B(r_0, \Sigma)$ provided by the estimate (11) blows up. This is precisely because T becomes null on \mathcal{H}^+ and thus its control over derivatives of ψ degenerates. Thus, one cannot prove uniform boundedness holding up to the horizon by the above.

Let us examine more carefully this degeneration on various hypersurfaces. On Σ_{τ} , we have only

$$J_{\mu}^{T}(\psi)n_{\Sigma_{\tau}}^{\mu} \ge B(\Sigma_{\tau})((\partial_{t^{*}}\psi)^{2} + (1 - 2M/r)(\partial_{r}\psi)^{2} + |\nabla\psi|^{2}). \tag{12}$$

We see the degeneration in the presence of the factor (1 - 2M/r). Note that (**Exercise**) 1 - 2M/r vanishes to first order on $\mathcal{H}^+ \setminus \mathcal{H}^-$. Alternatively, one can examine the flux on the horizon \mathcal{H}^+ itself. For definiteness, let us choose $n_{\mathcal{H}^+} = T$ in $\mathcal{R} \cap \mathcal{H}^+$. We have

$$J_{\mu}^{T}(\psi)T^{\mu} = (T\psi)^{2}. \tag{13}$$

Comparing with the analogous computation on a null cone in Minkowski space, one sees that a term $|\nabla \psi|^2$ is "missing".

Are estimates of the terms (12), (13) enough to control ψ ? It is a good idea to play with these estimates on your own, allowing yourself to commute the equation with T and Ω_i to obtain higher order estimates. **Exercise**: Why does this not lead to an estimate as in (11)?

It turns out that there is a way around this problem and the degeneration on the horizon is suggestive. For suppose there existed a $\tilde{\psi}$ such that

$$\Box_g \tilde{\psi} = 0, \qquad T\tilde{\psi} = \psi. \tag{14}$$

Let us see immediately how one can obtain estimates on the horizon itself. For this, we note that

$$J^T_\mu(\tilde\psi)T^\mu + J^T_\mu(\psi)T^\mu = \psi^2 + (T\psi)^2.$$

Commuting now with the whole Lie algebra of isometries, we obtain

$$\begin{split} J_{\mu}^{T}(\tilde{\psi})T^{\mu} + J_{\mu}^{T}(\psi)T^{\mu} + \sum_{i} J_{\mu}^{T}(\Omega_{i}\tilde{\psi})T^{\mu} + J_{\mu}^{T}(T\psi)T^{\mu} \cdots \\ &= \psi^{2} + (T\psi)^{2} + \sum_{i} (\Omega_{i}\psi)^{2} + (T^{2}\psi)^{2} + \cdots. \end{split}$$

³⁷Easier, but not necessarily better...

Clearly, by a Sobolev estimate applied on the horizon, together with the estimate

$$\int_{\mathcal{H}^+ \cap \mathcal{R}} J_{\mu}^T (\Gamma^{(\alpha)} \tilde{\psi}) n_{\mathcal{H}}^{\mu} \leq \int_{\Sigma_0} J_{\mu}^T (\Gamma^{(\alpha)} \tilde{\psi}) n_{\Sigma_0}^{\mu}$$

for $\Gamma = T, \Omega_i$ (here (α) denotes a multi-index of arbitrary order), we would obtain

$$|\psi|^2 \le B \sum_{\Gamma = T, \Omega_s} \sum_{|(\alpha)| \le 2} \int_{\Sigma_0} J_{\mu}^T (\Gamma^{(\alpha)} \tilde{\psi}) n_{\Sigma_0}^{\mu} \tag{15}$$

on $\mathcal{H}^+ \cap \mathcal{R}$.

It turns out that the estimate (15) can be extended to points not on the horizon by considering t = c surfaces. Note that these hypersurfaces all meet at $\mathcal{H}^+ \cap \mathcal{H}^-$. It is an informative calculation to examine the nature of the degeneration of estimates on such hypersurfaces because it is of a double nature, since, in addition to T becoming null, the limit of (subsets of) these spacelike hypersurfaces approaches the null horizon \mathcal{H}^+ . We leave the details as an exercise.

3.2.5 Inverting an elliptic operator

So can a $\tilde{\psi}$ satisfying (14) actually be constructed? We have

Proposition 3.2.1. Suppose m is sufficiently high, ψ , ψ' decay suitably at i^0 , and $\psi|_{\mathcal{H}^+\cap\mathcal{H}^-} = 0$, $\Xi\psi|_{\mathcal{H}^+\cap\mathcal{H}^-} = 0$ for some spherically symmetric timelike vector field Ξ defined along $\mathcal{H}^+\cap\mathcal{H}^-$. Then there exists a $\tilde{\psi}$ satisfying $\Box_g\tilde{\psi} = 0$ with $T\tilde{\psi} = \psi$ in \mathcal{D} , and moreover, the right hand side of (15) is finite.

Formally, one sees that on t=c say, if we let \bar{g} denote the induced Riemannian metric, and if we impose initial data

$$T\tilde{\psi}|_{t=c} = \psi,$$

$$\tilde{\psi}|_{t=c} = \triangle_{(1-2M/r)^{-1}\bar{q}}^{-1} T\psi,$$

and let $\tilde{\psi}$ solve the wave equation with this data, then

$$T\tilde{\psi} = \psi$$

as desired.

So to use the above, it suffices to ask whether the initial data for $\tilde{\psi}$ above can be constructed and have sufficient regularity so as for the right hand side of (15) to be defined. To impose the first condition, since T=0 along $\mathcal{H}^+ \cap \mathcal{H}^-$, one must have that ψ vanish there to some order. For the second condition, note first that the metric $(1-2M/r)^{-1}\bar{g}$ has an asymptotically hyperbolic end and an asymptotically flat end. Thus, to construct $\Delta_{(1-2M/r)^{-1}\bar{g}}^{-1}T\psi$ suitably well-behaved³⁸, one must have that $T\psi$ decays appropriately towards the ends. We leave to the reader the task of verifying that the assumptions of the Proposition are sufficient.

³⁸so that we may apply to this quantity the arguments of Section 3.2.4.

3.2.6 The discrete isometry

Proposition 3.2.1, together with estimates (15) and (11), yield the proof of Theorem 3.1 in the special case that the conditions of Proposition 3.2.1 happen to be satisfied. In the original paper of Wald [151], one took Σ_0 to coincide with t = 0 and restricted to data ψ , ψ' which were supported in a compact region not containing $\mathcal{H}^+ \cap \mathcal{H}^-$. Clearly, however, this is a deficiency, as general solutions will be supported in $\mathcal{H}^+ \cap \mathcal{H}^-$. (See also the last exercise below.)

It turns out, however, that one can overcome the restriction on the support by the following trick: Note that the previous proposition produces a $\tilde{\psi}$ such that $T\tilde{\psi} = \psi$ on all of \mathcal{D} . We only require however that $T\tilde{\psi} = \psi$ on \mathcal{R} . The idea is to define a new $\bar{\psi}$, $\bar{\psi}'$ on Σ , such that $\bar{\psi} = \psi$, $\bar{\psi}' = \psi'$ on Σ_0 and, denoting by $\bar{\psi}$ the solution to the Cauchy problem with the new data, $\bar{\psi}|_{\mathcal{H}^+ \cap \mathcal{H}^-} = 0$, $\Xi \bar{\psi}|_{\mathcal{H}^+ \cap \mathcal{H}^-} = 0$. By the previous proposition and the domain of dependence property of Proposition 3.1.1, we will have indeed constructed a $\tilde{\psi}$ with $T\tilde{\psi} = \psi$ in \mathcal{R} for which the right hand side of (15) is finite.

Remark that Schwarzschild admits a discrete symmetry generated by the map $X \to -X$ in Kruskal coordinates. Define $\bar{\psi}$, $\bar{\psi}'$ so that $\bar{\psi}(X,\cdot) = -\bar{\psi}(-X,\cdot)$, $\bar{\psi}'(X,\cdot) = -\bar{\psi}'(-X,\cdot)$.

Proposition 3.2.2. Under the above assumptions, it follows that

$$\bar{\psi}(X,\cdot) = -\bar{\psi}(-X,\cdot).$$

The proof of the above is left as an exercise in preservation of symmetry for solutions of the wave equation. It follows immediately that

$$\bar{\psi}|_{\mathcal{H}^+ \cap \mathcal{H}^-} = 0$$

and that

$$\partial_U \bar{\psi} = -\partial_V \bar{\psi},$$

and thus $(\partial_U + \partial_V)\bar{\psi} = 0$. In view of the above remarks and Proposition 3.2.1 with $\Xi = \partial_U + \partial_V$, we have shown the full statement of Theorem 3.1.

Exercise: Work out explicit regularity assumptions and quantitative dependence on initial data in Theorem 3.1, describing in particular decay assumptions necessary at i_0 .

Exercise: Prove the analogue of Theorem 3.1 on the Oppenheimer-Snyder spacetime discussed previously. Hint: One need not know the explicit form of the metric, the statement given about the Penrose diagram suffices. Convince yourself that the original restricted version of Theorem 3.1 due to Wald [151], where the support of ψ is restricted near $\mathcal{H}^+ \cap \mathcal{H}^-$, is not sufficient to yield this result.

3.2.7 Remarks

The clever proof described above successfully obtains pointwise boundedness for ψ up to the horizon \mathcal{H}^+ . Does this really close the book, however, on the boundedness question? From various points of view, it may be desirable to go further.

1. Even though one obtains the "correct" pointwise result, one does not obtain boundedness at the horizon for the energy measured by a local observer, that is to say, bounds for

$$\int_{\Sigma_{\tau}} J_{\mu}^{n_{\Sigma_{\tau}}}(\psi) n_{\Sigma_{\tau}}^{\mu}.$$

This indicates that it would be difficult to use this result even for the simplest non-linear problems.

- 2. One does not obtain boundedness for transverse derivatives to the horizon, i.e. in (t^*, r) coordinates, $\partial_r \psi$, $\partial_r^2 \psi$, etc. (Exercise: Why not?)
- 3. The dependence on initial data is somewhat unnatural. (Exercise: Work out explicitly what it is.)

As far as the method of proof is concerned, there are additional short-comings when the proof is viewed from the standpoint of possible future generalisations:

- 4. To obtain control at the horizon, one must commute (see (15)) with all angular momentum operators Ω_i . Thus the spherical symmetry of Schwarzschild is used in a fundamental way.
- 5. The exact staticity is fundamental for the construction of $\tilde{\psi}$. It is not clear how to generalise this argument in the case say where T is not hypersurface orthogonal and Killing but one assumes merely that its deformation tensor ${}^{T}\pi_{\mu\nu}$ decays. This would be the situation in a bootstrap setting of a nonlinear stability problem.
- 6. The construction of $\bar{\psi}$ requires the discrete isometry of Schwarzschild, which again, cannot be expected to be stable.

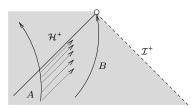
3.3 The red-shift and a new proof of boundedness

We give in this section a new proof of boundedness which overcomes the short-comings outlined above. In essence, the previous proof limited itself by relying solely on Killing fields as multipliers and commutators. It turns out that there is an important physical aspect of Schwarzschild which can be captured by other vector-field multipliers and commutators which are not however Killing. This is related to the celebrated red-shift effect.

3.3.1 The classical red-shift

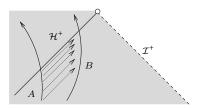
The red-shift effect is one of the most celebrated aspects of black holes. It is classically described as follows: Suppose two observers, A and B are such that A crosses the event horizon and B does not. If A emits a signal at constant

frequency as he measures it, then the frequency at which it is received by B is "shifted to the red".



The consequences of this for the appearance of a collapsing star to far-away observers were first explored in the seminal paper of Oppenheimer-Snyder [125] referred to at length in Section 2. For a nice discussion, see also the classic textbook [117].

The red-shift effect as described above is a global one, and essentially depends only on the fact that the proper time of B is infinite whereas the proper time of A before crossing \mathcal{H}^+ is finite. In the case of the Schwarzschild black hole, there is a "local" version of this red-shift: If B also crosses the event horizon but at advanced time later than A:



then the frequency at which B receives at his horizon crossing time is shifted to the red by a factor depending exponentially on the advanced time difference of the crossing points of A and B.

The exponential factor is determined by the so-called *surface gravity*, a quantity that can in fact be defined for all so-called Killing horizons. This localised red-shift effect depends only on the positivity of this quantity. We shall understand this more general situation in Section 7. Let us for now simply explore how we can "capture" the red-shift effect in the Schwarzschild geometry.

3.3.2 The vector fields N, Y, and \hat{Y}

It turns out that a "vector field multiplier" version of this localised red-shift effect is captured by the following

Proposition 3.3.1. There exists a φ_t -invariant smooth future-directed timelike vector field N on \mathcal{R} and a positive constant b > 0 such that

$$K^N(\psi) \ge bJ_\mu^N(\psi)N^\mu$$

on \mathcal{H}^+ for all solutions ψ of $\Box_q \psi = 0$.

(See Appendix D for the J^N , K^N notation.)

Proof. Note first that since T is tangent to \mathcal{H}^+ , it follows that given any $\sigma < \infty$, there clearly exists a vector field Y on \mathcal{R} such that

- 1. Y is φ_t invariant and spherically symmetric.
- 2. Y is future-directed null on \mathcal{H}^+ and transverse to \mathcal{H}^+ , say g(T,Y) = -2.
- 3. On \mathcal{H}^+ ,

$$\nabla_Y Y = -\sigma \left(Y + T \right). \tag{16}$$

Since T is tangent to \mathcal{H}^+ , along which Y is null, we have

$$g(\nabla_T Y, Y) = 0. (17)$$

From properties 1 and 2, and the form of the Schwarzschild metric, one computes (Exercise)

$$g(\nabla_T Y, T) \doteq 2\kappa > 0 \tag{18}$$

on \mathcal{H}^+ . Defining a local frame E_1 , E_2 for the SO(3) orbits, we note

$$g(\nabla_{E_i}Y,Y)=\frac{1}{2}E_ig(Y,Y)=0,$$

$$g(\nabla_{E_1}Y, E_2) = -g(Y, \nabla_{E_1}E_2) = -g(Y, \nabla_{E_2}E_1) = g(\nabla_{E_2}Y, E_1).$$

Writing thus

$$\nabla_T Y = -\kappa Y + a^1 E_1 + a^2 E_2 \tag{19}$$

$$\nabla_Y Y = -\sigma T - \sigma Y \tag{20}$$

$$\nabla_{E_1} Y = h_1^1 E_1 + h_1^2 E_2 - \frac{1}{2} a^1 Y$$
 (21)

$$\nabla_{E_2} Y = h_2^1 E_1 + h_2^2 E_2 - \frac{1}{2} a^2 Y$$
 (22)

with $(h_1^2 = h_2^1)$, we now compute

$$K^{Y} = -\frac{1}{2} \left(\mathbf{T}(Y,Y)(-\kappa) + \frac{1}{2} \mathbf{T}(T,Y)(-\sigma) + \frac{1}{2} \mathbf{T}(T,T)(-\sigma) \right)$$
$$-\frac{1}{2} (\mathbf{T}(E_{1},Y)a^{1} + \mathbf{T}(E_{2},Y)a^{2})$$
$$+ \mathbf{T}(E_{1},E_{1})h_{1}^{1} + \mathbf{T}(E_{2},E_{2})h_{2}^{1} + \mathbf{T}(E_{1},E_{2})(h_{1}^{2} + h_{2}^{1})$$

where we denote the energy momentum tensor by \mathbf{T} , to prevent confusion with T. (Note that, in view of the fact that \mathcal{Q} imbeds as a totally geodesic submanifold of \mathcal{M} , we have in fact $a^1 = a^2 = 0$. This is of no importance in our

computations, however.) It follows immediately in view again of the algebraic properties of \mathbf{T} , that

$$K^{Y} \geq \frac{1}{2}\kappa \mathbf{T}(Y,Y) + \frac{1}{4}\sigma \mathbf{T}(T,Y+T) - c\mathbf{T}(T,Y+T) - c\sqrt{\mathbf{T}(T,Y+T)\mathbf{T}(Y,Y)}$$

where c is independent of the choice of σ . It follows that choosing σ large enough, we have

$$K^Y \ge b \, J_{\mu}^{T+Y} \big(T+Y\big)^{\mu}.$$

So just set N = T + Y, noting that $K^N = K^T + K^Y = K^Y$.

The computation (18) represents a well known property of stationary black holes and the constant κ is the so-called *surface gravity*. (See [148].) Note that since Y is φ_t -invariant and T is Killing, we have

$$g(\nabla_T Y, T) = g(\nabla_Y T, T) = -g(\nabla_T T, Y)$$

on \mathcal{H}^+ . On the other hand

$$g(\nabla_T T, E_i) = -g(\nabla_{E_i} T, T) = 0,$$

since T is null on \mathcal{H}^+ . Thus, κ is alternatively characterized by

$$\nabla_T T = \kappa T$$

on \mathcal{H}^+ . We will elaborate on this in Section 7, where a generalisation of Proposition 3.3.1 will be presented.

Exercise: Relate the strength of the red-shift with the constant κ , for the case where observers A and B both cross the horizon, but B at advanced time v later than A.

If one desires an explicit form of the vector field, then one can argue as follows: Define first the vector field \hat{Y} by

$$\hat{Y} = \frac{1}{1 - 2M/r} \partial_u. \tag{23}$$

(See Appendix F.) Note that this vector field satisfies $g(\nabla_{\hat{Y}}\hat{Y},T)=0$. Define

$$Y = (1 + \delta_1(r - 2M))\hat{Y} + \delta_2(r - 2M)T.$$

It suffices to choose δ_1 , δ_2 appropriately.

The behaviour of N away from the horizon is of course irrelevant in the above proposition. It will be useful for us to have the following:

Corollary 3.1. Let Σ be as before. There exists a φ_t -invariant smooth future-directed timelike vector field N on \mathcal{R} , constants b > 0, B > 0, and two values $2M < r_0 < r_1 < \infty$ such that

- 1. $K^N \ge b J_{\mu}^N n_{\Sigma}^{\mu} \text{ for } r \le r_0,$
- 2. N = T for $r \ge r_1$,
- 3. $|K^N| \le BJ_\mu^T n_\Sigma^\mu$, and $J_\mu^N n_\Sigma^\mu \sim J^T n_\Sigma^\mu$ for $r_0 \le r \le r_1$.

3.3.3 N as a multiplier

Recall the definition of $\mathcal{R}(0,\tau)$. Applying the energy identity with the current J^N in this region, we obtain

$$\int_{\Sigma_{\tau}} J_{\mu}^{N} n_{\Sigma}^{\mu} + \int_{\mathcal{H}^{+}(0,\tau)} J_{\mu}^{N} n_{\mathcal{H}}^{\mu} + \int_{\{r \leq r_{0}\} \cap \mathcal{R}(0,\tau)} K^{N}
= \int_{\{r_{0} \leq r \leq r_{1}\} \cap \mathcal{R}(0,\tau)} (-K^{N}) + \int_{\Sigma_{0}} J_{\mu}^{N} n_{\Sigma}^{\mu}.$$
(24)

The reason for writing the above identity in this form will become apparent in what follows. Note that since N is timelike at \mathcal{H}^+ , we see all the "usual terms" in the flux integrals, i.e.

$$J^N_{\mu} n^{\mu}_{\mathcal{H}} \sim (\partial_{t^*} \psi)^2 + |\nabla \psi|^2,$$

and

$$J^N_{\mu} n^{\mu}_{\Sigma_{\tau}} \sim (\partial_{t^*} \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2.$$

The constants in the \sim depend as usual on the choice of the original Σ_0 and the precise choice of N.

Now the identity (24) also holds where Σ_0 is replaced by $\Sigma_{\tau'}$, $\mathcal{H}^+(0,\tau)$ is replaced by $\mathcal{H}^+(\tau',\tau)$, and $\mathcal{R}(0,\tau)$ is replaced by $\mathcal{R}(\tau',\tau)$, for an arbitrary $0 \le \tau' \le \tau$.

We may add to both sides of (24) an arbitrary multiple of the spacetime integral $\int_{\{r \geq r_0\} \cap \mathcal{R}(\tau',\tau)} J^T_{\mu} n^{\mu}_{\Sigma}$. In view of the fact that

$$\int_{\{r > r'\} \cap \mathcal{R}(\tau', \tau)} J^N_\mu n^\mu_\Sigma \sim \int_{\tau'}^\tau \left(\int_{\{r > r'\} \cap \Sigma_{\bar{\tau}}} J^N_\mu n^\mu_\Sigma \right) d\bar{\tau}$$

for any $r' \ge 2M$ (where \sim depends on Σ_0 , N), from the inequalities shown and property 3 of Corollary 3.1 we obtain

$$\int_{\Sigma_{\bar{\tau}}} J^N_\mu n^\mu_\Sigma + b \int_{\tau'}^{\bar{\tau}} \left(\int_{\Sigma_{\bar{\tau}}} J^N_\mu n^\mu_\Sigma \right) d\bar{\tau} \leq B \int_{\tau'}^{\bar{\tau}} \left(\int_{\Sigma_{\bar{\tau}}} J^T_\mu n^\mu_\Sigma \right) d\bar{\tau} + \int_{\Sigma_{\tau'}} J^N_\mu n^\mu_\Sigma.$$

On the other hand, in view of our previous (8), (9), we have

$$\int_{\tau'}^{\tau} \left(\int_{\Sigma_{\bar{\tau}}} J_{\mu}^{T} n_{\Sigma}^{\mu} \right) d\bar{\tau} \le (\tau - \tau') \int_{\Sigma_{0}} J_{\mu}^{T} n_{\Sigma}^{\mu}. \tag{25}$$

Setting

$$f(\tau) = \int_{\Sigma_{\tau}} J_{\mu}^{N} n_{\Sigma}^{\mu}$$

we have that

$$f(\tau) + b \int_{\tau'}^{\tau} f(\bar{\tau}) d\bar{\tau} \le BD(\tau - \tau') + f(\tau')$$
(26)

for all $\tau \geq \tau' \geq 0$, from which it follows (**Exercise**) that $f \leq B(D + f(0))$. (We use the inequality with $D = \int_{\Sigma_0} J_{\mu}^T n_{\Sigma_0}^{\mu}$.) In view of the trivial inequality

$$\int_{\Sigma_0} J_{\mu}^T n_{\Sigma_0}^{\mu} \le B \int_{\Sigma_0} J_{\mu}^N n_{\Sigma_0}^{\mu},$$

we obtain

$$\int_{\Sigma_{\tau}} J_{\mu}^{N} n_{\Sigma_{\tau}}^{\mu} \le B \int_{\Sigma_{0}} J_{\mu}^{N} n_{\Sigma_{0}}^{\mu}. \tag{27}$$

We have obtained a "local observer's" energy estimate. This addresses point 1 of Section 3.2.7.

3.3.4 \hat{Y} as a commutator

It turns out (**Exercise**) that from (27), one could obtain pointwise bounds as before on ψ by commuting with angular momentum operators Ω_i . No construction of $\tilde{\psi}$, $\bar{\psi}$, etc., would be necessary, and this would thus address points 3, 5, 6 of Section 3.2.7.

Commuting with Ω_i clearly would not address however point 4. Moreover, it would not address point 2. **Exercise**: Why not?

It turns out that one can resolve this problem by applying N not only as a multiplier, but also as a commutator. The calculations are slightly easier if we more simply commute with \hat{Y} defined in (23).

Proposition 3.3.2. Let ψ satisfy $\Box_g \psi = 0$. Then we may write

$$\Box_g(\hat{Y}\psi) = \frac{2}{r}\hat{Y}(\hat{Y}(\psi)) - \frac{4}{r}(\hat{Y}(T\psi)) + P_1\psi$$
 (28)

where P_1 is the first order operator $P_1\psi \doteq \frac{2}{r^2}(T\psi - \hat{Y}\psi)$.

This is proven easily with the help of Appendix E. As we shall see, the sign of the first term on the right hand side of (28) is important. We will interpret this computation geometrically in terms of the sign of the surface gravity in Theorem 7.2 of Section 7.

Let us first note that our boundedness result gives us in particular

$$\int_{\{r \le r_0\} \cap \mathcal{R}(0,\tau)} K^N(\psi) \le BD\tau \tag{29}$$

where D comes from initial data. (**Exercise**: Why?) Commute now the wave equation with T and apply the multiplier N. See Appendix E. One obtains in particular an estimate for

$$\int_{\{r \le r_0\} \cap \mathcal{R}(0,\tau)} (\hat{Y}T\psi)^2 \le B \int_{\{r \le r_0\} \cap \mathcal{R}(0,\tau)} K^N(T\psi) \le BD\tau, \tag{30}$$

where again D refers to a quantity coming from initial data. Commuting now the wave equation with \hat{Y} and applying the multiplier N, one obtains an energy

identity of the form

$$\begin{split} & \int_{\Sigma_{\tau}} J^{N}_{\mu}(\hat{Y}\psi) n^{\mu}_{\Sigma} + \int_{\mathcal{H}^{+}(0,\tau)} J^{N}_{\mu}(\hat{Y}\psi) n^{\mu}_{\mathcal{H}} + \int_{\{r \leq r_{0}\} \cap \mathcal{R}(0,\tau)} K^{N}(\hat{Y}\psi) \\ & = \int_{\{r_{0} \leq r \leq r_{1}\} \cap \mathcal{R}(0,\tau)} (-K^{N}(\hat{Y}(\psi))) \\ & + \int_{\{r \leq r_{0}\} \cap \mathcal{R}(0,\tau)} \mathcal{E}^{N}(\hat{Y}\psi) + \int_{\{r \geq r_{0}\} \cap \mathcal{R}(0,\tau)} \mathcal{E}^{N}(\hat{Y}\psi) \\ & + \int_{\Sigma_{0}} J^{N}_{\mu}(\hat{Y}\psi) n^{\mu}_{\Sigma}, \end{split}$$

where $J^N(\hat{Y}\psi)$, $K^N(\hat{Y}\psi)$ are defined by (121), (122), respectively, with $\hat{Y}\psi$ replacing ψ , and

$$\mathcal{E}^{N}(\hat{Y}\psi) = -(N\hat{Y}\psi) \left(\frac{2}{r}\hat{Y}(\hat{Y}(\psi)) - \frac{4}{r}(\hat{Y}(T\psi)) + P_{1}\psi\right)$$

$$= -\frac{2}{r}(\hat{Y}(\hat{Y}(\psi)))^{2}$$

$$-\frac{2}{r}((N-\hat{Y})\hat{Y}\psi)(\hat{Y}\hat{Y}\psi) + \frac{4}{r}(N\hat{Y}\psi)(\hat{Y}(T\psi))$$

$$-(N\hat{Y}\psi)P_{1}\psi.$$

The first term on the right hand side has a good sign! Applying Cauchy-Schwarz and the fact that $N-\hat{Y}=T$ on \mathcal{H}^+ , it follows that choosing r_0 accordingly, one obtains that the second two terms can be bounded in $r \leq r_0$ by

$$\epsilon K^N(\hat{Y}\psi) + \epsilon^{-1}(\hat{Y}T\psi)^2$$

whereas the last term can be bounded by

$$\epsilon K^N(\hat{Y}\psi) + \epsilon^{-1}K^N(\psi).$$

In view of (29) and (30), one obtains

$$\int_{\{r \le r_0\} \cap \mathcal{R}(0,\tau)} \mathcal{E}^N(\hat{Y}\psi) \le \epsilon \int_{\{r \le r_0\} \cap \mathcal{R}(0,\tau)} K^N(\hat{Y}\psi) + B\epsilon^{-1}D\tau.$$

Exercise: Show how from this one can arrive again at an inequality (26).

Commuting repeatedly with T, \hat{Y} , the above scheme plus elliptic estimates yield natural H^m estimates for all m. Pointwise estimates for all derivatives then follow by a standard Sobolev estimate.

3.3.5 The statement of the boundedness theorem

We obtain finally

Theorem 3.2. Let Σ be a Cauchy hypersurface for Schwarzschild such that $\Sigma \cap \mathcal{H}^- = \emptyset$, let $\Sigma_0 = \mathcal{D} \cap \Sigma$, let Σ_τ denote the translation of Σ_0 , let n_{Σ_τ} denote the future normal of Σ_τ , and let $\mathcal{R} = \cup_{\tau \geq 0} \Sigma_\tau$. Assume $-g(n_{\Sigma_0}, T)$ is uniformly

bounded. Then there exists a constant C depending only on Σ_0 such that the following holds. Let ψ , ψ , ψ' be as in Proposition 3.1.1, with $\psi \in H^{k+1}_{loc}(\Sigma)$, $\psi' \in H^k_{loc}(\Sigma)$, and

$$\int_{\Sigma_0} J_{\mu}^T (T^m \psi) n_{\Sigma_0}^{\mu} < \infty$$

for $0 \le m \le k$. Then

$$|\nabla^{\Sigma_{\tau}}\psi|_{H^{k}(\Sigma_{\tau})} + |n\psi|_{H^{k}(\Sigma_{\tau})} \leq C\left(|\nabla^{\Sigma_{0}}\psi|_{H^{k}(\Sigma_{0})} + |\psi'|_{H^{k}(\Sigma_{0})}\right).$$

If $k \ge 1$, then we have

$$\sum_{0 \leq m \leq k-1} \sum_{m_1 + m_2 = m, m_i \geq 0} \left| (\nabla^{\Sigma})^{m_1} n^{m_2} \psi \right| \leq C \left(\lim_{x \to i^0} |\psi| + |\nabla^{\Sigma(0)} \psi|_{H^k(\Sigma_0)} + |\psi'|_{H^k(\Sigma_0)} \right)$$

in \mathcal{R} .

Note that $(\nabla^{\Sigma})^{m_1} n^{m_2} \psi$ denotes an m_1 -tensor on the Riemannian manifold Σ_{τ} , and $|\cdot|$ on the left hand side of the last inequality above just denotes the induced norm on such tensors.

3.4 Comments and further reading

The first discussion of the wave equation on Schwarzschild is perhaps the work of Regge and Wheeler [131], but the true mathematical study of this problem was initiated by Wald [151], who proved Theorem 3.1 under the assumption that ψ vanished in a neighborhood of $\mathcal{H}^+ \cap \mathcal{H}^-$. The full statement of Theorem 3.1 and the proof presented in Section 3.2 is due to Kay and Wald [98]. The present notes owe a lot to the geometric view point emphasized in the works [151, 98].

Use of the vector field Y as a multiplier was first introduced in our [65], and its use is central in [66] and [67]. In particular, the property formalised by Proposition 3.3.1 was discovered there. It appears that this may be key to a stable understanding of black hole event horizons. See Section 3.5 below, as well as Section 7, for a generalisation of Proposition 3.3.1.

It is interesting to note that in [66, 67], Y had always been used in conjunction with vector fields X of the type to be discussed in the next section (which require a more delicate global construction) as well as T. This meant that one always had to obtain more than boundedness (i.e. decay!) in order to obtain the proper boundedness result at the horizon. Consequently, one had to use many aspects of the structure of Schwarzschild, particularly, the trapping to be discussed in later lectures. The argument given above, where boundedness is obtained using only N and T as multipliers is presented for the first time in a self-contained fashion in these lectures. The argument can be read off, however, from the more general argument of [68] concerning perturbations of Schwarzschild including Kerr. The use of \hat{Y} as a commutator to estimate higher order quantities also originates in [68]. The geometry behind this computation is further discussed in Section 7.

Note that the use of Y together with T is of course equivalent to the use of N and T. We have chosen to give a name to the vector field N = T + Y merely for convenience. Timelike vector fields are more convenient when perturbing...

Another remark on the use of \hat{Y} as a commutator: Enlarging the choice of commutators has proven very important in previous work on the global analysis of the wave equation. In a seminal paper, Klainerman [100] showed improved decay for the wave equation on Minkowski space in the interior region by commutation with scaling and Lorentz boosts. This was a key step for further developments for long time existence for quasilinear wave equations [101].

The distinct role of multipliers and commutators and the geometric considerations which enter into their construction is beautifully elaborated by Christodoulou [52].

3.5 Perturbing?

Can the proof of Theorem 3.2 be adapted to hold for spacetimes "near" Schwarzschild? To answer this, one must first decide what one means by the notion of "near". Perhaps the simplest class of perturbed metrics would be those that retain the same differentiable structure of \mathcal{R} , retain \mathcal{H}^+ as a null hypersurface, and retain the Killing field T. One infers (without computation!) that the statement of Proposition 3.3.1 and thus Corollary 3.1 is stable to such perturbations of the metric. Therein lies the power of that Proposition and of the multiplier N. (In fact, see Section 7.) Unfortunately, one easily sees that our argument proving Theorem 3.2 is still unstable, even in the class of perturbations just described. The reason is the following: Our argument relies essentially on an a priori estimate for $\int_{\Sigma_{\tau}} J_{\mu}^{T} n^{\mu}$ (see (25)), which requires T to be non-spacelike in \mathcal{R} . When one perturbs, T will in general become spacelike in a region of \mathcal{R} . (As we shall see in Section 5.1, this happens in particular in the case of Kerr. The region where T is spacelike is known as the ergoregion.)

There is a sense in which the above is the only obstruction to perturbing the above argument, i.e. one can solve the following

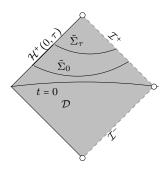
Exercise: Fix the differentiable structure of \mathcal{R} and the vector field T. Let g be a metric sufficiently close to Schwarzschild such that \mathcal{H}^+ is null, and suppose T is Killing and non-spacelike in \mathcal{R} , and T is null on \mathcal{H}^+ . Then Theorem 3.2 applies. (In fact, one need not assume that T is non-spacelike in \mathcal{R} , only that T is null on the horizon.) See also Section 7.

Exercise: Now do the above where T is not assumed to be Killing, but ${}^{T}\pi_{\mu\nu}$ is assumed to decay suitably. What precise assumptions must one impose?

This discussion may suggest that there is in fact no stable boundedness argument, that is to say, a "stable argument" would of necessity need to prove more than boundedness, i.e. decay. We shall see later that there is a sense in which this is true and a sense in which it is not! But before exploring this, let us understand how one can go beyond boundedness and prove quantitative decay for waves on Schwarzschild itself. It is quantitative decay after all that we must understand if we are to understand nonlinear problems.

4 The wave equation on Schwarzschild II: quantitative decay rates

Quantitative decay rates are central for our understanding of non-linear problems. To discuss energy decay for solutions ψ of $\Box_g \psi = 0$ on Schwarzschild, one must consider a different foliation. Let $\tilde{\Sigma}_0$ be a spacelike hypersurface terminating on null infinity and define $\tilde{\Sigma}_{\tau}$ by translation.



The main result of this section is the following

Theorem 4.1. There exists a constant C depending only on $\tilde{\Sigma}_0$ such that the following holds. Let $\psi \in H^4_{loc}$, $\psi' \in H^3_{loc}$, and suppose $\lim_{x \to i^0} \psi = 0$ and

$$E_1 = \sum_{|(\alpha)| \le 3} \sum_{\Gamma = \{\Omega_i\}} \int_{t=0} r^2 J_{\mu}^{n_0} (\Gamma^{(\alpha)} \psi) n_0^{\mu} < \infty$$

where n_0 denotes the unit normal of the hypersurface $\{t = 0\}$. Then

$$\int_{\tilde{\Sigma}_{\tau}} J_{\mu}^{N}(\psi) n_{\tilde{\Sigma}_{\tau}}^{\mu} \le C E_{1} \tau^{-2}, \tag{31}$$

where N is the vector field of Section 3.3.2. Now let $\psi \in H^7_{\rm loc}$, $\psi' \in H^6_{\rm loc}$, $\lim_{x \to i^0} \psi = 0$, and suppose

$$E_2 = \sum_{|(\alpha)| \le 6} \sum_{\Gamma = \{\Omega_i\}} \int_{t=0} r^2 J_{\mu}^{n_0} (\Gamma^{(\alpha)} \psi) n_0^{\mu} < \infty.$$

Then

$$\sup_{\tilde{\Sigma}_{\tau}} \sqrt{r} |\psi| \le C \sqrt{E_2} \tau^{-1}, \qquad \sup_{\tilde{\Sigma}_{\tau}} r |\psi| \le C \sqrt{E_2} \tau^{-1/2}. \tag{32}$$

The fact that (31) "loses derivatives" is a fundamental aspect of this problem related to the trapping phenomenon, to be discussed in what follows, although the precise number of derivatives lost above is wasteful. Indeed, there are several aspects in which the above results can be improved. See Proposition 4.2.1 and the exercise of Section 4.3.

We can also express the pointwise decay in terms of advanced and retarded null coordinates u and v. Defining³⁹ $v = 2(t + r^*) = 2(t + r + 2M \log(r - 2M))$, $u = 2(t - r^*) = 2(t - r - 2M \log(r - 2M))$, it follows in particular from (32) that

$$|\psi| \le CE_2(|v|+1)^{-1}, \qquad |r\psi| \le C(r_0)E_2u^{-\frac{1}{2}},$$
 (33)

where the first inequality applies in $\mathcal{D} \cap \operatorname{clos}(\{t \geq 0\})$, whereas the second applies only in $\mathcal{D} \cap \{t \geq 0\} \cap \{r \geq r_0\}$, with $C(r_0) \to \infty$ as $r_0 \to 2M$. See also Appendix F. Note that, as in Minkowski space, the first inequality of (33) is sharp as a uniform decay rate in v.

4.1 A spacetime integral estimate

The zero'th step in the proof of Theorem 4.1 is an estimate for a spacetime integral whose integrand should control the quantity

$$\chi J^N_{\mu}(\psi) n^{\mu}_{\tilde{\Sigma}_{\tau}} \tag{34}$$

where χ is a φ_t -invariant weight function such that χ degenerates only at infinity. Estimates of the spacetime integral (34) have their origin in the classical virial theorem, which in Minkowski space essentially arises from applying the energy identity to the current J^V with $V = \frac{\partial}{\partial r}$.

Naively, one might expect to be able to obtain an estimate of the form say

$$\int_{\tilde{\mathcal{R}}(0,\tau)} \chi J_{\mu}^{N}(\psi) n_{\tilde{\Sigma}_{\tau}}^{\mu} \le B \int_{\tilde{\Sigma}_{0}} J_{\mu}^{N} n_{\tilde{\Sigma}_{0}}^{\mu}, \tag{35}$$

for such a χ . It turns out that there is a well known high-frequency obstruction for the existence of an estimate of the form (35) arising from trapped null geodesics. This problem has been long studied in the context of the wave equation in Minkowski space outside of an obstacle, where the analogue of trapped null geodesics are straight lines which reflect off the obstacle's boundary in such a way so as to remain in a compact subset of space. In Schwarzschild, one can easily infer from a continuity argument the existence of a family of null geodesics with i^+ as a limit point.⁴⁰ But in view of the integrability of geodesic flow, one can in fact understand all such geodesics explicitly.

Exercise: Show that the hypersurface r = 3M is spanned by null geodesics. Show that from every point in \mathcal{R} , there is a codimension-one subset of future directed null directions whose corresponding geodesics approach r = 3M, and all other null geodesics either cross \mathcal{H}^+ or meet \mathcal{I}^+ .

The timelike hypersurface r = 3M is traditionally called the *photon sphere*. Let us first see how one can capture this high frequency obstruction.

³⁹The strange convention on the factor of 2 is chosen simply to agree with [65].

⁴⁰This can be thought of as a very weak notion of what it would mean for a null geodesic to be trapped from the point of view of decay results with respect to the foliation $\tilde{\Sigma}_{\tau}$.

4.1.1 A multiplier X for high angular frequencies

We look for a multiplier with the property that the spacetime integral it generates is positive definite. Since in Minkowski space, this is provided by the vector field ∂_r , we will look for simple generalizations. Calculations are slightly easier when one considers ∂_{r^*} associated to Regge-Wheeler coordinates (r^*,t) . See Appendix F.2 for the definition of this coordinate system.⁴¹ For $X = f(r^*)\partial_{r^*}$, where f is a general function, we obtain the formula

$$K^{X} = \frac{f'}{1 - 2M/r} (\partial_{r^{*}} \psi)^{2} + \frac{f}{r} \left(1 - \frac{3M}{r} \right) |\nabla \psi|^{2} - \frac{1}{4} \left(2f' + 4 \frac{r - 2M}{r^{2}} f \right) \nabla^{\alpha} \psi \nabla_{\alpha} \psi.$$

Here f' denotes $\frac{df}{dr^*}$. We can now define a "modified" current

$$J_{\mu}^{X,w} = J_{\mu}^{X}(\psi) + \frac{1}{8}w\partial_{\mu}(\psi^{2}) - \frac{1}{8}(\partial_{\mu}w)\psi^{2}$$

associated to the vector field X and the function w. Let

$$K^{X,w} = \nabla^{\mu} J_{\mu}^{X,w}.$$

Choosing

$$w = f' + 2\frac{r - 2M}{r^2}f + \frac{\delta(r - 2M)}{r^5}\left(1 - \frac{3M}{r}\right)f,$$

we have

$$K^{X,w} = \left(\frac{f'}{1 - 2M/r} - \frac{\delta f}{2r^4} \left(1 - \frac{3M}{r}\right)\right) (\partial_{r^*} \psi)^2 + \frac{f}{r} \left(1 - \frac{3M}{r}\right) \left(\left(1 - \frac{\delta(r - 2M)}{2r^4}\right) |\nabla \psi|^2 + \frac{\delta}{2r^3} (\partial_t \psi)^2\right) - \left(\frac{1}{8} \Box_g \left(2f' + 4\frac{r - 2M}{r^2} f + 2\frac{\delta(r - 2M)}{r^5} \left(1 - \frac{3M}{r}\right) f\right)\right) \psi^2.$$

Recall that in view of the spherical symmetry of \mathcal{M} , we may decompose

$$\psi = \sum_{\ell \ge 0, |m| \le \ell} \psi_{\ell}(r, t) Y_{m, \ell}(\theta, \phi)$$

where $Y_{m,\ell}$ are the so-called *spherical harmonics*, each summand satisfies again the wave equation, and the convergence is in L^2 of the SO(3) orbits.

Let us assume that ψ_{ℓ} = 0 for spherical harmonic number $\ell \leq L$ for some L to be determined. We look for $K^{X,w}$ such that $\int_{\mathbb{S}^2} K^{X,w} \geq 0$, but also $|J_{\mu}^{X,w} n^{\mu}| \leq 1$

 $^{^{41}}$ Remember, when considering coordinate vector fields, one has to specify the entire coordinate system. When considering ∂_r , it is here to be understood that we are using Schwarzschild coordinates, and when considering ∂_{r^*} , it is to be understood that we are using Regge-Wheeler coordinates. The precise choice of the angular coordinates is of course irrelevant.

 $BJ^N_\mu n^\mu$. Here $\int_{\mathbb{S}^2}$ denotes integration over group orbits of the SO(3) action. For such ψ , in view of the resulting inequality

$$\frac{L(L+1)}{r^2} \int_{\mathbb{S}^2} \psi^2 \le \int_{\mathbb{S}^2} |\nabla \psi|^2,$$

it follows that taking L sufficiently large and $0 < \delta < 1$ sufficiently small so that $1 - \frac{\delta(1-\mu)}{2r^3} \ge \frac{1}{2}$, it clearly suffices to construct an f with the following properties:

- 1. $|f| \leq B$,
- 2. $f' \ge B(1 2M/r)r^{-4}$,
- 3. f(r = 3M) = 0,

4.
$$-\frac{1}{8}\Box_g \left(2f' + 4\frac{r-2M}{r^2}f + 2\frac{\delta(r-2M)}{r^5}\left(1 - \frac{3M}{r}\right)f\right)(r = 3M) > 0$$
,

5.
$$\frac{1}{8} \Box_g \left(2f' + 4 \frac{r - 2M}{r^2} f + 2 \frac{\delta(r - 2M)}{r^5} \left(1 - \frac{3M}{r} \right) f \right) \le Br^{-3}$$

for some constant B. **Exercise**. Show that one can construct such a function. Note the significance of the photon sphere!

4.1.2 A multiplier X for all frequencies

Constructing a multiplier for all spherical harmonics, so as to capture in addition "low frequency" effects, is more tricky. It turns out, however, that one can actually define a current which does not require spherical harmonic decomposition at all. The current is of the form:

$$J_{\mu}^{\mathbf{X}}(\psi) = eJ_{\mu}^{N}(\psi) + J_{\mu}^{X^{a}}(\psi) + \sum_{i} J_{\mu}^{X^{b}, w^{b}}(\Omega_{i}\psi)$$
$$-\frac{1}{2} \frac{r(f^{b})'}{f^{b}(r - 2M)} \left(\frac{r - 2M}{r^{2}} - \frac{(r^{*} - \alpha - \alpha^{1/2})}{\alpha^{2} + (r^{*} - \alpha - \alpha^{1/2})^{2}} \right) X_{\mu}^{b}\psi^{2}.$$

Here, N is as in Section 3.3.2, $X^a = f^a \partial_{r^*}$, $X^b = f^b \partial_{r^*}$, the warped current $J^{X,w}$ is defined as in Section 4.1.1,

$$f^{a} = -\frac{C_{a}}{\alpha r^{2}} + \frac{c_{a}}{r^{3}},$$

$$f^{b} = \frac{1}{\alpha} \left(\tan^{-1} \frac{r^{*} - \alpha - \alpha^{1/2}}{\alpha} - \tan^{-1} (-1 - \alpha^{-1/2}) \right),$$

$$w^{b} = \frac{1}{8} \left((f^{b})' + 2 \frac{r - 2M}{r^{2}} f^{b} \right),$$

and e, C_a, c_a, α are positive parameters which must be chosen accordingly. With these choices, one can show (after some computation) that the divergence $K^{\mathbf{X}} = \nabla^{\mu} J_{\mu}^{\mathbf{X}}$ controls in particular

$$\int_{\mathbb{S}^2} K^{\mathbf{X}}(\psi) \ge b\chi \int_{\mathbb{S}^2} J^N_{\mu}(\psi) n^{\mu},\tag{36}$$

where χ is non-vanishing but decays (polynomially) as $r \to \infty$. Note that in view of the normalisation (123) of the r^* coordinate, $X^b = 0$ precisely at r = 3M. The left hand side of the inequality (36) controls also second order derivatives which degenerate however at r = 3M. We have dropped these terms. It is actually useful for applications that the $J^{X^a}(\psi)$ part of the current is not "modified" by a function w^a , and thus ψ itself does not occur in the boundary terms. That is to say

$$|J_{\mu}^{\mathbf{X}}(\psi)n^{\mu}| \le B\left(J_{\mu}^{N}(\psi)n^{\mu} + \sum_{i=1}^{3} J_{\mu}^{N}(\Omega_{i}\psi)n^{\mu}\right).$$
 (37)

On the event horizon \mathcal{H}^+ , we have a better one-sided bound

$$-J_{\mu}^{\mathbf{X}}(\psi)n_{\mathcal{H}^{+}}^{\mu} \leq B\left(J_{\mu}^{T}(\psi)n_{\mathcal{H}^{+}}^{\mu} + \sum_{i=1}^{3} J_{\mu}^{T}(\Omega_{i}\psi)n_{\mathcal{H}^{+}}^{\mu}\right). \tag{38}$$

For details of the construction, see [67].

In view of (36), (37) and (38), together with the previous boundedness result Theorem 3.2, one obtains in particular the estimate

$$\int_{\tilde{\mathcal{R}}(\tau',\tau)} \chi J_{\nu}^{N}(\psi) n_{\tilde{\Sigma}}^{\nu} \leq B \int_{\tilde{\Sigma}(\tau')} \left(J_{\mu}^{N}(\psi) + \sum_{i=1}^{3} J_{\mu}^{N}(\Omega_{i}\psi) \right) n_{\tilde{\Sigma}_{\tau}}^{\mu}, \tag{39}$$

for some nonvanishing φ_t -invariant function χ which decays polynomially as $r \to \infty$.

On the other hand, considering the current $J_{\mu}^{\mathbf{X}}(P_{\leq L}\psi) + J_{\mu}^{X,w}((I - P_{\leq L})\psi)$, where $J_{\mu}^{X,w}$ is the current of Section 4.1.1 and $P_{\leq L}\psi$ denotes the projection to the space spanned by spherical harmonics with $\ell \leq L$, we obtain the estimate

$$\int_{\tilde{\mathcal{R}}(\tau',\tau)} \chi h J_{\nu}^{N}(\psi) n_{\tilde{\Sigma}}^{\nu} \leq B \int_{\tilde{\Sigma}(\tau')} J_{\mu}^{N}(\psi) n_{\tilde{\Sigma}_{\tau}}^{\mu}, \tag{40}$$

where h is any smooth nonnegative function $0 \le h \le 1$ vanishing at r = 3M, and B depends also on the choice of function h.

4.2 The Morawetz conformal Z multiplier and energy decay

How does the estimate (39) assist us to prove decay?

Recall that energy decay can be proven in Minkowski space with the help of the so-called Morawetz current. Let

$$Z = u^2 \partial_u + v^2 \partial_v \tag{41}$$

and define

$$J_{\mu}^{Z,w}(\psi) = J_{\mu}^{Z}(\psi) + \frac{tr^{*}(1 - 2M/r)}{2r}\psi\partial_{\mu}\psi - \frac{r^{*}(1 - 2M/r)}{4r}\psi^{2}\partial_{\mu}t.$$

(Here (u, v), (r^*, t) are the coordinate systems of Appendix F.) Setting M = 0, this corresponds precisely to the current introduced by Morawetz [119] on Minkowski space.

It is a good **exercise** to show that (for M > 0!) the coefficients of this current are C^0 but not C^1 across $\mathcal{H}^+ \cup \mathcal{H}^-$.

To understand how one hopes to use this current, let us recall the situation in Minkowski space. There, the signifiance of (41) arises since it is a conformal Killing field. Setting M = 0, $r^* = r$ in the above one obtains⁴²

$$\int_{t=\tau} J_{\mu}^{Z,w} n^{\mu} \ge 0, \tag{42}$$

$$K^{Z,w} = 0. (43)$$

The inequality (42) remains true in the Schwarzschild case and one can obtain exactly as before

$$\int_{t=\tau} J_{\mu}^{Z,w} n^{\mu} \ge b \int_{t=\tau} u^{2} (\partial_{u} \psi)^{2} + v^{2} (\partial_{v} \psi)^{2} + \left(1 - \frac{2M}{r}\right) (u^{2} + v^{2}) |\nabla \psi|^{2}. \tag{44}$$

(In fact, we have dropped positive 0'th order terms from the right hand side of (44), which will be useful for us later on in Section 4.3.) Note that away from the horizon, we have that

$$\int_{t=\tau} J_{\mu}^{Z,w} n^{\mu} \ge b(r_0, R) \tau^2 \int_{\{t=\tau\} \cap \{r_0 \le r \le R\}} J_{\mu}^N n^{\mu}. \tag{45}$$

Thus, if the left hand side of (45) could be shown to be bounded, this would prove the first statement of Theorem 4.1 where $\tilde{\Sigma}_{\tau}$ is replaced however with $\{t = \tau\} \cap \{r_0 \le r \le R\}$.

In the case of Minkowski space, the boundedness of the left hand side of (44) follows immediately by (43) and the energy identity

$$\int_{t=\tau} J_{\mu}^{Z,w} + \int_{0 < t < \tau} K^{Z,w} = \int_{t=0} J_{\mu}^{Z,w}$$
 (46)

as long as the data are suitably regular and decay so as for the right hand side to be bounded. For Schwarzschild, one cannot expect (43) to hold, and this is why we have introduced the X-related currents.

First the good news: There exist constants $r_0 < R$ such that

$$K^{Z,w} > 0$$

for $r \leq r_0$, and in fact

$$K^{Z,w} \ge b \frac{t}{r^3} \psi^2 \tag{47}$$

 $^{^{42}}$ The reason for introducing the 0'th order terms is because the wave equation is not conformally invariant. It is remarkable that one can nonetheless obtain positive definite boundary terms, although a slightly unsettling feature is that this positivity property (42) requires looking specifically at constant $t = \tau$ surfaces and integrating.

for $r \ge R$ and some constant b. These terms have the "right sign" in the energy identity (46). In $\{r_0 \le r \le R\}$, however, the best we can do is

$$-K^{Z,w} \le B t (|\nabla \psi|^2 + |\psi|^2).$$

This is the bad news, although, in view of the presence of trapping, it is to be expected. Using also (47), we may estimate

$$\int_{0 \le t \le \tau} -K^{Z,w} \le B \int_{\{0 \le t \le \tau\} \cap \{r_0 \le r \le R\}} t J_{\mu}^N n^{\mu}$$

$$\le B \tau \int_{\{0 \le t \le \tau\} \cap \{r_0 \le r \le R\}} J_{\mu}^N n^{\mu}. \tag{48}$$

In view of the fact that the first integral on the right hand side of (48) is bounded by (39), and the weight τ^2 in (45), applying the energy identity of the current $J^{Z,w}$ in the region $0 \le t \le \tau$, we obtain immediately a preliminary version of the first statement of the Theorem 4.1, but with τ^2 replaced by τ , and the hypersurfaces $\tilde{\Sigma}_{\tau}$ replaced by $\{t = \tau\} \cap \{r' \le r \le R'\}$ for some constants r', R', but where B depends on these constants. (Note the geometry of this region. All $\{t = \text{constant}\}$ hypersurfaces have common boundary $\mathcal{H}^+ \cap \mathcal{H}^-$. Exercise: Justify the integration by parts (46), in view of the fact that Z and W are only C^0 at $\mathcal{H}^+ \cup \mathcal{H}^-$.)

Using the current J^T and an easy geometric argument, it is not difficult to replace the hypersurfaces $\{t=\tau\}\cap\{r'\leq r\leq R'\}$ above with $\tilde{\Sigma}_{\tau}\cap\{r\geq r'\}$, ⁴³ obtaining

$$\int_{\tilde{\Sigma}_{\tau} \cap \{r \geq r'\}} J_{\mu}^{N}(\psi) n^{\mu} \leq B \tau^{-1} \left(\int_{t=0} J_{\mu}^{Z,w}(\psi) n^{\mu} + \int_{\tilde{\Sigma}_{0}} J_{\mu}^{N}(\psi) n^{\mu} + \sum_{i=1}^{3} J_{\mu}^{N}(\Omega_{i}\psi) n^{\mu} \right). \tag{49}$$

To obtain decay for the nondegenerate energy near the horizon, note that by the pigeonhole principle in view of the boundedness of the left hand side of (39) and what has just been proven, there exists (**exercise**) a dyadic sequence $\tilde{\Sigma}_{\tau_i}$ for which the first statement of Theorem 4.1 holds, with τ^{-2} replaced by τ_i^{-1} . Finally, by Theorem 3.2, we immediately (**exercise**: why?) remove the restriction to the dyadic sequence.

We have thus obtained

$$\int_{\tilde{\Sigma}_{\tau}} J_{\mu}^{N}(\psi) n^{\mu} \leq B \tau^{-1} \left(\int_{t=0} J_{\mu}^{Z,w}(\psi) n^{\mu} + \int_{\tilde{\Sigma}_{0}} J_{\mu}^{N}(\psi) n^{\mu} + \sum_{i=1}^{3} J_{\mu}^{N}(\Omega_{i}\psi) n^{\mu} \right). \tag{50}$$

The statement (50) loses one power of τ in comparison with the first statement of Theorem 3.2. How do we obtain the full result? First of all, note that, commuting once again with Ω_j , it follows that (50) holds for ψ replaced with $\Omega_i \psi$. Now we may partition $\tilde{\mathcal{R}}(0,\tau)$ dyadically into subregions $\tilde{\mathcal{R}}(\tau_i,\tau_{i+1})$ and

⁴³Hint: Use (44) to estimate the energy on $\{t=t_0\} \cap J^+(\tilde{\Sigma}_{\tau})$ with weights in τ . Send $t_0 \to \infty$ and estimate backwards to $\tilde{\Sigma}_{\tau}$ using conservation of the J^T flux.

revisit the X-estimate (39) on each such region. In view of (50) applied to both ψ and $\Omega_i \psi$, the estimate (39) gives

$$\int_{\tilde{\mathcal{R}}(\tau_i, \tau_{i+1})} \chi J_{\nu}^{N}(\psi) n_{\tilde{\Sigma}}^{\nu} \leq BD\tau_i^{-1}, \tag{51}$$

where D is a quantity coming from data. Summing over i, this gives us that

$$\int_{\mathcal{R}(0,\tau)} t \, \chi J_{\nu}^{N}(\psi) n_{\tilde{\Sigma}}^{\nu} \leq BD(1 + \log|\tau + 1|)$$

This estimates in particular the first term on the right hand side of the first inequality of (48). Applying this inequality, we obtain as before (49), but with $\tau^{-2}(1 + \log |\tau + 1|)$ replacing τ . Using (51) and a pigeonhole principle, one improves this to (50), with $\tau^{-2}(1 + \log |\tau + 1|)$ now replacing τ . Iterating this argument again one removes the log (exercise).

Note that this loss of derivatives in (31) simply arises from the loss in (39). If Ω_i could be replaced by Ω_i^{ϵ} in (39), then the loss would be 3ϵ . The latter refinement can in fact be deduced from the original (31) using in addition work of Blue-Soffer [21]. Running the argument of this section with the ϵ -loss version of (31), we obtain now

Proposition 4.2.1. For any $\epsilon > 0$, statement (31) holds with 3 replaced by ϵ in the definition of E_1 and C replaced by C_{ϵ} .

4.3 Pointwise decay

To derive pointwise decay for ψ itself, we should remember that we have in fact dropped a good 0'th order term from the estimate (44). In particular, we have also

$$\int_{t=\tau} J^{Z,w}_{\mu}(\psi) n^{\mu} \ge b \int_{\{t=\tau\} \cap \{r \ge r_0\}} (\tau^2 r^{-2} + 1) \psi^2.$$

From this and the previously derived bounds, pointwise decay can be shown easily by applying Ω_i as commutators and Sobolev estimates. See [65] for details.

Exercise: Derive pointwise decay for all derivatives of ψ , including transverse derivatives to the horizon of any order, by commuting in addition with \hat{Y} as in the proof of Theorem 3.2.

4.4 Comments and further reading

4.4.1 The X-estimate

The origin of the use of vector field multipliers of the type X (as in Section 4.1) for proving decay for solutions of the wave equation goes back to Morawetz. (These identities are generalisations of the classical virial identity, which has itself a long and complicated history.) In the context of Schwarzschild black holes, the first results in the direction of such estimates were in Laba and Soffer [110] for a certain "Schrödinger" equation (related to the Schwarzschild t-function),

and, for the wave equation, in Blue and Soffer [19]. These results were incomplete (see [20]), however, and the first estimate of this type was actually obtained in our [65], motivated by the original calculations of [19, 110]. This estimate required decomposition of ψ into individual spherical harmonics ψ_{ℓ} , and choosing the current $J^{X,w}$ separately for each ψ_{ℓ} . A slightly different approach to this estimate is provided by [20]. A somewhat simpler choice of current $J^{X,w}$ which provides an estimate for all sufficiently high spherical harmonics was first presented by Alinhac [1]. Our Section 4.1.1 is similar in spirit. The first estimate not requiring a spherical harmonic decomposition was obtained in [67]. This is the current of Section 4.1.2. The problem of reducing the loss of derivatives in (39) has been addressed in Blue-Soffer [21]. The results of [21] in fact also apply to the Reissner-Nordström metric.

A slightly different construction of a current as in Section 4.1.2 has been given by Marzuola and collaborators [116]. This current does not require commuting with Ω_i . In their subsequent [115], the considerations of [116] are combined with ideas from [65, 67] to obtain an estimate which does not degenerate on the horizon: One includes a piece of the current J^N of Section 3.3.2 and exploits Proposition 3.1.

4.4.2 The Z-estimate

The use of vector-field multipliers of the type Z also goes back to celebrated work of Morawetz, in the context of the wave equation outside convex obstacles [119]. The geometric interpretation of this estimate arose later, and the use of Zadapted to the causal geometry of a non-trivial metric first appears perhaps in the proof of stability of Minkowski space [51]. The decay result Theorem 4.1 was obtained in our [65]. A result yielding similar decay away from the horizon (but weaker decay along the horizon) was proven independently in a nice paper of Blue and Sterbenz [22]. Both [22] and [65] make use of a current based on the vector field Z. In [22], the error term analogous to $K^{Z,w}$ of Section 4.2 was controlled with the help of an auxiliary collection of multipliers with linear weights in t, chosen at the level of each spherical harmonic, whereas in [65], these error terms are controlled directly from (39) by a dyadic iteration scheme similar to the one we have given here in Section 4.2. The paper [22] does not obtain estimates for the non-degenerate energy flux (31); moreover, a slower pointwise decay rate near the horizon is achieved in comparison to Theorem 4.1. Motivated by [65], the authors of [22] have since given a different argument [23] to obtain iust the pointwise estimate (32) on the horizon, exploiting the "good" term in $K^{Z,w}$ near the horizon. The proof of Theorem 4.1 presented in Section 4.2 is a slightly modified version of the scheme in [65], avoiding spherical harmonic decompositions (for obtaining (39)) by using in particular the result of [67].

 $^{^{44}}$ A related refinement, where h of (40) is replaced by a function vanishing logarithmically at 3M, follows from [115] referred to below.

4.4.3 Other results

Statement (32) of Theorem 4.1 has been generalised to the Maxwell case by Blue [18]. In fact, the Maxwell case is much "cleaner", as the current J^Z need not be modified by a function w, and its flux is pointwise positive through any spacelike hypersurface. The considerations near the horizon follow [23] and thus the analogue of (31) is not in fact obtained, only decay for the degenerate flux of J^T . Nevertheless, the non-degenerate (31) for Maxwell can be proven following the methods of this section, using in particular currents associated to the vector field Y (Exercise).

To our knowledge, the above discussion exhausts the quantitative pointwise and energy decay-type statements which are known for general solutions of the wave equation on Schwarzschild.⁴⁵ The best previously known results on general solutions of the wave equation were non-quantitative decay type statements which we briefly mention. A pointwise decay without a rate was first proven in the thesis of Twainy [149]. Scattering and asymptotic completeness statements for the wave, Klein-Gordon, Maxwell and Dirac equations have been obtained by [72, 73, 5, 4, 122]. These type of statements are typically insensitive to the amount of trapping. See the related discussion of Section 4.6, where the statement of Theorem 4.1 is compared to non-quantitative statements heuristically derived in the physics literature.

4.5 Perturbing?

Use of the J^N current "stabilises" the proof of Theorem 4.1 with respect to considerations near the horizon. There is, however, a sense in which the above argument is still fundamentally attached to Schwarzschild. The approach taken to derive the multiplier estimate (36) depends on the structure of the trapping set, in particular, the fact that trapped null geodesics approach a codimension-1 subset of spacetime, the photon sphere. Overcoming the restrictiveness of this approach is the fundamental remaining difficulty in extending these techniques to Kerr, as will be accomplished in Section 5.3. Precise implications of this fact for multiplier estimates are discussed further in [1].

4.6 Aside: Quantitative vs. non-quantitative results and the heuristic tradition

The study of wave equations on Schwarzschild has a long history in the physics literature, beginning with the pioneering Regge and Wheeler [131]. These studies have all been associated with showing "stability".

A seminal paper is that of Price [130]. There, insightful heuristic arguments were put forth deriving the asymptotic tail of each spherical harmonic ψ_{ℓ} evolving from compactly supported initial data, suggesting that for r > 2M,

$$\psi_{\ell}(r,t) \sim C_{\ell} t^{-(3+2\ell)}$$
. (52)

 $^{^{45}}$ For fixed spherical harmonic ℓ = 0, there is also the quantitative result of [63], to be mentioned in Section 4.6.

These arguments were later extended by Gundlach et al [88] to suggest

$$\psi_{\ell}|_{\mathcal{H}^+} \sim C_{\ell} v^{-(3+2\ell)}, \qquad r\psi_{\ell}|_{\mathcal{I}^+} \sim \bar{C}_{\ell} u^{-(2+\ell)}.$$
 (53)

Another approach to these heuristics via the analytic continuation of the Green's function was followed by [31]. The latter approach in principle could perhaps be turned into a rigorous proof, at least for solutions not supported on $\mathcal{H}^+ \cap \mathcal{H}^-$. See [114, 106] for just (52) for the $\ell = 0$ case.

Statements of the form (52) are interesting because, if proven, they would give the fine structure of the tail of the solution. However, it is important to realise that statements like (52) in of themselves would not give quantitative bounds for the size of the solution at all later times in terms of initial data. In fact, the above heuristics do not even suggest what the best such quantitative result would be, they only give a heuristic *lower* bound on the best possible quantitative decay rate in a theorem like Theorem 4.1.

Let us elaborate on this further. For fixed spherical harmonic, by compactness a statement of the form (52) would immediately yield some bound

$$|\psi_{\ell}|(r,t) \le D(r,\psi_{\ell})t^{-3},\tag{54}$$

for some constant D depending on r and on the solution itself. It is not clear, however, what the sharp such quantitative inequality of the form (54) is supposed to be when the constant is to depend on a natural quantity associated to data. It is the latter, however, which is important for the nonlinear stability problem.

There is a setting in which a quantitative version of (54) has indeed been obtained: The results of [63] (which apply to the nonlinear problem where the scalar field is coupled to the Einstein equation, but which can be specialised to the decoupled case of the $\ell=0$ harmonic on Schwarzschild) prove in particular that

$$|n_{\Sigma_{-}}\psi_{0}| + |\psi_{0}| \le C_{\epsilon}D(\psi, \psi')\tau^{-3+\epsilon}, \qquad |r\psi_{0}| \le CD(\psi, \psi')\tau^{-2}$$
 (55)

where C_{ϵ} depends only on ϵ , and $D(\psi, \psi')$ is a quantity depending only on the initial J^T energy and a pointwise weighted C^1 norm. In view of the relation between τ , u, and v, (55) includes also decay on the horizon and null infinity as in the heuristically derived (53). The fact that the power 3 indeed appears in both in the quantitative (55) and in (54) may be in part accidental. See also [15].

For general solutions, i.e. for the sum over spherical harmonics, the situation is even worse. In fact, a statement like (52) a priori gives no information whatsoever of any sort, even of the non-quantitative kind. It is in principle compatible with $\limsup_{t\to\infty}\psi(r,t)=\infty$.⁴⁶ It is well known, moreover, that to understand quantitative decay rates for general solutions, one must quantify trapping. This is not, however, captured by the heuristics leading to (52), essentially because for fixed ℓ , the effects of trapping concern an intermediate

⁴⁶Of course, given the quantitative result of Theorem 3.2 and the statement (52), one could then infer that for each r > 2M, then $\lim_{t\to\infty} \phi(r,t) = 0$, without however a rate (exercise).

time interval not reflected in the tail. It should thus not be surprising that these heuristics do not address the fundamental problem at hand.

Another direction for heuristic work has been the study of so-called quasinormal modes. These are solutions with time dependence $e^{-i\omega t}$ for ω with negative imaginary part, and appropriate boundary conditions. These occur as poles of the analytic continuation of the resolvent of an associated elliptic problem, and in the scattering theory literature are typically known as resonances. Quasinormal modes are discussed in the nice survey article of Kokkotas and Schmidt [104]. Rigorous results on the distribution of resonances have been achieved in Bachelot–Motet-Bachelot [7] and Sá Barreto-Zworski [135]. The asymptotic distribution of the quasi-normal modes as $\ell \to \infty$ can be thought to reflect trapping. On the other hand, these modes fo not reflect the "low-frequency" effects giving rise to tails. Thus, they too tell only part of the story. See, however, the case of Schwarzschild-de Sitter in Section 6.

Finally, we should mention Stewart [144]. This is to our knowledge the first clear discussion in the physics literature of the relevance of trapping on the Schwarzschild metric in this context and the difference between quantitative and non-quantitative decay rates. It is interesting to compare Section 3 of [144] with what has now been proven: Although the predictions of [144] do not quite match the situation in Schwarzschild (it is in particular incompatible with (52)), they apply well to the Schwarzschild-de Sitter case developed in Section 6.

The upshot of the present discussion is the following: Statements of the form (52), while interesting, may have little to do with the problem of non-linear stability of black holes, and are perhaps more interesting for the lower bounds that they suggest.⁴⁷ In fact, in view of their non-quantitative nature, these results are less relevant for the stability problem than the quantitative boundedness theorem of Kay and Wald. Even the statement of Section 3.2.3 cannot be derived as a corollary of the statement (52), nor would knowing (52) simplify in any way the proof of Section 3.2.3.

5 Perturbing Schwarzschild: Kerr and beyond

We now turn to the problem of perturbing the Schwarzschild metric and proving boundedness and decay for the wave equation on the backgrounds of such perturbed metrics. Let us recall our dilemma: The boundedness argument of Section 3 required that T remains causal everwhere in the exterior. In view of the comments of Section 3.5, this is clearly unstable. On the other hand, the decay argument of Section 4 requires understanding the trapped set and in particular, uses the fact that in Schwarzschild, a certain codimension-1 subset of spacetime—the photon sphere—plays a special role. Again, as discussed in Section 4.5, this special structure is unstable.

It turns out that nonetheless, these issues can be addressed and both boundedness (see Theorem 5.1) and decay (see Theorem 5.2) can be proven for the wave equation on suitable perturbations of Schwarzschild. As we shall see, the

⁴⁷See for instance the relevance of this in [59].

boundedness proof (See Section 5.2) turns out to be more robust and can be applied to a larger class of metrics—but it too requires some insight from the Schwarzschild decay argument! The decay proof (See Section 5.3) will require us to restrict to exactly Kerr spacetimes.

Without further delay, perhaps it is time to introduce the Kerr family...

5.1 The Kerr metric

The Kerr metric is a 2-parameter family of metrics first discovered [99] in 1963. The parameters are called $mass\ M$ and specific angular momentum a, i.e. angular momentum per unit mass. In so-called Boyer-Lindquist local coordinates, the metric element takes the form:

$$-\left(1 - \frac{2M}{r\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)}\right)dt^2 + \frac{1 + \frac{a^2\cos^2\theta}{r^2}}{1 - \frac{2M}{r} + \frac{a^2}{r^2}}dr^2 + r^2\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)d\theta^2 + r^2\left(1 + \frac{a^2}{r^2} + \left(\frac{2M}{r}\right)\frac{a^2\sin^2\theta}{r^2\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)}\right)\sin^2\theta d\phi^2 - 4M\frac{a\sin^2\theta}{r\left(1 + \frac{a^2\cos^2\theta}{r^2}\right)}dt d\phi.$$

The vector fields ∂_t and ∂_{ϕ} are Killing. We say that the Kerr family is *stationary* and *axisymmetric*.⁴⁸ Traditionally, one denotes

$$\Delta = r^2 - 2Mr + a^2.$$

If a = 0, the Kerr metric clearly reduces to Schwarzschild (5).

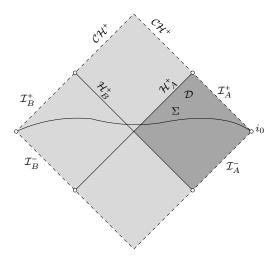
Maximal extensions of the Kerr metric were first constructed by Carter [29]. For parameter range $0 \le |a| < M$, these maximal extensions have black hole regions and white hole regions bounded by future and past event horizons \mathcal{H}^{\pm} meeting at a bifurcate sphere. The above coordinate system is defined in a domain of outer communications, and the horizon will correspond to the limit $r \to r_+$, where r_+ is the larger positive root of $\Delta = 0$, i.e.

$$r_+ = M + \sqrt{M^2 - a^2}.$$

Since the motivation of our study is the Cauchy problem for the Einstein equations, it is more natural to consider not maximal extensions, but maximal developments of complete initial data. (See Appendix B.) In the Schwarzschild case, the maximal development of initial data on a Cauchy surface Σ as described previously coincides with maximally-extended Schwarzschild. In Kerr, if we are to take an asymptotically flat (with two ends) hypersurface in a maximally extended Kerr for parameter range 0 < |a| < M, then its maximal development will have a smooth boundary in maximally-extended Kerr. This boundary is

 $^{^{48} \}rm There$ are various conventions on the meaning of the words "stationary" and "axisymmetric" depending on the context. Let us not worry about this here. . .

what is known as a Cauchy horizon. We have already discussed this phenomenon in Section 2.7.3 in the context of strong cosmic censorship. The maximally extended Kerr solutions are quite bizarre, in particular, they contain closed timelike curves. This is of no concern to us here, however. By definition, for us the term "Kerr metric $(\mathcal{M}, g_{M,a})$ " will always denote the maximal development of a complete asymptotically flat hypersurface Σ , as above, with two ends. One can depict the Penrose-diagramatic representation of a suitable two-dimensional timelike slice of this solution as below:



This depiction coincides with the standard Penrose diagram of the spherically symmetric Reissner-Nordström metric [91, 148].

With this convention in mind, we note that the dependence of $g_{M,a}$ on a is smooth in the range $0 \le |a| < M$. In particular, Kerr solutions with small $|a| \ll M$ can be viewed as close to Schwarzschild.

One can see this explicitly in the subregion of interest to us by passing to a new system of coordinates. Define

$$t^* = t + \bar{t}(r)$$

$$\phi^* = \phi + \bar{\phi}(r)$$

where

$$\frac{d\bar{t}}{dr}(r) = (r^2 + a^2)/\Delta^2, \qquad \frac{d\bar{\phi}}{dr}(r) = a/\Delta.$$

(These coordinates are often known as Kerr-star coordinates.) These coordinates are regular across $\mathcal{H}^+ \setminus \mathcal{H}^-$. We may finally define a coordinate $r_{\text{Schw}} = r_{\text{Schw}}(r, a)$ such that which takes $[r_+, \infty) \to [2M, \infty)$ with smooth dependence in a and such that $r_{\text{Schw}}(r, 0)$ is the identity map. In particular, if we define Σ_0 by $\mathcal{D} = \{t^* = 0\}$, and define $\mathcal{R} = \mathcal{D} \cap \{t^* \geq 0\}$, and fix $r_{\text{Schw}}, t^*, \phi^*$

⁴⁹Of course, one again needs two coordinate systems in view of the breakdown of spherical coordinates. We shall suppress this issue in the discussion that follows.

Schwarzschild coordinates, then the metric functions of $g_{M,a}$ written in terms of these coordinates as defined previously depend smoothly on a for $0 \le |a| < M$ in \mathcal{R} , and, for a = 0, reduce to the Schwarzschild metric form in (r, t^*, ϕ, θ) coordinates where t^* is defined from Schwarzschild t as above.

We note that $\partial_t = \partial_{t^*}$ in the intersection of the coordinate systems. We immediately note that ∂_t is spacelike on the horizon, except where $\theta = 0, \pi$, i.e. on the axis of symmetry. Note that we shall often abuse notation (as we just have done) and speak of ∂_t on the horizon or at $\theta = 0$, where of course the (r, t, θ, ϕ) coordinate system breaks down, and formally, this notation is meaningless.

In general, the part of the domain of outer communications plus horizon where ∂_t is spacelike in known as the *ergoregion*. It is bounded by a hypersurface known as the *ergosphere*. The ergosphere meets the horizon on the axis of symmetry $\theta = 0, \pi$.

The ergosphere allows for a particle "process", originally discovered by Penrose [127], for extracting energy out of a black hole. This came to be known as the *Penrose process*. In his thesis, Christodoulou [38] discovered the existence of a quantity—the so-called *irreducible mass* of the black hole—which he should to be always nondecreasing in a Penrose process. The analogy between this quantity and entropy led later to a subject known as "black hole thermodynamics" [8, 11]. This is currently the subject of intense investigation from the point of view of high energy physics.

In the context of the study of $\Box_g \psi = 0$, we have already discussed in Section 5 the effect of the ergoregion: It is precisely the presence of the ergoregion that makes our previous proof of boundedness for Schwarzschild not immediately generalise for Kerr. Moreover, in contrast to the Schwarzschild case, there is no "easy result" that one can obtain away from the horizon analogous to Section 3.2.3. In fact, the problem of proving any sort of boundedness statement for general solutions to $\Box_g \psi = 0$ on Kerr had been open until very recently. We will describe in the next section our recent resolution [68] of this problem.

5.2 Boundedness for axisymmetric stationary black holes

We will derive a rather general boundedness theorem for a class of axisymmetric stationary black hole exteriors near Schwarzschild. The result (Theorem 5.1) will include slowly rotating Kerr solutions with parameters $|a| \ll M$.

We have already explained in what sense the Kerr metric is "close" to Schwarzschild in the region \mathcal{R} . Let us note that with respect to the coordinates r_{Schw} , t^* , ϕ^* , θ in \mathcal{R} , then ∂_{t^*} and ∂_{ϕ^*} are Killing for both the Schwarzschild and the Kerr metric. The class of metrics which will concern us here are metrics defined on \mathcal{R} such that the metric functions are close to Schwarzschild in a suitable sense⁵⁰, and ∂_{t^*} , ∂_{ϕ^*} are Killing, where these are defined with respect to the ambient Schwarzschild coordinates.

There is however an additional geometric assumption we shall need, and this

 $^{^{50}}$ This requires moving to an auxiliary coordinate system. See [68].

is motivated by a geometric property of the Kerr spacetime, to be described in the section that follows immediately.

5.2.1 Killing fields on the horizon

Let us here remark a geometric property of the Kerr spacetime itself which turns out to be of utmost importance in what follows: Let V denote a null generator of \mathcal{H}^+ . Then

$$V \in \operatorname{Span}\{\partial_{t^*}, \partial_{\phi^*}\}. \tag{56}$$

There is a deep reason why this is true. For stationary black holes with non-degenerate horizons, a celebrated argument of Hawking retrieves a second Killing field in the direction of the null generator V. Thus, if ∂_{t^*} and ∂_{ϕ^*} span the complete set of Killing fields, then V must evidently be in their span.

In fact, choosing V accordingly we have

$$V = \partial_{t^*} + (a/2Mr_+)\partial_{\phi^*} \tag{57}$$

(For the Kerr solution, we have that there exists a timelike direction in the span of ∂_{t^*} and ∂_{ϕ^*} for all points outside the horizon. We shall not explicitly make reference to this property, although in view of Section 7, one can infer this property (**exercise**) for small perturbations of Schwarzschild of the type considered here, i.e., given any point p outside the horizon, there exists a Killing field V (depending on p) such that V(p) is timelike.)

5.2.2 The axisymmetric case

From (57), it follows that there is a constant $\omega_0 > 0$, depending only on the parameters a and M, such that if

$$|\partial_{t^*}\psi|^2 \ge \omega_0 |\partial_{\phi^*}\psi|^2,\tag{58}$$

on \mathcal{H}^+ , then the flux satisfies

$$J_{\mu}^{T}(\psi)n_{\mathcal{H}^{+}}^{\mu} \ge 0. \tag{59}$$

Note also that, for fixed M, we can take

$$\omega_0 \to 0$$
, as $a \to 0$. (60)

There is an immediate application of (58). Let us restrict for the moment to axisymmetric solutions, i.e. to ψ such that $\partial_{\phi}\psi = 0$. It follows that (58) trivially holds. As a result, our argument proving boundedness is stable, i.e. Theorem 3.2 holds for axisymmetric solutions of the wave equation on Kerr spacetimes with $|a| \ll M$. (See the exercise of Section 3.5.) In fact, the restriction on a can be be removed (**Exercise**, or go directly to Section 7).

Let us note that the above considerations make sense not only for Kerr but for the more general class of metrics on \mathcal{R} close to Schwarzschild such that ∂_{t^*} , ∂_{ϕ^*} are Killing, \mathcal{H}^+ is null and (56) holds. In particular, (58) implies (59),

where in (60), the condition $a \to 0$ is replaced by the condition that the metric is taken suitably close to Schwarzschild. The discussion which follows will refer to metrics satisfying these assumptions.⁵¹ For simplicity, the reader can specialise the discussion below to the case of a Kerr metric with $|a| \ll M$.

5.2.3 Superradiant and non-superradiant frequencies

There is a more general setting where we can make use of (58). Let us suppose for the time being that we could take the Fourier transform $\hat{\psi}(\omega)$ of our solution ψ in t^* and then expand in azimuthal modes ψ_m , i.e. modes associated to the Killing vector field ∂_{ϕ^*} .

If we were to restrict ψ to the frequency range

$$|\omega|^2 \ge \omega_0 m^2,\tag{61}$$

then (58) and thus (59) holds after integrating along \mathcal{H}^+ . In view of this, frequencies in the range (61) are known as nonsuperradiant frequencies. The frequency range

$$|\omega|^2 \le \omega_0 m^2 \tag{62}$$

determines the so-called *superradiant frequencies*. In the physics literature, the main difficulty of this problem has traditionally been perceived to "lie" with these frequencies.

Let us pretend for the time being that using the Fourier transform, we could indeed decompose

$$\psi = \psi_{\dagger} + \psi_{\flat} \tag{63}$$

where ψ_{\dagger} is supported in (61), whereas ψ_{\flat} is supported in (62).

In view of the discussion immediately above and the comments of Section 5.2.2, it is plausible to expect that one could indeed prove boundedness for ψ_{\parallel} in the manner of the proof of Theorem 3.2. In particular, if one could localise the integrated version of (59) to arbitrary sufficiently large subsegments $\mathcal{H}(\tau',\tau'')$, one could obtain

$$\int_{\Sigma_{\tau}} J_{\mu}^{n_{\Sigma_{\tau}}} (\psi_{\sharp}) n_{\Sigma_{\tau}}^{\mu} \leq B \int_{\Sigma_{0}} J_{\mu}^{n_{\Sigma_{0}}} (\psi_{\sharp}) n_{\Sigma_{0}}^{\mu}. \tag{64}$$

This would leave ψ_{\flat} . Since this frequency range does not suggest a direct boundedness argument, it is natural to revisit the decay mechanism of Schwarzschild. We have already discussed (see Section 4.5) the instability of the decay argument; this instability arose from the structure of the set of trapped null geodesics. At the heuristic level, however, it is easy to see that, if one can take ω_0 sufficiently small, then solutions supported in (62) cannot be trapped. In particular, for $|a| \ll M$, superradiant frequencies for $\Box_g \psi = 0$ on Kerr are not trapped. This will be the fundamental observation allowing for the boundedness theorem. Let us see how this statement can be understood from the point of view of energy currents.

 $^{^{51}\}mathrm{They}$ are summarised again in the formulation of Theorem 5.1.

5.2.4 A stable energy estimate for superradiant frequencies

We continue here our heuristic point of view, where we assume a decomposition (63) where ψ_b is supported in (62). In particular, one has an inequality

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} \omega_0^2 (\partial_\phi \psi_b)^2 d\phi^* dt^* \ge \int_{-\infty}^{\infty} \int_{0}^{2\pi} (\partial_t \psi_b)^2 d\phi^* dt^*$$
 (65)

for all (r, θ) . We shall see below that (65) allows us easily to construct a suitable stable current for Schwarzschild.

It may actually be a worthwhile **exercise** for the reader to come up with a suitable current for themselves. The choice is actually quite flexible in comparison with the considerations of Section 4.1. Our choice (see [68]) is defined by

$$J^{\mathbf{X}} = eJ^{N} + J^{X_a} + J^{X_b, w_b} \tag{66}$$

where here, N is the vector field of Section 3.3.2, $X_a = f_a \partial_{r^*}$, with

$$f_a = -r^{-4}(r_0)^4, \quad \text{for } r \le r_0$$

$$f_a = -1, \quad \text{for } r_0 \le r \le R_1,$$

$$f_a = -1 + \int_{R_1}^r \frac{d\tilde{r}}{4\tilde{r}} \quad \text{for } R_1 \le r \le R_2,$$

$$f_a = 0 \text{ for } r \ge R_2,$$

 $X_b = f_b \partial_{r^*}$ with

$$f_b = \chi(r^*)\pi^{-1} \int_0^{r^*} \frac{\alpha}{x^2 + \alpha^2}$$

and $\chi(r^*)$ is a smooth cutoff with $\chi=0$ for $r^*\leq 0$ and $\chi=1$ for $r^*\geq 1$. Here r and r^* are Schwarzschild coordinates. ⁵² The function w_b is given by

$$w_b = f_b' + \frac{2}{r}(1 - 2M/r)(1 - M/r)f_b.$$

The parameters e, α , r_0 , R_1 , R_2 must be chosen accordingly!

Restricting to the range (62), using (65), with some computation we would obtain

$$\int_{-\infty}^{\infty} \int_{0}^{2\pi} K^{\mathbf{X}}(\psi_{\flat}) d\phi^* dt^* \ge b \int_{-\infty}^{\infty} \int_{0}^{2\pi} \chi J_{\mu}^{n_{\Sigma}}(\psi_{\flat}) n_{\Sigma}^{\mu} d\phi^* dt^*, \tag{67}$$

for all (r, θ) .

The above inequality can immediately be seen to be stable to small⁵³ axisymmetric, stationary perturbations of the Schwarzschild metric. That is to

 $^{^{52}}$ Since we are dealing now with general perturbations of Schwarzschild, we shall now use r for what we previously denoted by $r_{\rm Schw}$. Note that in the special case that our metric is Kerr, this r is different from the Boyer-Lindquist r.

 $^{^{53}}$ Of course, in view of the degeneration towards i^0 , it is important that smallness is understood in a weighted sense.

say, for such metrics, if ψ_{\flat} is supported in (62) (where frequencies here are defined by Fourier transform in coordinates t^* , ϕ^*), then the inequality (67) holds as before. In particular, (67) holds for Kerr for small $|a| \ll M$.

How would (67) give boundedness for ψ_{\flat} ? We need in fact to suppose something slightly stronger, namely that (67) holds localised to $\mathcal{R}(0,\tau)$. Consider the currents

$$J = J^T + e_2 J^{\mathbf{X}}, \qquad K = \nabla^{\mu} J_{\mu},$$

where e_2 is a positive parameters, and J^N is the current of Section 3.3.2. Then, for metrics g close enough to Schwarzschild, and for e_2 sufficiently small, we would have from a localised (67) that

$$\int_{\mathcal{R}(0,\tau)} K(\psi_{\flat}) \ge 0,$$

$$\int_{\mathcal{H}(0,\tau)} J_{\mu}(\psi_{\flat}) n_{\mathcal{H}}^{\mu} \ge 0,$$

and thus

$$\int_{\Sigma_{\tau}} J_{\mu}(\psi_{\flat}) n_{\Sigma_{\tau}}^{\mu} \leq \int_{\Sigma_{0}} J_{\mu}(\psi_{\flat}) n_{\Sigma_{0}}^{\mu}.$$

Moreover, for g sufficiently close to Schwarzschild and e_1 , e_2 suitably defined, we also have (**exercise**)

$$\int_{\Sigma_{\tau}} J_{\mu}^{n_{\Sigma_{\tau}}}(\psi_{\flat}) n^{\mu} \leq B \int_{\Sigma_{\tau}} J_{\mu}(\psi_{\flat}) n_{\Sigma_{\tau}}^{\mu}.$$

We thus would obtain

$$\int_{\Sigma_{\tau}} J_{\mu}^{n_{\Sigma_{\tau}}}(\psi_{\flat}) n^{\mu} \leq B \int_{\Sigma_{0}} J_{\mu}^{n_{\Sigma_{0}}}(\psi_{\flat}) n^{\mu}. \tag{68}$$

Adding (68) and (64), we would obtain

$$\int_{\Sigma_{\tau}} J_{\mu}^{n_{\Sigma_{\tau}}}(\psi) n^{\mu} \leq B \int_{\Sigma_{0}} J_{\mu}^{n_{\Sigma_{0}}}(\psi) n^{\mu}$$

provided that we could also estimate say

$$\int_{\Sigma_0} J_{\mu}^{n_{\Sigma_0}} (\psi_{\sharp}) n^{\mu} \le B \int_{\Sigma_0} J_{\mu}^{n_{\Sigma_0}} (\psi) n^{\mu}. \tag{69}$$

5.2.5 Cutoff and decomposition

Unfortunately, things are not so simple!

For one thing, to take the Fourier transform necessary to decompose in frequency, one would need to know a priori that $\psi(t^*,\cdot)$ is in $L^2(t^*)$. What we want to prove at this stage is much less. A priori, ψ can grow exponentially in t^* . In order to apply the above, one must cut off the solution appropriately in time

This is achieved as follows. For definiteness, define Σ_0 to be $t^* = 0$, and Σ_{τ} as before. We will also need two auxiliary families of hypersurfaces defined as

follows. (The motivation for considering these will be discussed in Section 5.2.6.) Let χ be a cutoff such that $\chi(x) = 0$ for $x \ge 0$ and $\chi = 1$ for $x \le -1$, and define t^{\pm} by

$$t^+ = t^* - \chi(-r + R)(1 + r - R)^{1/2}$$

and

$$t^{-} = t^{*} + \chi(-r + R)(1 + r - R)^{1/2}$$

where R is a large constant, which must be chosen appropriately. Let us define then

$$\Sigma^{+}(\tau) \doteq \{t^{+} = \tau\}, \qquad \Sigma^{-}(\tau) \doteq \{t^{-} = \tau\}.$$

Finally, we define

$$\mathcal{R}(\tau_1, \tau_2) = \bigcup_{\tau_1 \le \tau \le \tau_2} \Sigma(\tau),$$

$$\mathcal{R}^+(\tau_1, \tau_2) = \bigcup_{\tau_1 \le \tau \le \tau_2} \Sigma^+(\tau),$$

$$\mathcal{R}^-(\tau_1, \tau_2) = \bigcup_{\tau_1 \le \tau \le \tau_2} \Sigma^-(\tau).$$

Let ξ now be a cutoff function such that $\xi = 1$ in $J^+(\Sigma_1^-) \cap J^-(\Sigma_{\tau-1}^+)$, and $\xi = 0$ in $J^+(\Sigma_{\tau}^+) \cap J^-(\Sigma_0^-)$. We may finally define

$$\psi_{>\!\!<} = \xi \psi$$
.

The function ψ_{\approx} is a solution of the inhomogeneous equation

$$\Box_{a}\psi \approx = F, \qquad F = 2\nabla^{\alpha}\xi \nabla_{\alpha}\psi + \Box_{a}\xi \psi.$$

Note that F is supported in $\mathcal{R}^-(0,1) \cup \mathcal{R}^+(\tau-1,\tau)$.

Another problem is that sharp cutoffs in frequency behave poorly under localisation. We thus do the following: Let ζ be a smooth cutoff supported in [-2,2] with the property that $\zeta=1$ in [-1,1], and let $\omega_0>0$ be a parameter to be determined later. For an arbitrary Ψ of compact support in t^* , define

$$\Psi_{\dagger}(t^*,\cdot) \doteq \sum_{m\neq 0} e^{im\phi^*} \int_{-\infty}^{\infty} \zeta((\omega_0 m)^{-1}\omega) \,\hat{\Psi}_m(\omega,\cdot) \, e^{i\omega t^*} d\omega,$$

$$\Psi_{\sharp}(t^*,\cdot) \doteq \Psi_0 + \sum_{m\neq 0} e^{im\phi^*} \int_{-\infty}^{\infty} \left(1 - \zeta((\omega_0 m)^{-1}\omega)\right) \,\hat{\Psi}_m(\omega,\cdot) \, e^{i\omega t^*} d\omega.$$

Note of course that $\Psi_{\sharp} + \Psi_{\flat} = \Psi$. We shall use the notation ψ_{\flat} for $(\psi_{\thickapprox})_{\flat}$ and ψ_{\sharp} for $(\psi_{శ})_{\sharp}$. Note that ψ_{\flat} , ψ_{\sharp} satisfy

$$\Box_g \psi_{\flat} = F_{\flat}, \qquad \Box_g \psi_{\sharp} = F_{\sharp}. \tag{70}$$

5.2.6 The bootstrap

With ψ_{\flat} , ψ_{\sharp} well defined, we now try to fill in the argument heuristically outlined before.

We wish to show the boundedness of

$$\mathbf{q} \doteq \sup_{0 \le \bar{\tau} \le \tau} \int_{\Sigma_{\bar{\tau}}} J^N_{\mu} n^{\mu}. \tag{71}$$

We will argue by continuity in τ . We have already seen heuristically how to obtain a bound for \mathbf{q} in Sections 5.2.3 and 5.2.4. When interpreted for the ψ_{\flat} , ψ_{\dagger} defined above, these arguments produce error terms from:

- the inhomogeneous terms F_{\flat}, F_{\sharp} from (70)
- the fact that we wish to localise estimates (59) and (65) to subregions $\mathcal{H}^+(\tau',\tau'')$ and $\mathcal{R}(\tau',\tau'')$ resepectively
- the fact that (69) is not exactly true.

These error terms can be controlled by \mathbf{q} itself. For this, one studies carefully the time-decay of F_{\flat} , F_{\sharp} away from the cutoff region $\mathcal{R}^{-}(0,1) \cup \mathcal{R}^{+}(\tau-1,\tau)$ using classical properties of the Fourier transform. An important subtlety arises from the presence of 0'th order terms in ψ , and it is here that the divergence of the region \mathcal{R}^{\pm} from $\mathcal{R}(0,\tau)$ is exploited to exchange decay in τ and r.

To close the continuity argument, it is essential not only that the error terms be controlled by \mathbf{q} itself, but that a small constant is retrieved, i.e. that the error terms are controlled by $\epsilon \mathbf{q}$, so that they can be absorbed. For this, use is made of the fact that for metrics in the allowed class sufficiently close to Schwarzschild (in the Kerr case, for $|a| \ll M$), one can control a priori the exponential growth rate of (71) to be small. See [68].

5.2.7 Pointwise bounds

Having proven the uniform boundedness of (71), one argues as in the proof of Theorem 3.2 to obtain higher order energy and pointwise bounds. In particular, the positivity property in the computation of Proposition 3.3.2 is stable. (It turns out that this positivity property persists in fact for much more general black hole spacetimes and there is in fact a geometric reason for this! See Chapter 7.)

5.2.8 The boundedness theorem

We have finally

Theorem 5.1. Let g be a metric defined on the differentiable manifold \mathcal{R} with stratified boundary $\mathcal{H}^+ \cup \Sigma_0$, and let T and $\Phi = \Omega_1$ be Schwarzschild Killing fields. Assume

1. g is sufficiently close to Schwarzschild in an appropriate sense

- 2. T and Φ are Killing with respect to g
- 3. \mathcal{H}^+ is null with respect to q and T and Φ span the null generator of \mathcal{H}^+ .

Then the statement of Theorem 3.2 holds.

See [68] for the precise formulation of the closeness assumption 1.

Corollary 5.1. The result applies to Kerr, and to the more general Kerr-Newman family (solving Einstein-Maxwell), for parameters $|a| \ll M$ (and also $|Q| \ll M$ in the Kerr-Newman case).

Thus, we have quantitative pointwise and energy bounds for ψ and arbitrary derivatives on slowly rotating Kerr and Kerr-Newman exteriors.

5.3 Decay for Kerr

To obtain decay results analogous to Theorem 4.1, one needs to understand trapping. For general perturbations of Schwarzschild of the class considered in Theorem 5.1, it is not a priori clear what stability properties one can infer about the nature of the trapped set, and how these can be exploited. But for the Kerr family itself, the trapping structure can easily be understood, in view of the complete integrability of geodesic flow discovered by Carter [29]. The codimensionality of the trapped set persists, but in contrast to the Schwarzschild case where trapped null geodesics all approach the codimensional-1 subset r = 3M of spacetime, in Kerr, this codimensionality must be viewed in phase space.

5.3.1 Separation

There is a convenient way of doing phase space analysis in Kerr spacetimes, namely, as discovered by Carter [30], the wave equation can be separated. Walker and Penrose [153] later showed that both the complete integrability of geodesic flow and the separability of the wave equation have their fundamental origin in the presence of a *Killing tensor*.⁵⁴ In fact, as we shall see, in view of its intimate relation with the integrability of geodesic flow, Carter's separation of \square_q immediately captures the codimensionality of the trapped set.

The separation of the wave equation requires taking the Fourier transform, and then expanding into oblate spheroidal harmonics. As before, taking the Fourier transform requires cutting off in time. We shall here do the cutoff, however, in a somewhat different fashion.

Let Σ_{τ} be defined specifically as $t^* = \tau$. Given $\tau' < \tau$, define $\mathcal{R}(\tau', \tau)$ as before, and let ξ be a cutoff function as in Section 5.2.5, but with $\Sigma_{\tau'+1}$ replacing Σ_1^- , $\Sigma_{\tau'}$ replacing Σ_0^- , and Σ_{τ} replacing Σ_{τ}^+ , $\Sigma_{\tau-1}$ replacing $\Sigma_{\tau-1}^+$. Define as before

$$\psi_{>\!\!\!<}$$
 = $\xi\psi$.

⁵⁴See [32, 108] for recent higher-dimensional generalisations of these properties.

The function ψ_{\approx} is a solution of the inhomogeneous equation

$$\Box_a \psi = F, \qquad F = 2 \nabla^\alpha \xi \nabla_\alpha \psi + \Box_a \xi \psi.$$

Note that F is supported in $\mathcal{R}(\tau', \tau' + 1) \cup \mathcal{R}(\tau - 1, \tau)$.

Since ψ_{\succeq} is compactly supported in t^* we may consider its Fourier transform $\hat{\psi}_{\succeq} = \hat{\psi}_{\succeq}(\omega,\cdot)$. We may now decompose

$$\hat{\psi}_{\gg}(\omega,\cdot) = \sum_{m,\ell} R_{m\ell}^{\omega}(r) S_{m\ell}(a\omega,\cos\theta) e^{im\phi^*},$$

$$\hat{F}(\omega,\cdot) = \sum_{m,\ell} F_{m\ell}^{\omega}(r) S_{m\ell}(a\omega,\cos\theta) e^{im\phi^*},$$

where $S_{m\ell}$ are the oblate spheroidal harmonics. For each $m \in \mathbb{Z}$, and fixed ω , these are a basis of eigenfunctions $S_{m\ell}$ satisfying

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}S_{m\ell}\right) + \frac{m^2}{\sin^2\theta}S_{m\ell} - a^2\omega^2\cos^2\theta S_{m\ell} = \lambda_{m\ell}S_{m\ell},$$

and, in addition, satisfying the orthogonality conditions with respect to the θ variable.

$$\int_0^{2\pi} d\varphi \int_{-1}^1 d(\cos\theta) e^{im\phi^*} S_{m\ell}(a\omega, \cos\theta) e^{-im'\phi^*} S_{m'\ell'}(a\omega, \cos\theta) = \delta_{mm'} \delta_{\ell\ell'}.$$

Here, the $\lambda_{m\ell}(\omega)$ are the eigenvalues associated with the harmonics $S_{m\ell}$. Each of the functions $R_{m\ell}^{\omega}(r)$ is a solution of the following problem

$$\Delta \frac{d}{dr} \left(\Delta \frac{R_{m\ell}^{\omega}}{dr} \right) + \left(a^2 m^2 + (r^2 + a^2)^2 \omega^2 - \Delta (\lambda_{m\ell} + a^2 \omega^2) \right) R_{m\ell}^{\omega} = (r^2 + a^2) \Delta F_{m\ell}^{\omega}.$$

Note that if a = 0, we typically label $S_{m\ell}$ by $\ell \ge |m|$ such that

$$\lambda_{m\ell}(\omega) = \ell(\ell+1)/2.$$

With this choice, $S_{m\ell}$ coincides with the standard spherical harmonics $Y_{m\ell}$. Given any $\omega_1 > 0$, $\lambda_1 > 0$ then we can choose a such that for $|\omega| \leq \omega_1$, $\lambda_{m\ell} \leq \lambda_1$, then

$$|\lambda_{m\ell} - \ell(\ell+1)/2| \le \epsilon$$
.

Rewriting the equation for the oblate spheroidal function

$$-\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} S_{m\ell} \right) + \frac{m^2}{\sin^2\theta} S_{m\ell} = \lambda_{m\ell} S_{m\ell} + a^2 \omega^2 \cos^2\theta S_{m\ell},$$

the smallest eigenvalue of the operator on the left hand side of the above equation is m(m+1). This implies that

$$\lambda_{m\ell} \ge m(m+1) - a^2 \omega^2. \tag{72}$$

This will be all that we require about $\lambda_{m\ell}$. For a more detailed analysis of $\lambda_{m\ell}$, see [81].

5.3.2 Frequency decomposition

Let ζ be a sharp cutoff function such that $\zeta = 1$ for $|x| \le 1$ and $\zeta = 0$ for |x| > 1. Note that

$$\zeta^2 = 1. \tag{73}$$

Let ω_1 , λ_1 be (potentially large) constants to be determined, and λ_2 be a (potentially small) constant to be determined.

Let us define

$$\psi_{\natural} = \int_{-\infty}^{\infty} \zeta(\omega/\omega_{1}) \sum_{m,\ell:\lambda_{m\ell}(\omega) \leq \lambda_{1}} R_{m\ell}^{\omega}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi^{*}} e^{i\omega t^{*}} d\omega,$$

$$\psi_{\natural} = \int_{-\infty}^{\infty} \zeta(\omega/\omega_{1}) \sum_{m,\ell:\lambda_{m\ell}(\omega) > \lambda_{1}} R_{m\ell}^{\omega}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi^{*}} e^{i\omega t^{*}} d\omega,$$

$$\psi_{\natural} = \int_{-\infty}^{\infty} (1 - \zeta(\omega/\omega_{1})) \sum_{m,\ell:\lambda_{m\ell}(\omega) \geq \lambda_{2}\omega^{2}} R_{m\ell}^{\omega}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi^{*}} e^{i\omega t^{*}} d\omega,$$

$$\psi_{\parallel} = \int_{-\infty}^{\infty} (1 - \zeta(\omega/\omega_1)) \sum_{m,\ell:\lambda_{m\ell}(\omega) < \lambda_2 \omega^2} R_{m\ell}^{\omega}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi^*} e^{i\omega t^*} d\omega.$$

We have clearly

For quick reference, we note:

- ψ_b is supported in $|\omega| \le \omega_1$, $\lambda_{m\ell} \le \lambda_1$,
- ψ_{d} is supported in $|\omega| \leq \omega_{1}$, $\lambda_{m\ell} > \lambda_{1}$,
- $\psi_{\mbox{$\downarrow$}}$ is supported in $|\omega| \ge \omega_1, \ \lambda_{m\ell} \ge \lambda_2 \omega^2$ and
- ψ_{\parallel} is supported in $|\omega| \ge \omega_1$, $\lambda_{m\ell} < \lambda_2 \omega^2$.

5.3.3 The trapped frequencies

Trapping takes place in $\psi_{\mbox{$\downarrow$}}$. We show here how to construct a multiplier for this frequency range.

Defining a coordinate r^* by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}$$

and setting

$$u(r) = (r^2 + a^2)^{1/2} R_{m\ell}^{\omega}(r), \qquad H(r) = \frac{\Delta F_{m\ell}^{\omega}(r)}{(r^2 + a^2)^{1/2}},$$

then u satisfies

$$\frac{d^2}{(dr^*)^2}u + (\omega^2 - V_{m\ell}^{\omega}(r))u = H$$

where

$$V_{m\ell}^{\omega}(r) = \frac{4Mram\omega - a^2m^2 + \Delta(\lambda_{m\ell} + \omega^2 a^2)}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2 r^2}{(r^2 + a^2)^4}.$$

Consider the following quantity

$$Q = f\left(\left|\frac{du}{dr^*}\right|^2 + (\omega^2 - V)|u|^2\right) + \frac{df}{dr^*} \operatorname{Re}\left(\frac{du}{dr^*}\bar{u}\right) - \frac{1}{2} \frac{d^2f}{dr^{*2}}|u|^2.$$

Then, with the notation $' = \frac{d}{dr^*}$,

$$Q' = 2f'|u'|^2 - fV'|u|^2 + \text{Re}(2f\bar{H}u' + f'\bar{H}u) - \frac{1}{2}f'''|u|^2.$$
 (74)

For ψ_{\flat} , we have

$$\lambda_{m\ell} + \omega^2 a^2 \ge (\lambda_2 + a^2)\omega^2 \ge (\lambda_2 + a^2)\omega_1^2.$$
 (75)

We set

$$V_0 = (\lambda_{m\ell} + \omega^2 a^2) \frac{r^2 - 2Mr}{(r^2 + a^2)^2}$$

so that

$$V_1 = V - V_0 = \frac{4Mram\omega - a^2m^2 + a^2(\lambda_{m\ell} + \omega^2a^2)}{(r^2 + a^2)^2} + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2r^2}{(r^2 + a^2)^4}.$$

Using (72), (75), we easily see that

$$|r^{3}|V_{1}'| + \left| \left(\frac{(r^{2} + a^{2})^{4}}{\Delta r^{2}} V_{1}' \right)' \right| \leq C \Delta r^{-2} \left(|am\omega| + a^{2} (\lambda_{m\ell} + a^{2} \omega^{2}) + 1 \right)$$

$$\leq \epsilon \Delta r^{-2} (\lambda_{m\ell} + a^{2} \omega^{2}), \tag{76}$$

where ϵ can be made arbitrarily small, if ω_1 is chosen sufficiently large, and a is chosen $a < \epsilon$. On the other hand

$$V_0' = 2 \frac{\Delta}{(r^2 + a^2)^4} (\lambda_{m\ell} + \omega^2 a^2) \left((r - M)(r^2 + a^2) - 2r(r^2 - 2Mr) \right)$$
$$= -2 \frac{\Delta r^2}{(r^2 + a^2)^4} \left(\lambda_{m\ell} + \omega^2 a^2 \right) \left(r - 3M + a^2 \frac{r - M}{r^2} \right). \tag{77}$$

This computation implies that V'_0 has a simple zero in the a^2 neighborhood of r = 3M. Furthermore,

$$\left(\frac{(r^2+a^2)^4}{\Delta r^2}V_0'\right)' \le -\Delta r^{-2}(\lambda_{m\ell} + \omega^2 a^2).$$

From the above and (76), it follows that for ω_1 sufficiently large and a sufficiently small, we have

$$\left(\frac{(r^2+a^2)^4}{\Delta r^2}V'\right)' \le -\frac{1}{2}\Delta r^{-2}(\lambda_{m\ell}+\omega^2 a^2).$$

This alone implies that V' has at most a simple zero.

To show that V' indeed has a zero we examine the boundary values at r_+ and ∞ . From (77) we see that

$$\frac{(r^2+a^2)^4}{\Delta r^2}V_0' \sim C(\lambda_{m\ell} + \omega^2 a^2)$$

for some positive constant C on the horizon and

$$\frac{(r^2+a^2)^4}{\Delta r^2}V_0' \sim -2r(\lambda_{m\ell}+\omega^2 a^2)$$

near $r = \infty$. On the other hand, from the inequality as applied to the first term on the right hand side of (76), it follows that

$$\left| \frac{(r^2 + a^2)^4}{\Delta r^2} V_1' \right| \le \epsilon r (\lambda_{m\ell} + \omega^2 a^2),$$

where ϵ can be chosen arbitrarily small if ω_1 is chosen sufficiently large and a sufficiently small. Thus, for suitable choice of ω_1 , it follows that

$$\frac{(r^2 + a^2)^4}{\Delta r^2} V' \Big|_{r_+} = \frac{(r^2 + a^2)^4}{\Delta r^2} (V'_0 + V'_1) \Big|_{r_+}
> 0 > \frac{(r^2 + a^2)^4}{\Delta r^2} (V'_0 + V'_1) \Big|_{\infty} = \frac{(r^2 + a^2)^4}{\Delta r^2} V' \Big|_{\infty},$$

and thus V' has a unique zero. Let us denote the r-value of this zero by $r_{m\ell}^{\omega}$. We now choose f so that

- 1. $f' \ge 0$,
- 2. $f \le 0$ for $r \le r_{m\ell}^{\omega}$ and $f \ge 0$ for $r \ge r_{m\ell}^{\omega}$
- 3. $-fV' \frac{1}{2}f''' \ge c$.

Property 3 can be verified by ensuring that $f'''(r_{m\ell}^{\omega}) < 0$ as well as requiring that f''' < 0 at the horizon. We may moreover normalise f to -1 on the horizon. Finally, we may assume that there exists an R such that for all $r \ge R$, f is of the form:

$$f = \tan^{-1} \frac{r^* - \alpha - \sqrt{\alpha}}{\alpha} - \tan^{-1} (-1 - \alpha^{-1/2})$$

In particular, for $r \geq R$, the function f will not depend on ω , ℓ , m.

Note the similarity of this construction with that of Section 4.1.1, modulo the need for complete separation to centre the function f appropriately.

Integrating the identity (74) and using that $u \to 0$ as $r \to \infty$ we obtain that for any compact set K_1 in r^* and a certain compact set K_2 (which in particular does not contain r = 3M), there exists a positive constant b > 0 so that

$$b \int_{K_1} (|u'|^2 + |u|^2) dr + b(\lambda_{m\ell} + \omega^2) \int_{K_2} |u|^2 dr$$

$$\leq (|u'|^2 + (\omega^2 - V)|u|^2) (r_+) + \int \operatorname{Re}(2f\bar{H}u' + f'\bar{H}u) dr.$$

On the horizon $r = r_+$, we have $u' = (i\omega + (iam/2Mr_+))u$ and

$$V(r_{+}) = \frac{4Mram\omega - a^{2}m^{2}}{(r_{+}^{2} + a^{2})^{2}}.$$

Therefore, we obtain

$$b \int_{K_1} (|u'|^2 + |u|^2) dr^* + b(\lambda_{m\ell} + \omega^2) \int_{K_2} |u|^2 dr^*$$

$$\leq (\omega^2 + \epsilon m^2) |u|^2 (r_+) + \int \operatorname{Re}(2f \bar{H} u' + f' \bar{H} u) dr^*. \tag{78}$$

We now wish to reinstate the dropped indices m, ℓ, ω , and sum over m, ℓ and integrate over ω . Note that by the orthogonality of the $S_{m\ell}^{\omega}$, it follows that for any functions α and β with coefficients defined by

$$\hat{\alpha}(\omega,\cdot) = \sum_{m,\ell} \alpha_{m\ell}^{\omega}(r) S_{m\ell}(a\omega,\cos\theta) e^{im\phi^*}, \qquad \hat{\beta}(\omega,\cdot) = \sum_{m,\ell} \beta_{m\ell}^{\omega}(r) S_{m\ell}(a\omega,\cos\theta) e^{im\phi^*},$$

we have

$$\int \alpha^{2}(t^{*}, r, \theta, \varphi) \sin \theta d\varphi d\theta dt^{*} = \int_{-\infty}^{\infty} \sum_{m, \ell} |\alpha_{m\ell}^{\omega}(r)|^{2} d\omega,$$
$$\int \alpha \cdot \beta \sin \theta d\varphi d\theta dt^{*} = \int_{-\infty}^{\infty} \sum_{m, \ell} \alpha_{m\ell}^{\omega} \cdot \bar{\beta}_{m\ell}^{\omega} d\omega.$$

Clearly, the summed and integrated left hand side of (78) bounds

$$b \int_{-\infty}^{\infty} dt^* \int_{K_1} \left((\partial_r \psi_{\natural})^2 + \psi_{\natural}^2 \right) dV_g + b \int_{K_2} \sum_i (\partial_i \psi_{\natural})^2 dV_g.$$

Similarly, we read off immediately that the first term on the right hand side of (78) upon summation and integration yields precisely

$$\int_{\mathcal{H}_{+}} \left((T\psi_{\natural})^{2} + \epsilon (\partial_{\phi^{*}} \psi_{\natural})^{2} \right).$$

Note that we can bound

$$\int_{\mathcal{H}_{+}} \left((T\psi_{\natural})^{2} + \epsilon (\partial_{\phi^{*}}\psi_{\natural})^{2} \right) \leq \int_{\mathcal{H}^{+}} \left((T\psi_{\bowtie})^{2} + \epsilon (\partial_{\phi^{*}}\psi_{\bowtie})^{2} \right) \\
\leq B \int_{\Sigma_{\tau'}} J_{\mu}^{N}(\psi) n_{\Sigma}^{\mu} + \epsilon \int_{\mathcal{H}(\tau',\tau)} (\partial_{\phi^{*}}\psi)^{2}$$

(Exercise: Why?)

The "error term" of the right hand side of (78) is more tricky. To estimate

the second summand of the integrand, note that

$$\begin{split} \int_{-\infty}^{\infty} & \sum_{m,\ell:\lambda_{m\ell}(\omega) \geq \lambda_2 \omega^2} (f')_{m\ell}^{\omega}(r) \bar{F}_{m\ell}^{\omega}(r) \psi_{m\ell}^{\omega}(r) d\omega \\ & \leq & \int_{-\infty}^{\infty} \sum_{m,\ell:\lambda_{m\ell}(\omega) \geq \lambda_2 \omega^2} \delta^{-1} |(f')_{m\ell}^{\omega} F_{m\ell}^{\omega}|^2(r) + \delta |\psi_{m\ell}^{\omega}|^2 d\omega \\ & \leq & \int_{-\infty}^{\infty} \sum_{m,\ell:\lambda_{m\ell}(\omega) \geq \lambda_2 \omega^2} \delta^{-1} B |F_{m\ell}^{\omega}|^2(r) + \delta |\psi_{m\ell}^{\omega}|^2 d\omega \\ & = & \delta^{-1} B \int (F_{\natural})^2 \sin\theta \, d\phi \, d\theta \, dt^* + \delta \int (\psi_{\natural})^2 \sin\theta \, d\phi \, d\theta \, dt^* \\ & \leq & \delta^{-1} B \int F^2 \sin\theta \, d\phi \, d\theta \, dt^* + \delta \int \psi^2 \sin\theta \, d\phi \, d\theta \, dt^*, \end{split}$$

where δ can be chosen arbitrarily. In particular, this estimate holds for $r \leq R$. For $r \geq R$, in view of the fact that f is independent of ω , m, ℓ , we have in fact

$$\int_{-\infty}^{\infty} \sum_{m,\ell:\lambda_{m\ell}(\omega) \ge \lambda_2 \omega^2} (f')(r) \bar{F}_{m\ell}^{\omega}(r) \psi_{m\ell}^{\omega}(r) d\omega$$

$$= f'(r) \int_{-\infty}^{\infty} \sum_{m,\ell:\lambda_{m\ell}(\omega) \ge \lambda_2 \omega^2} \bar{F}_{m\ell}^{\omega}(r) \psi_{m\ell}^{\omega}(r) d\omega$$

$$= f'(r) \int F_{\natural} \psi_{\natural} \sin \theta \, d\phi \, d\theta \, dt^*$$

$$= f'(r) \int F_{\natural} \psi_{\Bbbk} \sin \theta \, d\phi \, d\theta \, dt^*,$$

where for the last line we have used (73). The first summand of the error integrand of (78) can be estimated similarly.

We thus obtain

$$b \int_{\mathcal{R}} \chi \left((\partial_{r} \psi_{\natural})^{2} + \psi_{\natural}^{2} \right) + b \int_{\mathcal{R}} \chi h J_{\mu}^{N} (\psi_{\natural}) N^{\mu}$$

$$\leq B \int_{\Sigma_{\tau'}} J_{\mu}^{N} (\psi) n_{\Sigma}^{\mu} + \epsilon \int_{\mathcal{H}(\tau',\tau)} (\partial_{\phi} \psi)^{2} + \delta^{-1} B \int_{\mathcal{R} \cap \{r \leq R\}} F^{2}$$

$$+ \delta \int_{\mathcal{R} \cap \{r \leq R\}} \psi^{2} + (\partial_{r} \psi)^{2}$$

$$+ \int_{-\infty}^{\infty} dt^{*} \int_{r \geq R} \left(2f(r^{2} + a^{2})^{1/2} F_{\natural} \partial_{r^{*}} ((r^{2} + a^{2})^{1/2} \psi_{\Longrightarrow}) \right)$$

$$+ f'(r^{2} + a^{2}) F_{\natural} \psi_{\Longrightarrow} \right) \frac{\Delta}{r^{2} + a^{2}} \sin \theta \, d\phi^{*} \, d\theta \, dr^{*}, \tag{79}$$

where χ is a cutoff which degenerates at infinity and h is a function $0 \le h \le 1$ which vanishes in a suitable neighborhood of r = 3M.

5.3.4 The untrapped frequencies

Given λ_2 sufficiently small and any choice of ω_1 , λ_1 , then, for a sufficiently small (where sufficiently small depends on these latter two constants), it follows that

for h = \flat , d, \sharp , we may produce currents of type $J_{\mu}^{\mathbf{X}, \mathbf{h}}$ as in Section 5.2.4 such that

 $b \int_{\mathcal{R}} \chi J_{\mu}^{N}(\psi_{\uparrow}) N^{\mu} + \tilde{\chi} \psi_{\uparrow} \leq \int_{\mathcal{R}} K^{\mathbf{X}_{\uparrow}}(\psi_{\uparrow})$

for χ a suitable cutoff function degenerating at infinity, and $\tilde{\chi}$ a suitable cutoff function degenerating at infinity and vanishing in a neighborhood of \mathcal{H}^+ . These currents can in fact be chosen independently of a for such small a, and moreover, they can be chosen so that, defining

$$\mathcal{E}^{\mathbf{X}} \mathbf{J} \doteq \nabla^{\mu} J_{\mu}^{\mathbf{X}} \mathbf{J} - K^{\mathbf{X}} \mathbf{J}$$

we have on the one hand

$$\int_{\mathcal{R} \cap \{r \ge R\}} \mathcal{E}^{\mathbf{X}} = \int_{-\infty}^{\infty} dt^* \int_{r \ge R} \left(2f(r^2 + a^2)^{1/2} F_{\mathbf{y}} \partial_{r^*} ((r^2 + a^2)^{1/2} \psi_{> <}) + f'(r^2 + a^2) F_{\mathbf{y}} \psi_{> <} \right) \frac{\Delta}{r^2 + a^2} \sin \theta \, d\phi^* \, d\theta \, dr^*$$

for the f of Section 5.3.3, and on the other hand, for the region $r \leq R$, we have

$$\int_{\mathcal{R} \cap \{r \le R\}} \mathcal{E}^{\mathbf{X}} \mathbf{1} \le B \delta^{-1} \int_{\mathcal{R} \cap \{r \le R\}} F^2 + B \delta \int_{\mathcal{R} \cap \{r \le R\}} \psi_{\mathbf{x}}^2 + (\partial_r \psi_{\mathbf{x}})^2 + \chi J_{\mu}^N (\psi_{\mathbf{x}}) n^{\mu} dx$$

where χ is supported near the horizon and away from a neighborhood of r = 3M. Moreover, one can show as in Section 5.2.6 that

$$-\int_{\mathcal{H}} J_{\mu}^{\mathbf{X}} (\psi_{\mathbf{J}}) n^{\mu} \leq -\int_{\mathcal{H}} J_{\mu}^{T} (\psi_{\mathbf{J}}) n^{\mu}$$

$$\leq -\int_{\mathcal{H}} J_{\mu}^{T} (\psi_{\mathbf{X}}) n^{\mu}$$

$$\leq B \int_{\Sigma_{\tau'}} J_{\mu}^{N} (\psi) n^{\mu}.$$

(Exercise: Prove the last inequality.)

From the identity

$$\int_{\mathcal{H}^+} J_{\mu}^{\mathbf{X}, \mathbf{I}}(\psi_{\mathbf{I}}) n_{\mathcal{H}}^{\mu} + \int_{\mathcal{R}} K^{\mathbf{X}, \mathbf{I}}(\psi_{\mathbf{I}}) = \int_{\mathcal{R}} \mathcal{E}^{\mathbf{X}, \mathbf{I}}(\psi_{\mathbf{I}})$$

and the above remarks, one obtains finally an estimate

$$\int_{\mathcal{R}} \chi(J_{\mu}^{N}(\psi_{\flat}) + J_{\mu}^{N}(\psi_{\dagger}) + J_{\mu}^{N}(\psi_{\sharp})) n_{\Sigma_{\tau}}^{\mu} \\
\leq B \int_{\Sigma_{\tau'}} J_{\mu}^{N}(\psi) n^{\mu} + B \delta^{-1} \int_{\mathcal{R} \cap \{r \leq R\}} F^{2} \\
+ B \delta \int_{\mathcal{R} \cap \{r \leq R\}} \psi^{2} + (\partial_{r} \psi)^{2} + \chi J_{\mu}^{N}(\psi) N^{\mu} \\
+ \int_{-\infty}^{\infty} dt^{*} \int_{r \geq R} \left(2f(r^{2} + a^{2})^{1/2} (F_{\flat} + F_{\dagger} + F_{\sharp}) \partial_{r^{*}} ((r^{2} + a^{2})^{1/2} \psi_{\thickapprox}) \right. \\
+ f'(r^{2} + a^{2}) (F_{\flat} + F_{\dagger} + F_{\sharp}) \psi_{\thickapprox} \left) \frac{\Delta}{r^{2} + a^{2}} \sin \theta \, d\phi^{*} \, d\theta \, dr^{*}. \tag{80}$$

5.3.5 The integrated decay estimates

Now, we will add (79), (80) and the energy identity of $eJ^Y(\psi)$

$$\int_{\tilde{\Sigma}_{\tau}} J_{\mu}^{N}(\psi) n_{\tilde{\Sigma}_{\tau}}^{\mu} + \int_{\tilde{\mathcal{R}}(\tau',\tau) \cap \{r \leq r_{0}\}} eK^{Y}(\psi)$$

$$= -\int_{\mathcal{H}(\tau',\tau)} eJ_{\mu}^{Y}(\psi) n_{\mathcal{H}}^{\mu} + \int_{\tilde{\mathcal{R}}(\tau',\tau) \cap \{r_{0} \leq r_{1} \leq r_{0}\}} eK^{Y}(\psi) + \int_{\tilde{\Sigma}_{\tau'}} J_{\mu}^{N}(\psi) n_{\tilde{\Sigma}_{\tau'}}^{\mu} \tag{81}$$

for a small e with $\epsilon \ll e$, and where $r_0 < r_1 < 3M$ are as in Corollary 3.1, and r_1 is in the support of K_2 .

In the resulting inequality, the left hand side bounds in particular

$$\int_{\mathcal{R}(\tau'+1,\tau-1)} \chi(hJ^N_\mu(\psi)N^\mu + (\partial_\tau\psi)^2) \tag{82}$$

where χ is a cutoff decaying at infinity, $\tilde{\chi}$ is a cutoff decaying at infinity and vanishing at \mathcal{H}^+ and h is a function with $0 \le h \le 1$ such that h vanishes precisely in a neighborhood of r = 3M. (As $a \to 0$, this neighborhood can be chosen smaller and smaller in the sense of the coordinate r.)

Let us examine the right hand side of the resulting inequality.

The second term of the first line of the right hand side of (79) is absorbed by the first term on the right hand side of (81) provided that $\epsilon \ll e$.

The third term of the first line of the right hand side of (79) and the second term of (80) are bounded by

$$B\delta^{-1} \int_{\Sigma_{\tau'}} J^N_{\mu}(\psi) n^{\mu}_{\Sigma_{\tau'}}$$

in view of Theorem 5.1.

The second line of the right side of (79) and the third term of (80) can be absorbed by (82), provided that δ is chosen suitably small, whereas the second term of the right hand side of (81) can be absorbed by (82), provided that e is sufficiently small.

The fourth terms of the right hand sides of (79) and (80) combine to yield

$$\int_{-\infty}^{\infty} dt^* \int_{r \ge R} \left(2f(r^2 + a^2)^{1/2} F \partial_{r^*} ((r^2 + a^2)^{1/2} \psi_{> <}) + f'(r^2 + a^2) F \psi_{> <} \right) \frac{\Delta}{r^2 + a^2} \sin \theta \, d\phi^* \, d\theta \, dr^*.$$

Note where F is supported and how it decays. Using our boundedness Theorem 5.1, a Hardy inequality and integration by parts we may now bound this term by

$$B\int_{\Sigma_{-\prime}}J^N_\mu(\psi)n^\mu_{\Sigma_\tau}.$$

But the remaining terms on the right hand side of (79), (80) and (81) are also of this form! We thus obtain

Proposition 5.3.1. There exists a φ_t -invariant weight χ , degenerating only at i_0 , a second φ_t -invariant weight $\tilde{\chi}$, degenerating at i_0 and vanishing at \mathcal{H}^+ , a third φ_t -invariant weight h, which vanishes on a neighborhood of r = 3M, and a constant B > 0 such that the following estimates hold for all $\tau' \leq \tau$,

$$\int_{\mathcal{R}(\tau',\tau)} \chi h J_{\mu}^{N}(\psi) N^{\mu} + \tilde{\chi} \psi^{2} \leq B \int_{\Sigma_{\tau'}} J_{\mu}^{N}(\psi) n_{\Sigma_{\tau'}}^{\mu}$$

$$\int_{\mathcal{R}(\tau',\tau)} \chi J_{\mu}^{N}(\psi) N^{\mu} + \tilde{\chi} \psi^{2} \leq B \int_{\Sigma_{\tau'}} (J_{\mu}^{N}(\psi) + J_{\mu}^{N}(T\psi)) n_{\Sigma_{\tau'}}^{\mu}$$

for all solutions $\square_q \psi = 0$ on Kerr.

Similar estimates could be shown on regions $\tilde{\mathcal{R}}(\tau',\tau)$, $\tilde{\Sigma}'_{\tau}$, after having derived a priori suitable decay of ψ in r.⁵⁵

5.3.6 The Z-estimate

To turn integrated decay as in Proposition 5.3.1 into decay of energy and pointwise decay, we must adapt the argument of Section 4.2.

Let V be a ϕ_t -invariant vector field such that $V = \partial_{t^*}$ for $r \geq r_+ + c_2$ and $V = \partial_{t^*} + (a/2Mr_+)\partial_{\phi^*}$ for $f \leq r_+ + c_1$ for some $c_1 < c_2$, and such that V is timelike in $\mathcal{R} \setminus \mathcal{H}^+$. Note that V is Killing except in $r_+ + c_1 \leq r \leq r_+ + c_2$. As $a \to 0$, we can construct such a V with c_2 arbitrarily small.

Now let us define u and v to be the Schwarzschild⁵⁶ coordinates

$$u = t - r_{\text{Schw}}^*,$$
$$v = t + r_{\text{Schw}}^*.$$

With respect to the coordinates (u, v, ϕ^*, θ) , defining $\underline{L} = \partial_u$, then \underline{L} vanishes smoothly along the horizon. Define $\overline{L} = V - \underline{L}$. Finally, define the vector field

$$Z = u^2 L + v^2 L.$$

Note that under these choices Z is null on \mathcal{H}^+ . With w as before, the currents $J^{Z,w}$ together with J^N can be used to control the energy fluxes on Σ_{τ} with weights. Use of the energy identities of $J^{Z,w}$ and J^N leads to estimates of the form

$$\int_{\Sigma_{\tau}} \chi \psi^2 + \int_{\Sigma_{\tau} \cap \{r \lesssim \tau\}} J^N_{\mu}(\psi) n^{\mu}_{\tilde{\Sigma}_{\tau}} \le B D \tau^{-2} + B \tau^{-2} \int_{\mathcal{R}(0,\tau)} \mathcal{E}, \tag{83}$$

where χ is a cutoff function supported suitably, and where \mathcal{E} is an error term arising from the part of $K^{Z,w}$ which has the "wrong" sign; D arises from data.

We may partition

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

where

 $^{^{55}}$ In the section that follows, we shall in fact localise the above estimate in a different way applying a cutoff function. The resulting 0'th order terms which arise can be controlled using the "good" 0'th order term in the boundary integrals of $J^{Z,w}$.

 $^{^{56} \}text{Recall}$ that we are considering both the Kerr and Schwarzschild metric on the fixed differentiable structure $\mathcal R$ as described in Section 5.1.

- \mathcal{E}_1 is supported in some region $r_0 \le r \le R_0$,
- \mathcal{E}_2 is supported in $r \leq r_0$ and
- \mathcal{E}_3 is supported in $r \geq R_0$.

Recall that $L+\underline{L}$ is Killing for $r \geq 2M+c_2$. It follows (**Exercise**) that choosing $c_2 < r_0$, there are no terms growing quadratically in t for \mathcal{E}_1 , \mathcal{E}_3 . Moreover, by our construction, Z depends smoothly on a away from the horizon. The behaviour near the horizon is more subtle as Z itself is not smooth! We shall return to this when discussing \mathcal{E}_2 .

In view of our above remarks. we have that

$$\mathcal{E}_1 \le B t(J_\mu^N(\psi)N^\mu + \psi^2),$$

just like in the case of Schwarzschild. In view of Proposition 5.3.1, this leads to the following estimate: If $\hat{\psi} = \psi$ in $\mathcal{R}(\tau', \tau'') \cap \{r \leq R_0\}$, where $\hat{\psi}$ solves again $\Box_a \hat{\psi} = 0$, then

$$\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}_1(\psi) = \int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}_1(\hat{\psi}) \le B\tau' \int_{\Sigma_{\tau'}} (J^N_{\mu}(\hat{\psi}) + J^N_{\mu}(T\hat{\psi})) n^{\mu}_{\tilde{\Sigma}_{\tau'}}. \tag{84}$$

The introduction of $\hat{\psi}$ is related to our localisation procedure we shall carry out in what follows.

Recall that in the Schwarzschild case, for R_0 suitably chosen, there is no \mathcal{E}_3 term, as the term $K^{Z,w}$ has a good sign in that region. (See Section 4.2.) Examining the r-decay of error terms in the smooth dependence of Z in a, we obtain

$$\mathcal{E}_3 \le \epsilon t r^{-2} J_{\mu}^N(\psi) N^{\mu}$$

where ϵ can be made arbitrarily small if a is small. If $\tau'' - \tau' \sim \tau' \sim t$, this leads to an estimate

$$\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}_{3}(\psi) \leq \epsilon(\tau'' - \tau')(\tau'' + \tau') \int_{\Sigma_{\tau'} \cap \{r \lesssim \tau'' - \tau'\}} J_{\mu}^{N}(\psi) n_{\Sigma_{\tau'}}^{\mu}
+ \epsilon \log |\tau'' - \tau'| \int_{\Sigma_{\tau'}} J_{\mu}^{N}(\psi) n_{\Sigma_{\tau'}}^{\mu}.$$
(85)

In the region $r_+ + c_1 \le r \le r_+ + c_2$, then, choosing r_0 such that \mathcal{E}_2 is absent in Schwarzschild, we can argue without computation from the smooth dependence on a that

$$\mathcal{E}_2 \le \epsilon t^2 (J_\mu^N(\psi) N^\mu + \psi^2)$$

where ϵ can be made arbitrarily small by choosing a small. The necessity of a quadratically growing error term arises from the fact that $L+\underline{L}$ is not Killing in this region.⁵⁷

As we have already mentioned, an important subtely occurs near the horizon \mathcal{H}^+ where Z fails to be C^1 . This means that \mathcal{E}_2 is not necessarily small in local coordinates, and one must understand how to bound the singular terms. It turns out that these singular terms have a structure:

 $^{^{57}}$ Alternatively, one can keep $L+\underline{L}$ Killing at the expense of Z failing to be causal on the horizon. This would lead to errors of a similar nature.

Proposition 5.3.2. Let \hat{V} , \hat{Y} , E_1 , E_2 extend V to a null frame in $r \leq r_+ + c_1$. We have

$$\mathcal{E}_2 \le \epsilon v |\log(r - r_+)|^p (\mathbf{T}(\hat{Y}, \hat{V}) + \mathbf{T}(\hat{V}, \hat{V})) + \epsilon v J_u^N(\psi) N^{\mu}.$$

Proof. The warping function w can be chosen as in Schwarzschild near \mathcal{H}^+ , and thus, the extra terms it generates are harmless. For the worst behaviour, it suffices to examine now K^Z itself. We must show that terms of the form:

$$|\log(r-r_+)|^p(T(\hat{Y},\hat{Y}))$$

do not appear in the computation for K^Z .

The relevant property follows from examining the covariant derivative of Z with respect to the null frame:

$$\nabla_{\hat{V}}Z = 2u(\hat{V}u)\underline{L} + 2v\hat{V}(v)L + v^2\nabla_{\hat{V}}V - 4r^*v\nabla_{\hat{V}}\underline{L} + 4(r^*)^2\nabla_{\hat{V}}\underline{L},$$

$$\nabla_{\hat{Y}}Z = 2u(\hat{Y}u)\underline{L} + 2v(\hat{Y}v)L + v^2\nabla_{\hat{Y}}V - 4r^*v\nabla_{\hat{Y}}\underline{L} + 4(r^*)^2\nabla_{\hat{Y}}\underline{L},$$

$$\nabla_{E_1}Z = 2u(E_1u)\underline{L} + 2v(E_1v)L + v^2\nabla_{E_1}V - 4r^*v\nabla_{E_1}\underline{L} + 4(r^*)^2\nabla_{E_1}\underline{L},$$

$$\nabla_{E_2}Z = 2u(E_2u)\underline{L} + 2v(E_2v)L + v^2\nabla_{E_2}V - 4r^*v\nabla_{E_2}\underline{L} + 4(r^*)^2\nabla_{E_2}\underline{L}.$$

To estimate now \mathcal{E}_2 , we first remark that with Proposition 5.3.1, we can obtain the following refinement of the red-shift multiplier construction of Corollary 3.1:

Proposition 5.3.3. If we weaken the requirement that N be smooth in Corollary 3.1 with the statement that N is C^0 at \mathcal{H}^+ and smooth away from \mathcal{H}^+ , then given $p \geq 0$, we may construct an N as in Corollary 3.1 where property 1 is replaced by the stronger inequality:

$$K^{N}(\psi) \geq b_{p} |\log(r - r_{+})|^{p} (\mathbf{T}(\hat{Y}, \hat{V}) + \mathbf{T}(\hat{V}, \hat{V}))$$

for $r \leq r_0$.

It now follows immediately from Proposition 5.3.1 that with ψ and $\hat{\psi}$ as before, we have

$$\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}_2(\psi) \le \epsilon(\tau')^2 \int_{\Sigma_{\tau'}} J^N_{\mu}(\hat{\psi}) n^{\mu}_{\Sigma_{\tau'}}.$$
 (86)

To obtain energy decay from (83), (85), (84) and (86), we argue now by continuity. Introduce the bootstrap assumptions

$$\int_{\Sigma_{\tau} \cap \{r \lesssim \tau\}} J_{\mu}^{N}(\psi) N^{\mu} + \chi \psi^{2} \le C D \tau^{-2+2\delta}, \tag{87}$$

$$\int_{\Sigma_{\tau} \cap \{r \preceq \tau\}} J_{\mu}^{N}(T\psi) N^{\mu} \le C' D\tau^{-1+2\delta}$$
(88)

for a $\delta > 0$.

Now dyadically decompose the interval $[0,\tau]$ by $\tau_i < \tau_{i+1}$. Using (84) and the above, we obtain

$$\int_{\mathcal{R}(0,\tau)} \mathcal{E}_{1}(\psi) \leq \sum_{i} \int_{\mathcal{R}(\tau_{i},\tau_{i+1})} \mathcal{E}_{1}(\psi)$$

$$\leq \sum_{i} \tau_{i} \int_{\Sigma_{\tau_{i}}} (J_{\mu}^{N}(\hat{\psi}) + J_{\mu}^{N}(T\hat{\psi})) N^{\mu}$$

$$\leq \sum_{i} \tau_{i} \int_{\Sigma_{\tau_{i}} \cap \{r \lesssim \tau_{i+1} - \tau_{i}\}} (J_{\mu}^{N}(\psi) + J_{\mu}^{N}(T\psi)) N^{\mu} + \chi \psi^{2}$$

$$\leq \sum_{i} \tau_{i} (\tau_{i}^{-2+2\delta} CD + \tau_{i}^{-1+2\delta} C'D)$$

$$\leq \delta^{-1} (CD\tau^{-1+2\delta} + C'D\tau^{2\delta}). \tag{89}$$

Here, $\hat{\psi}$ is constructed separately on each dyadic region $\mathcal{R}(\tau_i, \tau_{i+1})$ by throwing a cutoff on $\psi|_{\Sigma_{\tau_i}}$ equal to 1 in $r \lesssim \tau_{i+1} - \tau_i$ and vanishing in $\tau_{i+1} - \tau_i \lesssim r$, solving again the initial value problem in $\mathcal{R}(\tau_i, \tau_{i+1})$, and exploiting the domain of dependence property. See the original [65] for this localisation scheme. The parameters of the "dyadic" decomposition must be chosen accordingly for the constants to work out. Similarly, using (86) we obtain

$$\int_{\mathcal{R}(0,\tau)} \mathcal{E}_{2}(\psi) \leq \sum_{i} \int_{\mathcal{R}(\tau_{i},\tau_{i+1})} \mathcal{E}_{2}(\psi)$$

$$\leq \epsilon \sum_{i} \tau_{i}^{2} \int_{\Sigma_{\tau_{i}}} J_{\mu}^{N}(\hat{\psi}) N^{\mu}$$

$$\leq \epsilon \sum_{i} \tau_{i}^{2} \int_{\Sigma_{\tau_{i}} \cap \{r \lesssim \tau_{i+1} - \tau_{i}\}} J_{\mu}^{N}(\psi) N^{\mu} + \chi \psi^{2}$$

$$\leq \epsilon \sum_{i} \tau_{i}^{2} \tau_{i}^{-2+2\delta} CD$$

$$\leq \epsilon \delta^{-1} \tau^{2\delta} CD \tag{90}$$

and using (85)

$$\int_{\mathcal{R}(0,\tau)} \mathcal{E}_{3}(\psi) \leq \sum_{i} \int_{\mathcal{R}(\tau_{i},\tau_{i+1})} \mathcal{E}_{3}(\psi)$$

$$\leq \epsilon \sum_{i} \left(\tau_{i}^{2} \int_{\Sigma_{\tau_{i}}} J_{\mu}^{N}(\psi) N^{\mu} + \int_{\Sigma_{\tau_{i}}} J_{\mu}^{N}(\psi) n_{\Sigma_{\tau_{i}}}^{\mu} \right)$$

$$\leq \epsilon \sum_{i} \left(\tau_{i}^{2} \tau_{i}^{-2+2\delta} CD + D \log \tau' \right)$$

$$\leq \epsilon \delta^{-1} \tau^{2\delta} CD. \tag{91}$$

For $T\psi$ we obtain

$$\int_{\mathcal{R}(0,\tau)} \mathcal{E}_1(T\psi) \le BD\tau,\tag{92}$$

$$\int_{\mathcal{R}(0,\tau)} \mathcal{E}_{2}(T\psi) \leq \sum_{i} \int_{\mathcal{R}(\tau_{i},\tau_{i+1})} \mathcal{E}_{2}(T\psi)$$

$$\leq \epsilon \sum_{i} \tau_{i}^{2} \int_{\Sigma_{\tau_{i}}} J_{\mu}^{N}(T\hat{\psi}) N^{\mu}$$

$$\leq \epsilon \sum_{i} \tau_{i}^{2} \int_{\Sigma_{\tau_{i}} \cap \{r \lesssim \tau_{i+1} - \tau_{i}\}} J_{\mu}^{N}(T\psi) N^{\mu} + \chi(T\psi)^{2}$$

$$\leq \epsilon \sum_{i} \tau_{i}^{2} (\tau_{i}^{-1+2\delta} C'D + \tau_{i}^{-2+2\delta} CD)$$

$$\leq \epsilon \delta^{-1} \tau^{1+2\delta} C'D + \epsilon \delta^{-1} \tau^{2\delta} CD, \tag{93}$$

$$\int_{\mathcal{R}(0,\tau)} \mathcal{E}_{3}(T\psi) \leq \sum_{i} \int_{\mathcal{R}(\tau_{i},\tau_{i+1})} \mathcal{E}_{3}(T\psi)$$

$$\leq \epsilon \sum_{i} \left(\tau_{i}^{2} \int_{\Sigma_{\tau_{i}}} J_{\mu}^{N}(T\psi) N^{\mu} + \int_{\Sigma_{\tau_{i}}} J_{\mu}^{N}(T\psi) n_{\Sigma_{\tau_{i}}}^{\mu} \right)$$

$$\leq \epsilon \sum_{i} \left(\tau_{i}^{2} \tau_{i}^{-1+2\delta} C' D + D \log \tau_{i} \right)$$

$$\leq \epsilon \delta^{-1} \tau^{1+2\delta} C' D. \tag{94}$$

We use here the algebra of constants where $B\epsilon = \epsilon$. The constant D is a quantity coming from data. **Exercise**: What is D and why is (92) true?

For $\epsilon \ll \delta$ and C' sufficiently large, we see that from (83) applied to $T\psi$ in place of ψ , using (92), (93), we improve (88).

On the other hand choosing $C' \ll C$ and then τ sufficiently large, we have

$$\tau^{-2}\delta^{-1}(CD\tau^{-1+2\delta} + C'D\tau^{2\delta}) \le \frac{1}{2}CD\tau^{-2+2\delta}$$

and thus, again for $\epsilon \ll \delta$, using (89), (90) we can improve (87) from (83).

Once one obtains (87), then decay can be extended to decay in $\tilde{\Sigma}_{\tau}$ by the argument of Section 4.2, by applying conservation of the J^T flux backwards.⁵⁸

5.3.7 Pointwise bounds

In any region $r \leq R$, we may now obtain pointwise decay bounds simply by further commutation with T, N as in Section 3.3.4. To obtain the correct pointwise decay statement towards null infinity, one must also commute the equation with a basis Ω_i for the Lie algebra of the *Schwarzschild* metric, exploiting the r-weights of these vector fields. Defining $\tilde{\Omega}_i = \zeta(r)\Omega_i$, where ζ is a cutoff which vanishes for $r \leq R_0$, where $3M \ll R_0$, and, setting $\tilde{\psi} = \tilde{\Omega}\psi$, we have

$$\Box_g \tilde{\psi} = F_1 \partial^2 \psi + F_2 \partial \psi$$

⁵⁸Note that in view of the fact that we argued by continuity to obtain (87), we could not obtain this extended decay through $\tilde{\Sigma}_{\tau}$ earlier. This is why we have localised as in [65], not as in Section 4.2.

where $F_1 = O(r^{-2})$ and $F_2 = O(r^{-3})$. Having estimates already for ψ , $T\psi$, one can may apply the X and Z estimates as before for $\tilde{\psi}$, only, in view of the F_2 term, now one must exploit also the X-estimate in $D^+(\Sigma_{\tau_i} \cap \{r \lesssim \tau_{i+1} - \tau_i\}) \cap J^-(\Sigma_{\tau_{i+1}})$. We leave this as an **exercise**.

5.3.8 The decay theorem

We have obtained thus

Theorem 5.2. Let (\mathcal{M},g) be Kerr for $|a| \ll M$, \mathcal{D} be the closure of its domain of dependence, let Σ_0 be the surface $\mathcal{D} \cap \{t^* = 0\}$, let ψ , ψ' be initial data on Σ_0 such that $\psi \in H^s_{loc}(\Sigma)$, $\psi' \in H^{s-1}_{loc}(\Sigma)$ for $s \geq 1$, and $\lim_{x \to i^0} \psi = 0$, and let ψ be the corresponding unique solution of $\Box_g \psi = 0$. Let φ_τ denote the 1-parameter family of diffeomorphisms generated by T, let $\tilde{\Sigma}_0$ be an arbitrary spacelike hypersurface in $J^+(\Sigma_0 \setminus \mathcal{U})$ where \mathcal{U} is an open neighborhood of the asymptotically flat end⁵⁹, and define $\tilde{\Sigma}_\tau = \varphi_\tau(\tilde{\Sigma}_0)$. Let $s \geq 3$ and assume

$$E_1 \doteq \int_{\Sigma_0} r^2 (J_{\mu}^{n_0}(\psi) + J_{\mu}^{n_0}(T\psi) + J_{\mu}^{n_0}(TT\psi)) n_0^{\mu} < \infty.$$

Then there exists a $\delta>0$ depending on a (with $\delta\to 0$ as $a\to 0$) and a B depending only on $\tilde{\Sigma}_0$ such that

$$\int_{\tilde{\Sigma}_{\tau}} J^{N}(\psi) n_{\tilde{\Sigma}_{\tau}}^{\mu} \leq B E_{1} \tau^{-2+2\delta}.$$

Now let $s \ge 5$ and assume

$$E_2 \doteq \sum_{|\alpha| \le 2} \sum_{\Gamma = \{T, N, \Omega_i\}} \int_{\Sigma_0} r^2 (J_{\mu}^{n_0}(\Gamma^{\alpha}\psi) + J_{\mu}^{n_0}(\Gamma^{\alpha}T\psi) + J_{\mu}^{n_0}(\Gamma^{\alpha}TT\psi)) n_0^{\mu} < \infty$$

where Ω_i are the Schwarzschild angular momentum operators. Then

$$\sup_{\tilde{\Sigma}_{\tau}} \sqrt{r} |\psi| \le B \sqrt{E_2} \, \tau^{-1+\delta}, \qquad \sup_{\tilde{\Sigma}_{\tau}} r |\psi| \le B \sqrt{E_2} \, \tau^{(-1+\delta)/2}.$$

One can obtain decay for arbitrary derivatives, including transversal derivatives to \mathcal{H}^+ , using additional commutation by N. See [69].

5.4 Black hole uniqueness

In the context of the vacuum equations (4), the Kerr solution plays an important role not only because it is believed to be stable, but because it is believed to be the only stationary black hole solution.⁶⁰ This is the celebrated *no-hair* "theorem". In the case of the Einstein-Maxwell equations, there is an analogous no-hair "theorem" stating uniqueness for Kerr-Newman. A general reference is [92].

 $^{^{59} \}mathrm{This}$ is just the assumption that $\tilde{\Sigma}_0$ "terminates" on null infinity

 $^{^{60}}$ A further extrapolation leads to the "belief" that all vacuum solutions eventually decompose into n Kerr solutions moving away from each other.

Neither of these results is close to being a theorem in the generality which they are often stated. Reasonably definitive statements have only been proven in the much easier static case, and in the case where axisymmetry is assumed a priori and the horizon is assumed connected, i.e. that there is one black hole. Axisymmetry can be inferred from stationarity under various special assumptions, including the especially restrictive assumption of analyticity. See [57] for the latest on the analytic case, and [96] for new interesting results in the direction of removing the analyticity assumption in inferring axisymmetry from stationarity.

Nonetheless, the expectation that black hole uniqueness is true reasonably raises the question: why the interest in more general black holes, allowed in Theorem 5.1?

For a classical "astrophysical" motivation, note that black hole solutions can in principle exist in the presence of persistent atmospheres. Perhaps the simplest such constructions would involve solutions of the Einstein-Vlasov system, where matter is described by a distribution function on phase space invariant under geodesic flow. These black hole spacetimes would in general not be Kerr even in their vacuum regions. Recent speculations in high energy physics yield other possible motivations: There are now a variety of "hairy black holes" solving Einstein-matter systems for non-classical matter, like Yang-Mills fields [141], and a large variety of vacuum black holes in higher dimensions [77], many of which are currently the topic of intense study.

There is, however, a second type of reason, which is relevant even when we restrict our attention to the vacuum equations (4) in dimension 4. The less information one must use about the spacetime to obtain quantitative control on fields, the better chance one has at obtaining a stability theorem. The essentially non-quantitative⁶¹ aspect of our current limited understanding of black hole uniqueness should make it clear that these arguments probably will not have a place in a stability proof. Indeed, it would be an interesting problem to explore the possibility of obtaining a more quantitative version of uniqueness theorems (in a neighborhood of Kerr) following ideas in this section.

5.5 Comments and further reading

Theorem 5.1 was proven in [68]. In particular, this provided the first global result of any kind for general solutions of the Cauchy problem on a (non-Schwarzschild) Kerr background. Theorem 5.2 was first announced at the Clay Summer School where these notes were lectured. Results in the direction of Proposition 5.3.1 are independently being studied in work in progress by Tataru-Tohaneanu⁶² and Andersson-Blue⁶³.

The best previous results concerning Kerr had been obtained by Finster and collaborators in an important series of papers culminating in [79]. See also [80].

⁶¹As should be apparent by the role of analyticity or Carleman estimates.

 $^{^{62}\}mathrm{communication}$ from Mihai Tohaneanu, a summer school participant who attended these lectures

⁶³lecture of P. Blue, Mittag-Leffler, September 2008

The methods of [79] are spectral theoretic, with many pretty applications of contour integration and o.d.e. techniques. The results of [79] do not apply to general solutions of the Cauchy problem, however, only to individual azimuthal modes, i.e. solutions ψ_m of fixed m. In addition, [79] imposes the restrictive assumption that $\mathcal{H}^+ \cap \mathcal{H}^-$ not be in the support of the modes. (Recall the discussion of Section 3.2.6.) Under these assumptions, the main result stated in [79] is that

$$\lim_{t \to \infty} \psi_m(r, t) = 0 \tag{95}$$

for any $r > r_+$. Note that the reason that (95) did not yield any statement concerning general solutions, i.e. the sum over m-not even a non-quantitative one—is that one did not have a quantitative boundedness statement as in Theorem 5.1. Moreover, one should mention that even for fixed m, the results of [79] are in principle compatible with the statement

$$\sup_{\mathcal{H}^+} \psi_m = \infty,$$

i.e. that the azimuthal modes blow up along the horizon. See the comments in Section 4.6. It is important to note, however, that the statement of [79] need not restrict to $|a| \ll M$, but concerns the entire subextremal range |a| < M. Thus, the statement (95) of [79] is currently the only known global statement about azimuthal modes on Kerr spacetimes for large but subextremal a.

There has also been interesting work on the Dirac equation [78, 90], for which superradiance does not occur, and the Klein-Gordon equation [89]. For the latter, see also Section 8.3.

5.6 The nonlinear stability problem for Kerr

We have motivated these notes with the nonlinear stability problem of Kerr. Let us give finally a rough formulation.

Conjecture 5.1. Let (Σ, \bar{g}, K) be a vacuum initial data set (see Appendix B.2) sufficiently close (in a weighted sense) to the initial data on Cauchy hypersurface in the Kerr solution $(\mathcal{M}, g_{M,a})$ for some parameters $0 \leq |a| < M$. Then the maximal vacuum development (\mathcal{M}, g) possesses a complete null infinity \mathcal{I}^+ such that the metric restricted to $J^-(\mathcal{I}^+)$ approaches a Kerr solution $(\mathcal{M}, g_{M_f, a_f})$ in a uniform way (with respect to a foliation of the type $\tilde{\Sigma}_{\tau}$ of Section 4) with quantitative decay rates, where M_f , a_f are near M, a respectively.

Let us make some remarks concerning the above statement. Under the assumptions of the above conjecture, (\mathcal{M}, g) certainly contains a trapped surface S by Cauchy stability. By Penrose's incompleteness theorem (Theorem 2.2), this implies that (\mathcal{M}, g) is future causally geodesically incomplete. By the methods of the proof of Theorem 2.2, it is easy to see that $S \cap J^-(\mathcal{I}^+) = \emptyset$. Thus, as soon as \mathcal{I}^+ is shown to be complete, it would follow that the spacetime has a black hole region in the sense of Section 2.5.4.⁶⁴

⁶⁴Let us also remark the obvious fact that the above conjecture implies in particular that weak cosmic censorship holds in a neighborhood of Kerr data.

In view of this, one can also formulate the problem where the initial data are assumed close to Kerr initial data on an incomplete subset of a Cauchy hypersurface with one asymptotically flat end and bounded by a trapped surface. This is in fact the physical problem⁶⁵, but in view of Cauchy stability, it is equivalent to the formulation we have given above. Note also the open problem described in the last paragraph of Section 2.8.

In the spherically symmetric analogue of this problem where the Einstein equations are coupled with matter, or the Bianchi-triaxial IX vacuum problem discussed in Section 2.6.4, the completeness of null infinity can be inferred easily without detailed understanding of the geometry [60, 62]. One can view this as an "orbital stability" statement. In this spherically symmetric case, the asymptotic stability can then be studied a posteriori, as in [63, 94]. This latter problem is much more difficult.

In the case of Conjecture 5.1, in contrast to the symmetric cases mentioned above, one does *not* expect to be able to show any weaker stability statement than the asymptotic stability with decay rates as stated. Note that it is only the Kerr family as a whole–*not* the Schwarzschild subfamily–which is expected to be asymptotically stable: Choosing a=0 certainly does not imply that $a_f=0$. On the other hand, if $|a| \ll M$, then by the formulation of the above conjecture, it would follow that $|a_f| \ll M_f$. It is with this in mind that we have considered the $|a| \ll M$ case in this paper.

6 The cosmological constant Λ and Schwarzschildde Sitter

Another interesting setting for the study of the stability problem are black holes within cosmological spacetimes. Cosmological spacetimes—as opposed to asymptotically flat spacetimes (See Appendix B.2.3), which model spacetime in the vicinity of an isolated self-gravitating system—are supposed to model the whole universe. The working hypothesis of classical cosmology is that the universe is approximately homogeneous and isotropic (sometimes known as the Copernican principle [91]). In the Newtonian theory, it was not possible to formulate a cosmological model satisfying this hypothesis. ⁶⁶ One of the major successes of general relativity was that the theory allowed for such solutions, thus making cosmology into a mathematical science.

In the early years of mathematical cosmology, it was assumed that the universe should be static⁶⁷. To allow for such static cosmological solutions, Einstein

 $^{^{65}\}mathrm{Cf}.$ the comments on the relation between maximally-extended Schwarzschild and Oppenheimer-Snyder.

⁶⁶It is possible, however, if one geometrically reinterprets the Newtonian theory and allows space to be–say–the torus. See [132]. These reinterpretations, of course, postdate the formulation of general relativity.

⁶⁷much like in the early studies of asymptotically flat spacetimes discussed in Section 2.1

modified his equations (2) by adding a 0'th order term:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$
 (96)

Here Λ is a constant known as the cosmological constant. When coupled with a perfect fluid, this system admits a static, homogeneous, isotropic solution with $\Lambda > 0$ and topology $\mathbb{S}^3 \times \mathbb{R}$. This spacetime is sometimes called the *Einstein static universe*.

Cosmological solutions with various values of the parameter Λ were studied by Friedmann and Lemaitre, under the hypothesis of exact homogeneity and isotropy. Static solutions are in fact always unstable under perturbation of initial data. Typical homogeneous isotropic solutions expand or contract, or both, beginning and or ending in singular configurations. As with the early studies (referred to in Sections 2.2) illuminating the extensions of the Schwarzschild metric across the horizon, these were ahead of their time. (See the forthcoming book [123] for a history of this fascinating early period in the history of mathematical cosmology.) These predictions were taken more seriously with Hubble's observational discovery of the expansion of the universe, and the subsequent evolutionary theories of matter, but the relevance of the solutions near where they are actually singular was taken seriously only after the incompleteness theorems of Penrose and Hawking–Penrose were proven (see Section 2.7).

We shall not go into a general discussion of cosmology here, nor tell the fascinating story of the ups and downs of Λ -from its adoption by Einstein to his subsequent well-known rejection of it, to its later "triumphant" return in current cosmological models, taking a very small positive value, the "explanation" of which is widely regarded as one of the outstanding puzzles of theoretical physics. Rather, let us pass directly to the object of our study here, one of the simplest examples of an inhomogeneous "cosmological" spacetime, where nontrivial small scale structure occurs in an ambient expanding cosmology. This is the Schwarzschild–de Sitter solution.

6.1 The Schwarzschild-de Sitter geometry

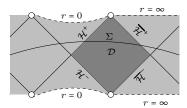
Again, this metric was discovered in local coordinates early in the history of general relativity, independently by Kottler [105] and Weyl [155]. Fixing $\Lambda > 0$, ⁶⁹ Schwarzschild-de Sitter is a one-parameter family of solutions of the from

$$-(1-2M/r-\Lambda r^3)dt^2 + (1-2M/r-\Lambda r^3)^{-1}dr^2 + r^2d\sigma_{\mathbb{S}^2}.$$
 (97)

⁶⁸In fact, the two are very closely related! The interior region of the Oppenheimer-Snyder collapsing star is precisely isometric to a region of a Friedmann universe. See [117].

⁶⁹The expression (97) with $\Lambda < 0$ defines Schwarzschild-anti-de Sitter. See Section 8.4.

The black hole case is the case where $0 < M < \frac{1}{3\sqrt{\Lambda}}$. A maximally-extended solution (see [28, 85]) then has as Penrose diagram the infinitely repeating chain:



To construct "cosmological solutions" one often takes spatially compact quotients. (One can also glue such regions into other cosmological spacetimes. See [56]. For more on the geometry of this solution, see [10].)

6.2 Boundedness and decay

The region "analogous" to the region studied previously for Schwarzschild and Kerr is the darker shaded region \mathcal{D} above. The horizon $\overline{\mathcal{H}}^+$ separates \mathcal{D} from an "expanding" region where the spacetime is similar to the celebrated de-Sitter space. If Σ is a Cauchy surface such that $\Sigma \cap \mathcal{H}^- = \Sigma \cap \overline{\mathcal{H}}^- = \emptyset$, then let us define $\Sigma_0 = \mathcal{D} \cap \Sigma$, and let us define Σ_τ to be the translates of Σ_0 by the flow φ_t generated by the Killing field $T = \frac{\partial}{\partial t}$. Note that, in contrast to the Schwarzschild or Kerr case, Σ_0 is compact.

We have

Theorem 6.1. The statement of Theorem 3.2 holds for these spacetimes, where Σ , Σ_0 , Σ_{τ} are as above, and $\lim_{x\to i^0} |\psi|$ is replaced by $\sup_{x\in\Sigma_0} |\psi|$.

Proof. The proof of the above theorem is as in the Schwarzschild case, except that in addition to the analogue of N, one must use a vector field \bar{N} which plays the role of N near the "cosmological horizon" $\bar{\mathcal{H}}^+$. It is a good **exercise** for the reader to think about the properties required to construct such a \bar{N} . A general construction of such a vector field applicable to all non-extremal stationary black holes is done in Section 7.

As for decay, we have

Theorem 6.2. For every $k \ge 0$, there exist constants C_k such that the following holds. Let $\psi \in H^{k+1}_{loc}$, $\psi' \in H^k_{loc}$, and define

$$E_k \doteq \sum_{|(\alpha)| \le k} \sum_{\Gamma = \{\Omega_i\}} \int_{\Sigma_0} J_{\mu}^{n_{\Sigma_{\tau}}} (\Gamma^{\alpha} \psi) n_{\Sigma_{\tau}}^{\mu}.$$

Then

$$\int_{\Sigma_{\tau}} J_{\mu}^{n_{\Sigma_{\tau}}}(\psi) n_{\Sigma_{\tau}}^{\mu} \le C_k E_k \tau^{-k}. \tag{98}$$

For k > 1 we have

$$\sup_{\Sigma_{\tau}} |\psi - \psi_0| \le C_k \sqrt{E_k \tau^{\frac{-k+1}{2}}}, \tag{99}$$

where ψ_0 denotes the 0'th spherical harmonic, for which we have for instance the estimate

$$\sup_{\Sigma_{\tau}} |\psi_0| \le \sup_{x \in \Sigma_0} \psi_0 + C_0 \sqrt{E_0(\psi_0, \psi_0')}. \tag{100}$$

The proof of this theorem uses the vector fields T, Y and \bar{Y} (alternatively N, \bar{N}), together with a version of X as multipliers, and requires commutation of the equation with Ω_i to quantify the loss caused by trapping. (Like Schwarzschild, the Schwarzschild-de Sitter metric has a photon sphere which is at r = 3M for all values of Λ in the allowed range. See [86] for a discussion of the optical geometry of this metric and its importance for gravitational lensing.) An estimate analogous to (39) is obtained, but without the χ weight, in view of the compactness of Σ_0 . The result of the Theorem follows essentially immediately, in view of Theorem 6.1 and a pigeonhole argument. No use need be made of a vector field of the type Z as in Section 4.2. Note that for ψ = constant, $E_k = 0$, so removing the 0'th spherical harmonic in (99) is necessary. See [66] for details.

Note that if Ω_i can be replaced by Ω_i^{ϵ} in (39), then it follows that the loss in derivatives for energy decay at any polynomial rate k in (98) can be made arbitrarily small. If Ω_i could be replaced by $\log \Omega_i$, then what would one obtain? (**Exercise**)

It would be a nice **exercise** to commute with \hat{Y} as in the proof of Theorem 6.1, to obtain pointwise decay for arbitrary derivatives of k. See the related exercise in Section 4.3 concerning improving the statement of Theorem 4.1.

6.3 Comments and further reading

Theorem 6.2 was proven in [66]. Independently, the problem of the wave equation on Schwarzschild-de Sitter has been considered in a nice paper of Bony-Häfner [24] using methods of scattering theory. In that setting, the presence of trapping is manifest by the appearance of resonances, that is to say, the poles of the analytic continuation of the resolvent.⁷⁰ The relevant estimates on the distribution of these necessary for the analysis of [24] had been obtained earlier by Sá Barreto and Zworski [135].

In contrast to Theorem 6.2, the theorem of Bony-Häfner [24] makes the familiar restrictive assumption on the support of initial data discussed in Section 3.2.5. For these data, however, the results of [24] obtain better decay than Theorem 6.2 away from the horizon, namely exponential, at the cost of only an ϵ derivative. The decay results of [24] degenerate at the horizon, in particular, they do not retrieve even boundedness for ψ itself. However, using the result of [24] together with the analogue of the red-shift Y estimate as used in the proof of Theorem 6.2, one can prove exponential decay up to and including

 $^{^{70}{\}rm In}$ the physics literature, these are known as $quasi\text{-}normal\ modes.}$ See [104] for a nice survey, as well as the discussion in Section 4.6.

the horizon, i.e. exponential decay in the parameter τ (Exercise). This still requires, however, the restrictive hypothesis of [24] concerning the support of the data. It would be interesting to sort out whether the restrictive hypothesis can be removed from [24], and whether this fast decay is stable to perturbation. There also appears to be interesting work in progress by Sá Barreto, Melrose and Vasy [150] on a related problem.

One should expect that the statement of Theorem 6.1 holds for the wave equation on axisymmetric stationary perturbations of Schwarzschild-de Sitter, in particular, slowly rotating Kerr-de Sitter, in analogy to Theorem 5.1.

Finally, we note that in many context, more natural than the wave equation is the conformally covariant wave equation $\Box_g \psi - \frac{1}{6}R\psi = 0$. For Schwarzschildde Sitter, this is then a special case of Klein-Gordon (106) with $\mu > 0$. The analogue of Theorem 6.1 holds by virtue of Section 7.2. **Exercise**: Prove the analogue of Theorem 6.2 for this equation.

7 Epilogue: The red-shift effect for non-extremal black holes

We give in this section general assumptions for the existence of vector fields Y and N as in Section 3.3.2. As an application, we can obtain the boundedness result of Theorem 3.2 or Theorem 6.1 for all classical non-extremal black holes for general nonnegative cosmological constant $\Lambda \geq 0$. See [91, 148, 28] for discussions of these solutions.

7.1 A general construction of vector fields Y and N

Recall that a Killing horizon is a null hypersurface whose normal is Killing [92, 148]. Let \mathcal{H} be a sufficiently regular Killing horizon with (future-directed) generator the Killing field V, which bounds a spacetime \mathcal{D} . Let φ_t^V denote the one-parameter family of transformations generated by V, assumed to be globally defined for all $t \geq 0$. Assume there exists a spatial hypersurface $\Sigma \subset \mathcal{D}$ transverse to V, such that $\Sigma \cap \mathcal{H} = S$ is a compact 2-surface. Consider the region

$$\mathcal{R}' = \cup_{t \ge 0} \varphi_t^V(\Sigma)$$

and assume that $\mathcal{R}' \cap \mathcal{D}$ is smoothly foliated by $\varphi_t(\Sigma)$.

Note that

$$\nabla_V V = \kappa V$$

for some function $\kappa: \mathcal{H} \to \mathbb{R}$.

Theorem 7.1. Let \mathcal{H} , \mathcal{D} , \mathcal{R}' , Σ , V, φ_t^V be as above. Suppose $\kappa > 0$. Then there exists a ϕ_t^V -invariant future-directed timelike vector field N on \mathcal{R}' and a constant b > 0 such that

$$K^N \geq b\,J^N_\mu N^\mu$$

in an open φ_t -invariant (for $t \geq 0$) subset $\tilde{\mathcal{U}} \subset \mathcal{R}'$ containing $\mathcal{H} \cap \mathcal{R}'$.

Proof. Define Y on S so that Y is future directed null, say

$$g(Y,V) = -2, (101)$$

and orthogonal to S. Moreover, extend Y off S so that

$$\nabla_Y Y = -\sigma(Y + V) \tag{102}$$

on S. Now push Y forward by φ_t^V to a vector field on \mathcal{U} . Note that all the above relations still hold on \mathcal{H} .

It is easy to see that the relations (19)–(22) hold as before, where E_1 , E_2 are a local frame for $T_p\varphi_t^V(S)$. Now a^1 , a^2 are not necessarily 0, hence our having included them in the original computation! We define as before

$$N = V + Y$$
.

Note that it is the compactness of S which gives the uniformity of the choice of b in the statement of the theorem.

We also have the following commutation theorem

Theorem 7.2. Under the assumptions of the above theorem, if ψ satisfies $\Box_g \psi = 0$, then for all $k \ge 1$.

$$\Box_g(Y^k\psi) = \kappa_k Y^k \psi + \sum_{0 \le |m| \le k, \ 0 \le m_4 < k} c_m E_1^{m_1} E_2^{m_2} T^{m_3} Y^{m_4} \psi$$

on \mathcal{H}^+ , where $\kappa_k > 0$.

Proof. From (19)–(22), we deduce that relative to the null frame (on the horizon) V, Y, E_1, E_2 the deformation tensor $^Y \pi$ takes the form

$$^{Y}\pi_{YY}=2\sigma,\quad ^{Y}\pi_{VV}=2\kappa,\quad ^{Y}\pi_{VY}=\sigma\quad ^{Y}\pi_{YE_{i}}=0,\quad ^{Y}\pi_{VE_{i}}=a^{i},\quad ^{Y}\pi_{E_{i}E_{j}}=h_{i}^{j}$$

As a result the principal part of the commutator expression—the term $2^{Y}\pi^{\alpha\beta}D_{\alpha}D_{\beta}\psi$ can be written as follows

$$2^{Y}\pi^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\psi = \kappa\nabla_{YY}^{2}\psi + \sigma(\nabla_{VV}^{2} + \nabla_{YV}^{2})\psi - a^{i}\nabla_{YE_{i}}^{2}\psi + 2h_{j}^{i}\nabla_{E_{i}E_{j}}^{2}\psi.$$

The result now follows by induction on k.

7.2 Applications

The proposition applies in particular to sub-extremal Kerr and Kerr-Newman, as well as to both horizons of sub-extremal Kerr-de Sitter, Kerr-Newman-de Sitter, etc. Let us give the following general, albeit somewhat awkward statement:

Theorem 7.3. Let (\mathcal{R}, g) be a manifold with stratified boundary $\mathcal{H}^+ \cup \Sigma$, such that \mathcal{R} is globally hyperbolic with past boundary the Cauchy hypersurface Σ , where Σ and \mathcal{H} are themselves manifolds with (common) boundary S. Assume

$$\mathcal{H}^+ = \cup_{i=1}^n \mathcal{H}_i^+, \qquad S = \cup_{i=1}^n S_i,$$

where the unions are disjoint and each \mathcal{H}_i^+ , S_i is connected. Assume each \mathcal{H}_i^+ satisfies the assumptions of Theorem 7.1 with future-directed Killing field V_i , some subset $\Sigma_i \subset \Sigma$, and cross section a connected component S_i of S. Let us assume there exists a Killing field T with future complete orbits, and φ_t is the one-parameter family of transformations generated by T. Let $\tilde{\mathcal{U}}_i$ be given by Theorem 7.1 and assume that there exists a \mathcal{V} as above such that

$$\mathcal{R} = \varphi_t(\Sigma \setminus \mathcal{V}) \cup \cup_{i=1}^n \tilde{\mathcal{U}}_i.$$

and

$$-g((\varphi_t^{V_i})_* n_{\Sigma}, n_{\Sigma_{\tau}}) \le B$$

where $\Sigma_{\tau} = \varphi_{\tau}(\Sigma)$, $\varphi_t^{V_i}$ represents the one-parameter family of transformations generated by V^i , and the last inequality is assumed for all values of t, τ where the left hand side can be defined. Finally, let ψ be a solution to the wave equation and assume that for any open neighborhood \mathcal{V} of S in Σ , there exists a positive constant $b_{\mathcal{V}} > 0$ such that

$$J_{\mu}^{T}(T^{k}\psi)n_{\Sigma}^{\mu} \ge b_{\mathcal{V}}J_{\mu}^{n_{\Sigma}}(T^{k}\psi)n_{\Sigma}^{\mu} \tag{103}$$

in $\Sigma \setminus \mathcal{V}$ and

$$T\psi = c_i V_i \psi \tag{104}$$

on \mathcal{H}_{i}^{+} . It follows that the first statement of Theorem 3.2 holds for ψ .

Assume in addition that Σ is compact or asymptotically flat, in the weak sense of the validity of a Sobolev estimate (11) near infinity. Then the second statement of Theorem 3.2 holds for ψ .

In the case where T is assumed timelike in $\mathcal{R} \setminus \mathcal{H}^+$, then (104) is automatic, whereas (103) holds if

$$-q(T,T) \ge -b_{\mathcal{V}} q(n_{\mathcal{U}},T)$$

in $\Sigma \setminus \mathcal{V}$. Thus we have

Corollary 7.1. The above theorem applies to Reissner-Nordström, Reissner-Nordström-de Sitter, etc, for all subextremal range of parameters. Thus Theorem 3.2 holds for all such metrics.⁷¹

On the other hand, (104), (103) can be easily seen to hold for axisymmetric solutions ψ_0 of $\Box_g \psi = 0$ on backgrounds in the Kerr family (see Section 5.2). We thus have

Corollary 7.2. The statement of Theorem 3.2 holds for axisymmetric solutions ψ_0 of for Kerr-Newman and Kerr-Newman-de Sitter for the full subextremal range of parameters.⁷²

Let us also mention that the the theorems of this section apply to the Klein-Gordon equation $\Box_q \psi = \mu^2 \psi$, as well as to the Maxwell equations (**Exercise**).

⁷¹In the $\Lambda = 0$ case this range is M > 0, $0 \le |Q| < M$. **Exercise**: What is it for $\Lambda > 0$?

⁷²In the $\Lambda = 0$ case this range is M > 0, $0 \le |Q| < \sqrt{M^2 - a^2}$. Exercise: What is it for $\Lambda > 0$?

8 Open problems

We end these notes with a discussion of open problems. Some of these are related to Conjecture 5.1, but all have independent interest.

8.1 The wave equation

The decay rates of Theorem 4.1 are sharp as uniform decay rates in v for any nontrivial class of initial data. On the other hand, it would be nice to obtain more decay in the interior, possibly under a stronger assumption on initial data.

Open problem 1. Show that there exists a $\delta > 0$ such that (31) holds with τ replaced with $\tau^{-2(1+\delta)}$, for a suitable redefinition of E_1 . Show the same thing for Kerr spacetimes with $|a| \ll M$.

At the very least, it would be nice to obtain this result for the energy restricted to $\tilde{\Sigma}_{\tau} \cap \{r \leq R\}$.

Recall how the algebraic structure of the Kerr solution is used in a fundamental way in the proof of Theorem 5.2. On the other hand, one would think that the validity of the results should depend only on the robustness of the trapping structure. This suggests the following

Open problem 2. Show the analogue of Theorem 5.2 for the wave equation on metrics close to Schwarzschild with as few as possible geometric assumptions on the metric.

For instance, can Theorem 5.2 be proven under the assumptions of Theorem 5.1? Under even weaker assumptions?

Our results for Kerr require $|a| \ll M$. Of course, this is a "valid" assumption in the context of the nonlinear stability problem, in the sense that if this condition is assumed on the parameters of the initial reference Kerr solution, one expects it holds for the final Kerr solution. Nonetheless, one certainly would like a result for all cases. See the discussion in Section 5.5.

Open problem 3. Show the analogue of Theorem 5.2 for Kerr solutions in the entire subextremal range $0 \le |a| < M$.

The extremal case |a| = M may be quite different in view of the fact that Section 7 cannot apply:

Open problem 4. Understand the behaviour of solutions to the wave equation on extremal Reissner-Nordström, extremal Schwarzschild-de Sitter, and extremal Kerr.

Turning to the case of $\Lambda > 0$, we have already remarked that the analogue of Theorems 6.1 and 6.2 should certainly hold in the case of Kerr-de Sitter. In the case of both Schwarzschild-de Sitter and Kerr-de Sitter, another interesting problem is to understand the behaviour in the region $\mathcal{C} = J^+(\overline{\mathcal{H}}_A^+) \cap J^+(\overline{\mathcal{H}}_B^+)$, where \mathcal{H}_A^+ , \mathcal{H}_B^+ are two cosmological horizons meeting at a sphere:

Open problem 5. Understand the behaviour of solutions to the wave equation in region C of Schwarzschild-de Sitter and Kerr-de Sitter, in particular, their behaviour along $r = \infty$ as i^+ is approached.

Let us add that in the case of cosmological constant, in some contexts it is appropriate to replace \Box_g with the conformally covariant wave operator $\Box_g - \frac{1}{6}R$. In view of the fact that R is constant, this is a special case of the Klein-Gordon equation discussed in Section 8.3 below.

8.2 Higher spin

The wave equation is a "poor man's" linearisation of the Einstein equations (4). The role of linearisation in the mathematical theory of nonlinear partial differential equations is of a different nature than that which one might imagine from the formal "perturbation" theory which one still encounters in the physics literature. Rather than linearising the equations, one considers the solution of the nonlinear equation from the point of view of a related linear equation that it itself satisfies.

In the case of the simplest nonlinear equations (say (107) discussed in Section 8.6 below), typically this means freezing the right hand side, i.e. treating it as a given inhomogeneous term. In the case of the Einstein equations, the proper analogue of this procedure is much more geometric. Specifically, it amounts to looking at the so called Bianchi equations

$$\nabla_{[\mu} R_{\nu\lambda]\rho\sigma} = 0, \tag{105}$$

which are already linear as equations for the curvature tensor when g is regarded as fixed. For more on this point of view, see [51]. The above equations for a field $S_{\lambda\mu\nu\rho}$ with the symmetries of the Riemann curvature tensor are in general known as the spin-2 equations. This motivates:

Open problem 6. State and prove the spin-2 version of Theorems 5.1 or 5.2 (or Open problem 1) on Kerr metric backgrounds or more generally, metrics settling down to Kerr.

In addition to [51], a good reference for these problems is [50], where this problem is resolved just for Minkowski space. In contrast to the case of Minkowski space, an additional difficulty in the above problem for the black hole setting arises from the presence of nontrivial stationary solutions provided by the curvature tensor of the solutions themselves. This will have to be accounted for in the statement of any decay theorem. From the "linearisation" point of view, the existence of stationary solutions is of course related to the fact that it is the 2-parameter Kerr family which is expected to be stable, not an individual solution.

8.3 The Klein-Gordon equation

Another important problem is the Klein-Gordon equation

$$\Box_g \psi = \mu \psi. \tag{106}$$

A large body of heuristic studies suggest the existence of a sequence of quasinormal modes (see Section 4.6) approaching the real axis from below in the Schwarzschild case. When the metric is perturbed to Kerr, it is thought that essentially this sequence "moves up" and produces exponentially growing solutions. See [158, 71]. This suggests

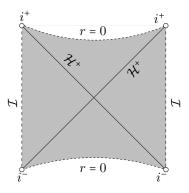
Open problem 7. Construct an exponentially growing solution of (106) on Kerr, for arbitrarily small $\mu > 0$ and arbitrary small a.

Interestingly, if one fixed m, then adapting the proof of Section 5.2, one can show that for $\mu > 0$ sufficiently small and a sufficiently small, depending on m, the statement of Theorem 5.1 holds for (106) for such Kerr's. This is consistent with the quasinormal mode picture, as one must take $m \to \infty$ for the modes to approach the real axis in Schwarzschild. This shows how misleading fixed-m results can be when compared to the actual physical problem.

8.4 Asymptotically anti-de Sitter spacetimes

In discussing the cosmological constant we have considered only the case $\Lambda > 0$. This is the case of current interest in cosmology. On the other hand, from the completely different viewpoint of high energy physics, there has been intense interest in the case $\Lambda < 0$. See [84].

The expression (97) for $\Lambda < 0$ defines a solution known as *Schwarzschild-anti-de Sitter*. A Penrose diagramme of this solution is given below.



The timelike character of infinity means that this solution is not globally hyperbolic. As with Schwarzschild-de Sitter, Schwarzschild-anti-de Sitter can be viewed as a subfamily of a larger Kerr-anti de Sitter family, with similar properties.

Again, as with Schwarzschild-de Sitter, the role of the wave equation is in some contexts replaced by the conformally covariant wave equation. Note that this corresponds to (106) with a negative $\mu = 2\Lambda/3 < 0$.

Even in the case of anti-de Sitter space itself (set M = 0 in (97)), the question of the existence and uniqueness of dynamics is subtle in view of the timelike character of the ideal boundary \mathcal{I} . It turns out that dynamics are unique for

(106) only if the $\mu \ge 5\Lambda/12$, whereas for the total energy to be nonnegative one must have $\mu \ge 3\Lambda/4$. Under our conventions, the conformally covariant wave equation lies between these values. See [6, 26].

Open problem 8. For suitable ranges of μ , understand the boundedness and blow-up properties for solutions of (106) on Schwarzschild-anti de Sitter and Kerr-anti de Sitter.

See [109, 27] for background.

8.5 Higher dimensions

All the black hole solutions described above have higher dimensional analogues. See [77, 120]. These are currently of great interest from the point of view of high energy physics.

Open problem 9. Study all the problems of Sections 8.1–8.4 in dimension greater than 4.

Higher dimensions also brings a wealth of explicit black hole solutions such that the topology of spatial sections of \mathcal{H}^+ is no longer spherical. In particular, in 5 spacetime dimensions there exist "black string" solutions, and much more interestingly, asymptotically flat "black ring" solutions with horizon topology $S^1 \times S^2$. See [77].

Open problem 10. Investigate the dynamics of the wave equation $\Box_g \psi = 0$ and related equations on black ring backgrounds.

8.6 Nonlinear problems

The eventual goal of this subject is to study the global dynamics of the Einstein equations (4) themselves and in particular, to resolve Conjecture 5.1.

It may be interesting, however, to first look at simpler non-linear equations on fixed black hole backgrounds and ask whether decay results of the type proven here are sufficient to show non-linear stability.

The simplest non-linear perturbation of the wave equation is

$$\Box_a \psi = V'(\psi) \tag{107}$$

where V = V(x) is a potential function. Aspects of this problem on a Schwarzschild background have been studied by [121, 64, 22, 115].

Open problem 11. Investigate the problem (107) on Kerr backgrounds.

In particular, in view of the discussion of Section 8.3, one may be able to construct solutions of (107) with $V = \mu \psi^2 + |\psi|^p$, for $\mu > 0$ and for arbitrarily large p, arising from arbitrarily small, decaying initial data, which blow up in finite time. This would be quite interesting.

A nonlinear problem with a stronger relation to (4) is the wave map problem. Wave maps are maps $\Phi : \mathcal{M} \to \mathcal{N}$ where \mathcal{M} is Lorentzian and \mathcal{N} is Riemannian, which are critical points of the Lagrangian

$$\mathcal{L}(\Phi) = \int |d\Phi|_{g_N}^2$$

In local coordinates, the equations take the form

$$\Box_{g_M} \Phi^k = -\Gamma^k_{ij} g_M^{\alpha\beta} (\partial_\alpha \Phi^i \partial_\beta \Phi^j),$$

where Γ_{ij}^k denote the Christoffel symbols of g_N . See the lecture notes of Struwe [145] for a nice introduction.

Open problem 12. Show global existence in the domain of outer communications for small data solutions of the wave map problem, for arbitrary target manifold \mathcal{N} , on Schwarzschild and Kerr backgrounds.

All the above problems concern fixed black hole backgrounds. One of the essential difficulties in proving Conjecture 5.1 is dealing with a black hole background which is not known a priori, and whose geometry must thus be recovered in a bootstrap setting. It would be nice to have more tractable model problems which address this difficulty. One can arrive at such problems by passing to symmetry classes. The closest analogue to Conjecture 5.1 in such a context is perhaps provided by the results of Holzegel [94], which concern the dynamic stability of the 5-dimensional Schwarzschild as a solution of (4), restricted under Triaxial Bianchi IX symmetry. See Section 2.6.4. In the symmetric setting, one can perhaps attain more insight on the geometric difficulties by attempting a large-data problem. For instance

Open problem 13. Show that the maximal development of asymptotically flat triaxial Bianchi IX vacuum initial data for the 5-dimensional vacuum equations containing a trapped surface settles down to Schwarzschild.

The analogue of the above statement has in fact been proven for the Einstein-scalar field system under spherical symmetry [40, 63]. In the direction of the above, another interesting set of problems is provided by the Einstein-Maxwell-charged scalar field system under spherical symmetry. For both the charged-scalar field system and the Bianchi IX vacuum system, even more ambitious than Open problem 13 would be to study the strong and weak cosmic censorship conjectures, possibly unifying the analysis of [45, 58, 59]. Discussion of these open problems, however, is beyond the scope of the present notes.

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A Lorentzian geometry

The reader who wishes a formal introduction to Lorentzian geometry can consult [91]. For the reader familiar with the concepts and notations of Riemannian geometry, the following remarks should suffice for a quick introduction.

A.1 The Lorentzian signature

Lorentzian geometry is defined as in Riemannian geometry, except that the metric g is not assumed positive definite, but of signature $(-,+,\ldots,+)$. That is to say, we assume that at each point $p \in \mathcal{M}^{n+1}$, 73 we may find a basis \mathbf{e}_i of the tangent space $T_p\mathcal{M}$, $i=0,\ldots,n$, such that

$$g = -\mathbf{e}_0 \otimes \mathbf{e}_0 + \mathbf{e}_1 \otimes \mathbf{e}_1 + \cdots + \mathbf{e}_n \otimes + \mathbf{e}_n.$$

In Riemannian geometry, the – in the first term on the right hand side would by +.

A non-zero vector $v \in T_p \mathcal{M}$ is called *timelike*, *spacelike*, or *null*, according to whether g(v,v) < 0, g(v,v) > 0, or g(v,v) = 0. Null and timelike vectors collectively are known as *causal*. There are various conventions for the 0-vector. Let us not concern ourselves with such issues here.

The appellations timelike, spacelike, null are inherited by vector fields and immersed curves by their tangent vectors, i.e. a vector field V is timelike if V(p) is timelike, etc., and a curve γ is timelike if $\dot{\gamma}$ is timelike, etc. On the other hand, a submanifold $\Sigma \subset \mathcal{M}$ is said to be spacelike if its induced geometry is Riemannian, timelike if its induced geometry is Lorentzian, and null if its induced geometry is degenerate. (Check that these two definitions coincide for embedded curves.) For a codimension-1 submanifold $\Sigma \subset \mathcal{M}$, at every $p \in \mathcal{M}$, there exists a non-zero normal n^{μ} , i.e. a vector in $T_p\mathcal{M}$ such that g(n,v) = 0 for all $v \in T_p\Sigma$. It is easily seen that Σ is spacelike iff n is timelike, Σ is timelike iff n is spacelike, and Σ is null iff n is null. Note that in the latter case $n \in T_p\Sigma$. The normal of Σ is thus tangent to Σ .

A.2 Time-orientation and causality

A time-orientation on (\mathcal{M}, g) is defined by an equivalence class [K] where K is a continuous timelike vector field, where $K_1 \sim K_2$ if $g(K_1, K_2) < 0$. A Lorentzian manifold admitting a time-orientation is called time-orientable, and a

⁷³It is conventional to denote the dimension of the manifold by n+1.

triple $(\mathcal{M}, g, [K])$ is said to be a *time-oriented* Lorentzian manifold. Sometimes one reserves the use of the word "spacetime" for such triples. In any case, we shall always consider time-oriented Lorentzian manifolds and often drop explicit mention of the time orientation.

Given this, we may further partition causal vectors as follows. A causal vector v is said to be future-pointing if g(v,K) < 0, otherwise past-pointing, where K is a representative for the time orientation. As before, these names are inherited by causal curves, i.e. we may now talk of a future-directed timelike curve, etc. Given p, we define the causal future $J^+(p)$ by

$$J^+(p) = p \cup \{q \in \mathcal{M} : \exists \gamma : [0,1] \to \mathcal{M} : \dot{\gamma} \text{ future-pointing, causal}\}$$

Similarly, we define $J^-(p)$ where future is replaced by past in the above. If $S \subset \mathcal{M}$ is a set, then we define

$$J^{\pm}(S) = \cup_{p \in S} J^{\pm}(p).$$

A.3 Covariant derivatives, geodesics, curvature

The standard local notions of Riemannian geometry carry over. In particular, one defines the Christoffel symbols

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\alpha}(\partial_{\nu}g_{\alpha\lambda} + \partial_{\lambda}g_{\nu\alpha} - \partial_{\alpha}g_{\nu\lambda}),$$

and geodesics $\gamma(t) = (x^{\alpha}(t))$ are defined as solutions to

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\lambda} \dot{x}^{\nu} \dot{x}^{\lambda} = 0.$$

Here $g_{\mu\nu}$ denote the components of g with respect to a local coordinate system x^{μ} , $g^{\mu\nu}$ denotes the components of the inverse metric, and we are applying the Einstein summation convention where repeated upper and lower indices are summed. The Christoffel symbols allow us to define the *covariant derivative* on (k,l) tensor fields by

$$\nabla_{\lambda}A^{\nu_{1}\dots\nu_{k}}_{\mu_{1}\dots\mu_{\ell}}=\partial_{\lambda}A^{\nu_{1}\dots\nu_{k}}_{\mu_{1}\dots\mu_{\ell}}+\sum_{i=1}^{k}\Gamma^{\nu_{i}}_{\lambda\rho}A^{\nu_{1}\dots\rho\dots\nu_{k}}_{\mu_{1}\dots\mu_{\ell}}-\sum_{i=1}^{l}\Gamma^{\rho}_{\lambda\mu_{i}}A^{\nu_{1}\dots\nu_{k}}_{\mu_{1}\dots\rho\dots\mu_{\ell}}$$

where it is understood that ρ replaces ν_i , μ_i , respectively in the two terms on the right. This defines (k, l+1) tensor. As usual, if we contract this with a vector v at p, then we will denote this operator as ∇_v and we note that this can be defined in the case that the tensor field is defined only on a curve tangent to v at p. We may thus express the geodesic equation as

$$\nabla_{\dot{\gamma}}\dot{\gamma}=0.$$

The Riemann curvature tensor is given by

$$R^{\mu}_{\nu\lambda\rho} \doteq \partial_{\lambda}\Gamma^{\mu}_{\rho\nu} - \partial_{\rho}\Gamma^{\mu}_{\lambda\nu} + \Gamma^{\alpha}_{\rho\nu}\Gamma^{\mu}_{\lambda\alpha} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\mu}_{\rho\alpha},$$

and the Ricci and scalar curvatures by

$$R_{\mu\nu} \doteq R^{\alpha}_{\mu\alpha\nu}, \qquad R \doteq g^{\mu\nu} R_{\mu\nu}.$$

Using the same letter R to denote all these tensors is conventional in relativity, the number of indices indicating which tensor is being referred to. For this reason we will avoid writing "the tensor R". The expression R without indices will always denote the scalar curvature. As usual, we shall also use the letter R with indices to denote the various manifestations of these tensors with indices raised and lowered by the inverse metric and metric, e.g.

$$R_{\mu\nu\lambda\rho} = g_{\mu\alpha} R^{\alpha}_{\nu\lambda\rho}$$

Note the important formula

$$\nabla_{\alpha}\nabla_{\beta}Z_{\mu} - \nabla_{\alpha}\nabla_{\beta}Z_{\mu} = R_{\sigma\mu\alpha\beta}Z^{\sigma}$$

We say that an immersed curve $\gamma: I \to \mathcal{M}$ is *inextendible* if there does not exist an immersed curve $\tilde{\gamma}: J \to \mathcal{M}$ where $J \supset I$ and $\tilde{\gamma}|_{I} = \gamma$.

We say that (\mathcal{M}, g) is geodesically complete if for all inextendible geodesics $\gamma: I \to \mathcal{M}$, then $I = \mathbb{R}$. Otherwise, we say that it is geodesically incomplete. We can similarly define the notion of spacelike geodesic (in)completeness, timelike geodesic completeness, causal geodesic completeness, etc, by restricting the definition to such geodesics. In the latter two cases, we may further specialise, e.g. to the notion of future causal geodesic completeness, by replacing the condition $I = \mathbb{R}$ with $I \supset (a, \infty)$ for some a.

We say that a spacelike hypersurface $\Sigma \subset \mathcal{M}$ is Cauchy if every inextendible causal curve in \mathcal{M} intersects it precisely once. A spacetime (\mathcal{M}, g) admitting such a hypersurface is called globally hyperbolic. This notion was first introduced by Leray [112].

B The Cauchy problem for the Einstein equations

We outline here for reference the basic framework of the Cauchy problem for the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$
 (108)

Here Λ is a constant known as the *cosmological constant* and $T_{\mu\nu}$ is the so-called energy momentum tensor of matter. We will consider mainly the vacuum case

$$R_{\mu\nu} = \Lambda g_{\mu\nu},\tag{109}$$

where the system closes in itself. If the reader wants to set $\Lambda=0$, he should feel free to do so. To illustrate the case of matter, we will consider the example of a scalar field.

B.1 The constraint equations

Let Σ be a spacelike hypersurface in (\mathcal{M}, g) , with future directed unit timelike normal N. By definition, Σ inherits a Riemannian metric from g. On the other hand, we can define the so-called second fundamental form of Σ to be the symmetric covariant 2-tensor in $T\Sigma$ defined by

$$K(u,v) = -g(\nabla_u V, N)$$

where V denotes an arbitrary extension of v to a vector field along Σ , and ∇ here denotes the connection of g. As in Riemannian geometry, one easily shows that the above indeed defines a tensor on $T\Sigma$, and that it is symmetric.

Suppose now (\mathcal{M}, g) satisfies (108) with some tensor $T_{\mu\nu}$. With Σ as above, let \bar{g}_{ab} , $\bar{\nabla}$, K_{ab} denote the induced metric, connection, and second fundamental form, respectively, of Σ . Let barred quantities and Latin indices refer to tensors, curvature, etc., on Σ , and let $\Pi_a^{\nu}(p)$ denote the components of the pullback map $T^*\mathcal{M} \to T^*\Sigma$. It follows that

$$\bar{R} + (K_a^a)^2 - K_b^a K_a^b = 16\pi T_{\mu\nu} n^{\mu} n^{\nu} + 2\Lambda, \tag{110}$$

$$\nabla_b K_a^b - \nabla_a K_b^b = 16\pi \,\Pi_a^{\nu} T_{\mu\nu} n^{\mu}. \tag{111}$$

To see this, one derives as in Riemannian geometry the Gauss and Codazzi equations, take traces, and apply (108).

B.2 Initial data

It is clear that (110), (111) are necessary conditions on the induced geometry of a spacelike hypersurface Σ so as to arise as a hypersurface in a spacetime satisfying (108). As we shall see, immediately, they will also be sufficient conditions for solving the initial value problem.

B.2.1 The vacuum case

Let Σ be a 3-manifold, \bar{g} a Riemannian metric on Σ , and K a symmetric covariant 2-tensor. We shall call (Σ, \bar{g}, K) a vacuum initial data set with cosmological constant Λ if (110)–(111) are satisfied with $T_{\mu\nu} = 0$. Note that in this case, equations (110)–(111) refer only to Σ, \bar{g}, K .

B.2.2 The case of matter

Let us here provide only the case for the Einstein-scalar field case. Here, the system is (108) coupled with

$$\Box_a \psi = 0, \tag{112}$$

$$T_{\mu\nu} = \partial_{\mu}\psi \partial_{\nu}\psi - \frac{1}{2}g_{\mu\nu}\nabla^{\alpha}\psi \nabla_{\alpha}\psi. \tag{113}$$

First note that were Σ a spacelike hypersurface in a spacetime (\mathcal{M}, g) satisfying the Einstein-scalar field system with massless scalar field ψ , and n^{μ} were the future-directed normal, then setting $\psi' = n^{\mu} \partial_{\mu} \phi$, $\psi = \phi|_{\Sigma}$ we have

$$T_{\mu\nu}n^{\mu}n^{\nu} = \frac{1}{2}((\psi')^2 + \bar{\nabla}^a\psi\bar{\nabla}_a\psi),$$
$$\Pi_a^{\nu}T_{\mu\nu}n^{\mu} = \psi'\bar{\nabla}_a\psi,$$

where latin indices and barred quantities refer to Σ and its induced metric and connection.

This motivates the following: Let Σ be a 3-manifold, \bar{g} a Riemannian metric on Σ , K a symmetric covariant 2-tensor, and $\psi : \Sigma \to \mathbb{R}$, $\psi' : \Sigma \to \mathbb{R}$ functions. We shall call (Σ, \bar{g}, K) an Einstein-scalar field initial data set with cosmological constant Λ if (110)–(111) are satisfied replacing $T_{\mu\nu}n^{\mu}n^{\nu}$ with $\frac{1}{2}((\psi')^2 + \bar{\nabla}^a\psi\bar{\nabla}_a\psi)$, and replacing $\Pi^{\nu}_a T_{\mu\nu}n^{\mu}$ with $\psi'\bar{\nabla}_a\psi$.

Note again that with the above replacements the equations (110)–(111) do not refer to an ambient spacetime \mathcal{M} . See [36] for the construction of solutions to this system.

B.2.3 Asymptotic flatness and the positive mass theorem

The study of the Einstein constraint equations is non-trivial!

Let us refer in this section to a triple (Σ, \bar{g}, K) where Σ is a 3-manifold, \bar{g} a Riemannian metric, and K a symmetric two-tensor on Σ as an *initial data* set, even though we have not specified a particular closed system of equations. An initial data set (Σ, \bar{g}, K) is strongly asymptotically flat with one end if there exists a compact set $K \subset \Sigma$ and a coordinate chart on $\Sigma \setminus K$ which is a diffeomorphism to the complement of a ball in \mathbb{R}^3 , and for which

$$g_{ab} = \left(1 + \frac{2M}{r}\right)\delta_{ab} + o_2(r^{-1}), \qquad k_{ab} = o_1(r^{-2}),$$

where δ_{ab} denotes the Euclidean metric and r denotes the Euclidean polar coordinate.

In appropriate units, M is the "mass" measured by asymptotic observers, when comparing to Newtonian motion in the frame δ_{ab} . On the other hand, under the assumption of a global coordinate system well-behaved at infinity, M can be computed by integration of the t_0^0 component of a certain pseudotensor⁷⁴ added to T_0^0 . In this manifestation, the quantity E = M is known as the total energy.⁷⁵ This relation was first studied by Einstein and is discussed in Weyl's

 $^{^{74}}$ This is subtle: The Einstein vacuum equations arise from the Hilbert Lagrangian $\mathcal{L}(g) = \int R$ which is 2nd order in the metric. In local coordinates, the highest order term is a divergence, and the Lagrangian can thus be replaced by a new Lagrangian which is 1st order in the metric. The resultant Lagrangian density, however, is no longer coordinate invariant. The quantity t_0^0 now arises from "Noether's theorem" [124]. See [49] for a nice discussion.

⁷⁵With the above asymptotics, the so-called linear momentum vanishes. Thus, in this case "mass" and energy are equivalent.

book Raum-Zeit-Materie [154]. In one looks at E for a family of hypersurfaces with the above asymptotics, then E is conserved.

A celebrated theorem of Schoen-Yau [137, 138] (see also [156]) states

Theorem B.1. Let (Σ, \bar{g}, K) be strongly asymptotically flat with one end and satisfy (110), (111) with $\Lambda = 0$, and where $T_{\mu\nu}n^{\mu}n^{\nu}$, $\Pi_a^{\nu}T_{\mu\nu}n^{\mu}$ are replaced by the scalar μ and the tensor J_a , respectively, defined on Σ , such that moreover $\mu \geq \sqrt{J^a J_a}$. Suppose moreover the asymptotics are strengthened by replacing $o_2(r^{-1})$ by $O_4(r^{-2})$ and $o_1(r^{-2})$ by $O_3(r^{-3})$. Then $M \geq 0$ and M = 0 iff Σ embeds isometrically into \mathbb{R}^{3+1} with induced metric \bar{g} and second fundamental form K.

The assumption $\mu \ge \sqrt{J^a J_a}$ holds if the matter satisfies the dominant energy condition [91]. In particular, it holds for the Einstein scalar field system of Section B.2.2, and (of course) for the vacuum case. The statement we have given above is weaker than the full strength of the Schoen-Yau result. For the most general assumptions under which mass can be defined, see [9].

One can define the notion of strongly asymptotically flat with k ends by assuming that there exists a compact \mathcal{K} such that $\Sigma \setminus \mathcal{K}$ is a disjoint union of k regions possessing a chart as in the above definition. The Cauchy surface Σ of Schwarzschild of Kerr with $0 \leq |a| < M$, can be chosen to be strongly asymptotically flat with 2-ends. The mass of both ends coincides with the parameter M of the solution.

The above theorem applies to this case as well for the parameter M associated to any end. If M=0 for one end, then it follows by the rigidity statement that there is only one end. Note why Schwarzschild with M<0 does not provide a counterexample.

The association of "naked singularities" with negative mass Schwarzschild gave the impression that the positive energy theorem protects against naked singularities. This is not true! See the examples discussed in Section 2.6.2.

In the presence of black holes, one expects a strengthening of the lower bound on mass in Theorem B.1 to include a term related to the square root of the area of a cross section of the horizon. Such inequalities were first discussed by Penrose [127] with the Bondi mass in place of the mass defined above. All inequalities of this type are often called *Penrose inequalities*. It is not clear what this term should be, as the horizon is only identifyable after global properties of the maximal development have been understood. Thus, one often replaces this area in the conjectured inequality with the area of a suitably defined apparent horizon. Such a statement has indeed been obtained in the so-called Riemannian case (corresponding to K=0) where the relevant notion of apparent horizon coincides with that of minimal surface. See the important papers of Huisken-Ilmanen [95] and Bray [25].

B.3 The maximal development

Let (Σ, \bar{g}, K) denote a smooth vacuum initial data set with cosmological constant Λ . We say that a smooth spacetime (\mathcal{M}, g) is a smooth development of

initial data if

- 1. (\mathcal{M}, g) satisfies the Einstein vacuum equations (4) with cosmological constant Λ .
- 2. There exists a smooth embedding $i: \Sigma \to \mathcal{M}$ such that (\mathcal{M}, g) is globally hyperbolic with Cauchy surface $i(\Sigma)$, and \bar{g} , K are the induced metric and second fundamental form, respectively.

The original local existence and uniqueness theorems were proven in 1952 by Choquet-Bruhat [33].⁷⁶ In modern language, they can be formulated as follows

Theorem B.2. Let (Σ, \bar{g}, K) be as in the statement of the above theorem. Then there exists a smooth development (\mathcal{M}, g) of initial data.

Theorem B.3. Let \mathcal{M} , $\widetilde{\mathcal{M}}$ be two smooth developments of initial data. Then there exists a third development \mathcal{M}' and isometric embeddings $j: \mathcal{M}' \to \mathcal{M}$, $\tilde{j}: \mathcal{M}' \to \widetilde{\mathcal{M}}$ commuting with i, \tilde{i} .

Application of Zorn's lemma, the above two theorems and simple facts about Lorentzian causality yields:

Theorem B.4. (Choquet-Bruhat-Geroch [35]) Let (Σ, \bar{g}, K) denote a smooth vacuum initial data set with cosmological constant Λ . Then there exists a unique development of initial data (\mathcal{M}, g) satisfying the following maximality statement: If $(\widetilde{\mathcal{M}}, \widetilde{g})$ satisfies (1), (2) with embedding \tilde{i} , then there exists an isometric embedding $j: \widetilde{\mathcal{M}} \to \mathcal{M}$ such that j commutes with \tilde{i} .

The spacetime (\mathcal{M}, g) is known as the maximal development of (Σ, \bar{g}, K) . The spacetime $\mathcal{M} \cap J^+(\Sigma)$ is known as the maximal future development and $\mathcal{M} \cap J^-(\Sigma)$ the maximal past development.

We have formulated the above theorems in the class of smooth initial data. They are of course proven in classes of finite regularity. There has been much recent work in proving a version of Theorem B.2 under minimal regularity assumptions. The current state of the art requires only $\bar{g} \in H^{2+\epsilon}$, $K \in H^{1+\epsilon}$. See [102].

We leave as an **exercise** formulating the analogue of Theorem B.4 for the Einstein-scalar field system (108), (112), (113), where the notion of initial data set is that given in Section B.2.2.

B.4 Harmonic coordinates and the proof of local existence

The statements of Theorems B.2 and B.3 are coordinate independent. Their proofs, however, require fixing a gauge which determines the form of the metric functions in coordinates from initial data. The classic gauge is the so-called harmonic gauge⁷⁷. Here the coordinates x^{μ} are required to satisfy

$$\Box_g x^\mu = 0. \tag{114}$$

 $^{^{76}\}mathrm{Then}$ called Fourès-Bruhat.

 $^{^{77}}$ also known as wave coordinates

Equivalently, this gauge is characterized by the condition

$$g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0. \tag{115}$$

A linearised version of these coordinates was used by Einstein [74] to predict gravitational waves. It appears that de Donder [70] was the first to consider harmonic coordinates in general. These coordinates are discussed extensively in the book of Fock [82].

The result of Theorem B.3 actually predates Theorem B.2, and in some form was first proven by Stellmacher [143]. Given two developments (\mathcal{M}, g) , $(\widetilde{\mathcal{M}}, \tilde{g})$ one constructs for each harmonic coordinates x^{μ} , \tilde{x}^{μ} adapted to Σ , such that $g_{\mu\nu} = \tilde{g}_{\mu\nu}$, $\partial_{\lambda}g_{\mu\nu} = \partial_{\lambda}\tilde{g}_{\mu\nu}$ along Σ . In these coordinates, the Einstein vaccum equations can be expressed as

$$\Box_g g^{\mu\nu} = Q^{\mu\nu,\alpha\beta}_{\iota\kappa\lambda\rho\sigma\tau} g^{\iota\kappa} \,\partial_\alpha g^{\lambda\rho} \,\partial_\beta g^{\sigma\tau} \tag{116}$$

for which uniqueness follows from general results of Schauder [136]. This theorem gives in addition a domain of dependence property.⁷⁸

Existence for solutions of the system (116) with smooth initial data would also follow from the results of Schauder [136]. This does not immediately yield a proof of Theorem B.2, because one does not have a priori the spacetime metric g so as to impose (114) or (115)! The crucial observation is that if (115) is true "to first order" on Σ , and g is defined to be the unique solution to (116), then (115) will hold, and thus, g will solve (108). Thus, to prove Theorem B.2, it suffices to show that one can arrange for (115) to be true "to first order" initially. Choquet-Bruhat [33] showed that this can be done precisely when the constraint equations (110)–(111) are satisfied with vanishing right hand side. Interestingly, to obtain existence for (116), Choquet-Bruhat's proof [33] does not in fact appeal to the techniques of Schauder [136], but, following Sobolev, rests on a Kirchhoff formula representation of the solution. Recently, new representations of this type have found applications to refined extension criteria [103].

An interesting feature of the classical existence and uniqueness proofs is that Theorem B.3 requires more regularity than Theorem B.2. This is because solutions of (114) are a priori only as regular as the metric. This difficulty has recently been overcome in [129].

B.5 Stability of Minkowski space

The most celebrated global result on the Einstein equations is the stability of Minkowski space, first proven in monumental work of Christodoulou and Klainerman [51]:

⁷⁸There is even earlier work on uniqueness in the analytic category going back to Hilbert, appealing to Cauchy-Kovalevskaya. Unfortunately, nature is not analytic; in particular, one cannot infer the domain of dependence property from those considerations.

Theorem B.5. Let (Σ, \bar{g}, K) be a strongly asymptotically flat vacuum initial data set, assumed sufficiently close to Minkowski space in a weighted sense. Then the maximal development is geodesically complete, and the spacetime approaches Minkowski space (with quantitative decay rates) in all directions. Moreover, a complete future null infinity \mathcal{I}^+ can be attached to the spacetime such that $J^-(\mathcal{I}^+) = \mathcal{M}$.

The above theorem also allows one to rigorously define the laws of gravitational radiation. These laws are nonlinear even at infinity. Theorem B.5 led to the discovery of Christodoulou's memory effect [42].

A new proof of a version of stability of Minkowski space using harmonic coordinates has been given in [113]. This has now been extended in various directions in [34]. The original result [51] was extended to the Maxwell case in the Ph.D. thesis of Zipser [157]. Bieri [13] has very recently given a proof of a version of stability of Minkowski space under weak asymptotics and regularity assumptions, following the basic setup of [51].

There was an earlier semi-global result of Friedrich [83] where initial data were prescribed on a hyperboloidal initial hypersurface meeting \mathcal{I}^+ .

A common misconception is that it is the positivity of mass which is somehow responsible for the stability of Minkowski space. The results of [113] for this are very telling, for they apply not only to the Einstein-vacuum equations, but also to the Einstein-scalar field system of Section B.2.2, including the case where the definition of the energy-momentum tensor (113) is replaced with its negative. Minkowski space is then not even a local minimizer for the mass functional in the class of perturbations allowed! Nonetheless, by the results of [113], Minkowski space is still stable in this context.

Another point which cannot be overemphasized: It is essential that the smallness in (B.5) concern a weighted norm. Compare with the results of Section 2.8.

Stability of Minkowski space is the only truly global result on the maximal development which has been obtained for asymptotically flat initial data without symmetry. There are a number of important results applicable in cosmological settings, due to Friedrich [83], Andersson-Moncrief [3], and most recently Ringstrom [134].

Other than this, our current global understanding of solutions to the Einstein equations (in particular all work on the cosmic censorship conjectures) has been confined to solutions under symmetry. We have given many such references in the asymptotically flat setting in the course of Section 2. The cosmological setting is beyond the scope of these notes, but we refer the reader to the recent review article and book of Rendall [132, 133] for an overview and many references.

C The divergence theorem

Let (\mathcal{M}, g) be a spacetime, and let Σ_0 , Σ_1 be homologous spacelike hypersurfaces with common boundary, bounding a spacetime region \mathcal{B} , with $\Sigma_1 \subset J^+(\Sigma_0)$.

Let n_0^{μ} , n_1^{μ} denote the future unit normals of Σ_0 , Σ_1 respectively, and let P_{μ} denote a one-form. Under our convention on the signature, the divergence theorem takes the form

 $\int_{\Sigma_1} P_{\mu} n_1^{\mu} + \int_{\mathcal{B}} \nabla^{\mu} P_{\mu} = \int_{\Sigma_0} P_{\mu} n_0^{\mu}, \tag{117}$

where all integrals are with respect to the induced volume form.

This is defined as follows. The volume form of spacetime is

$$\sqrt{-\det g}dx^0\dots dx^n$$

where det g denotes the determinant of the matrix $g_{\alpha\beta}$ in the above coordinates. The induced volume form of a spacelike hypersurface is defined as in Riemannian geometry.

We will also consider the case where (part of) Σ_1 is null. Then, we choose arbitrarily a future directed null generator n_1^{Σ} for Σ_1 arbitrarily and define the volume element so that the divergence theorem applies. For instance the divergence theorem in the region $\mathcal{R}(\tau',\tau'')$ (described in the lectures) for an arbitrary current P_{μ} then takes the form

$$\int_{\Sigma_{\tau''}} P_\mu n^\mu_{\Sigma_{\tau''}} + \int_{\mathcal{H}(\tau',\tau'')} P_\mu n^\mu_{\mathcal{H}} + \int_{\mathcal{R}(\tau',\tau'')} \nabla^\mu P_\mu = \int_{\Sigma_{\tau'}} P_\mu n^\mu_{\Sigma_{\tau'}},$$

where the volume elements are as described.

Note how the form of this theorem can change depending on sign conventions regarding the directions of the normal, the definition of the divergence and the signature of the metric.

D Vector field multipliers and their currents

Let ψ be a solution of

$$\Box_a \psi = 0 \tag{118}$$

on a Lorentzian manifold (\mathcal{M}, g) . Define

$$T_{\mu\nu}(\psi) = \partial_{\mu}\psi\partial_{\nu}\psi - \frac{1}{2}g_{\mu\nu}\partial^{\alpha}\psi\partial_{\alpha}\psi \tag{119}$$

We call $T_{\mu\nu}$ the energy-momentum tensor of ψ .⁷⁹ Note the symmetry property

$$T_{\mu\nu} = T_{\nu\mu}$$
.

The wave equation (118) implies

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{120}$$

Given a vector field V^{μ} , we may define the associated currents

$$J_{\mu}^{V}(\psi) = V^{\nu} T_{\mu\nu}(\psi) \tag{121}$$

⁷⁹Note that this is the same expression that appears on the right hand side of (108) in the Einstein-scalar field system. See Section B.2.2.

$$K^V = {}^V \pi_{\mu\nu} T^{\mu\nu}(\psi) \tag{122}$$

where π^X is the deformation tensor defined by

$$X_{\mu\nu} = \frac{1}{2} \nabla_{(\mu} X_{\nu)} = \frac{1}{2} (\mathcal{L}_X g)_{\mu\nu}.$$

The identity (120) gives

$$\nabla^{\mu} J_{\mu}^{V}(\psi) = K^{V}(\psi).$$

Note that $J^{V}_{\mu}(\psi)$ and $K^{V}(\psi)$ both depend only on the 1-jet of ψ , yet the latter is the divergence of the former. Applying the divergence theorem (117), this allows one to relate quantities of the same order.

The existence of a tensor $T_{\mu\nu}(\psi)$ satisfying (120) follows from the fact that equation (118) derives from a Lagrangian of a specific type. These issues were first systematically studied by Noether [124]. For more general such Lagrangian theories, two currents J_{μ} , K with $\nabla^{\mu}J_{\mu} = K$, both depending only on the 1-jet, but not necessarily arising from $T_{\mu\nu}$ as above, are known as *compatible currents*. These have been introduced and classified by Christodoulou [46].

E Vector field commutators

Proposition E.0.1. Let ψ be a solution of the equation of the scalar equation

$$\Box_g \psi = f$$
,

and X be a vectorfield. Then

$$\Box_{g}(X\psi) = X(f) - 2^{X} \pi^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \psi - 2 \left(2(\nabla^{\alpha X} \pi_{\alpha\mu}) - (\nabla_{\mu}^{X} \pi_{\alpha}^{\alpha}) \right) \nabla^{\mu} \psi.$$

Proof. To show this we write

$$X(\Box_g \psi) = \mathcal{L}_X(g^{\alpha\beta} \nabla_\alpha \nabla_\beta \psi) = 2^X \pi^{\alpha\beta} \nabla_\alpha \nabla_\beta \psi + g^{\alpha\beta} \mathcal{L}_X(\nabla_\alpha \nabla_\beta \psi).$$

Furthermore,

$$\mathcal{L}_{X}(\nabla_{\alpha}\nabla_{\beta}\psi) - \nabla_{\alpha}\mathcal{L}_{X}\nabla_{\beta}\psi = 2\left(\left(\nabla_{\beta}^{X}\pi_{\alpha\mu}\right) - \left(\nabla_{\mu}^{X}\pi_{\beta\alpha}\right) + \left(\nabla_{\alpha}^{X}\pi_{\mu\beta}\right)\right)\nabla^{\mu}\psi$$

and

$$\mathcal{L}_X \nabla_{\beta} \psi = \nabla_X \nabla_{\beta} \psi + \nabla_{\beta} X^{\mu} \nabla_{\mu} \psi = \nabla_{\beta} (X \psi).$$

F Some useful Schwarzschild computations

In this section, (\mathcal{M}, g) refers to maximal Schwarzschild with M > 0, $\mathcal{Q} = \mathcal{M}/SO(3)$, \mathcal{I}^{\pm} , $J^{\mp}(\mathcal{I}^{\pm})$ are as defined in Section 2.4.

F.1 Schwarzschild coordinates (r,t)

The coordinates are (r,t) and the metric takes the form

$$-(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2d\sigma_{\mathbb{S}^2}$$

These coordinates can be used to cover any of the four connected components of $Q \setminus \mathcal{H}^{\pm}$. In particular, the region $J^{-}(\mathcal{I}_{A}^{+}) \cap J^{+}(\mathcal{I}_{A}^{-})$ (where \mathcal{I}_{A}^{\pm} correspond to a pair of connected components of \mathcal{I}^{\pm} sharing a limit point in the embedding) is covered by a Schwarzschild coordinate system where $2M < r < \infty, -\infty < t < \infty$. Note that r has an invariant characterization namely $r(x) = \sqrt{\operatorname{Area}(S)/4\pi}$ where S is the unique group orbit of the SO(3) action containing x.

The hypersurface $\{t=c\}$ in the Schwarzschild coordinate region $J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$ extends regularly to a hypersurface with boundary in \mathcal{M} where the boundary is precisely $\mathcal{H}^+ \cap \mathcal{H}^-$.

The coordinate vector field ∂_t is Killing (and extends to the globally defined Killing field T).

In a slight abuse of notation, we will often extend Schwarzschild coordinate notation to \mathcal{D} , the closure of $J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$. For instance, we may talk of the vector field ∂_t "on" \mathcal{H}^{\pm} , or of $\{t=c\}$ having boundary $\mathcal{H}^+ \cap \mathcal{H}^-$, etc.

F.2 Regge-Wheeler coordinates (r^*, t)

Here t is as before and

$$r^* = r + 2M\log(r - 2M) - 3M - 2M\log M \tag{123}$$

and the metric takes the form

$$-(1-2M/r)(-dt^2+(dr^*)^2)+r^2d\sigma_{\mathbb{S}^2}$$

where r is defined implictly by (123). A coordinate chart defined in $-\infty < r^* < \infty$, $-\infty < t < \infty$ covers $J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$.

The constant renormalisation of the coordinate is taken so that $r^* = 0$ at the photon sphere, where r = 3M.

Note the explicit form of the wave operator

$$\Box_g \psi = -(1 - 2M/r)^{-1} (\partial_t^2 \psi - r^{-2} \partial_{r^*} (r^2 \partial_{r^*} \psi)) + \nabla^A \nabla_A \psi$$

where ∇ denotes the induced covariant derivative on the group orbit spheres.

Similar warnings of abuse of notation apply, for instance, we may write $\partial_t = \partial_{r*}$ on \mathcal{H}^+ .

F.3 Double null coordinates (u, v)

Our convention is to define

$$u = \frac{1}{2}(t - r^*),$$

 $^{^{80}\}mathrm{Compare}$ with the Minkowski case M=0 where the SO(3) action is of course not unique.

$$v = \frac{1}{2}(t + r^*).$$

The metric takes the form

$$-4(1-2M/r)dudv + r^2d\sigma_{\mathbb{S}^2}$$

and $J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$ is covered by a chart $-\infty < u < \infty, -\infty < v < \infty$.

The usual comments about abuse of notation hold, in particular, we may now parametrize $\mathcal{H}^+ \cap \mathcal{D}$ with $\{\infty\} \times [-\infty, \infty)$ and similarly $\mathcal{H}^- \cap \mathcal{D}$ with $(-\infty, \infty] \times \{-\infty\}$, and write $\partial_v(-\infty, v) = \partial_t(-\infty, v)$, $\partial_u(-\infty, v) = 0$.

Note that the vector field $(1 - 2M/r)^{-1}\partial_u$ extends to a regular vector null field across $\mathcal{H}^+ \setminus \mathcal{H}^-$. Thus, with the basis ∂_v , $(1 - 2M/r)^{-1}\partial_u$, one can choose regular vector fields near $\mathcal{H}^+ \setminus \mathcal{H}^-$ without changing to regular coordinates. In practice, this can be convenient.

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