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A Local Energy Estimate on Kerr Black Hole Backgrounds

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We study dispersive properties for the wave equation in the Kerr space–time with small angular momentum. The main result of this paper is to establish uniform energy bounds and local energy decay for such backgrounds. This follows similar results for the Schwarzschild space–time proved earlier in [3], [8], and [16] and extended in earlier work [29] of the authors and collaborators.

1 Introduction

The Schwarzschild space–times are the unique spherically symmetric solutions to the vacuum Einstein equations, modeling stationary black holes. This family of solutions is parametrized by the mass $M > 0$, which can be viewed as a scaling parameter. In the limit $M \to 0$, one obtains the Minkowski space–time.

The Kerr space–times are stationary axisymmetric solutions to the vacuum Einstein equation, modeling rotating black holes. They are parametrized by the mass M, which is again a scaling parameter, and by the angular momentum per unit mass a. When $a = 0$, one recovers the Schwarzschild space–time. If a is small, $|a| \ll M$, then one can formally think of the Kerr metric as a small perturbation of the Schwarzschild metric.

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There are also other related models which correspond to the vacuum Einstein equation with a nonzero cosmological constant, namely the Schwarzschild–de Sitter and Kerr–de Sitter space–times (with positive cosmological constant), respectively the Schwarzschild–anti-de Sitter and Kerr–anti-de Sitter space–times (with negative cosmological constant).

Geometrically, in each of these models one can identify an *exterior region*, which is bounded by the *event horizon*. The event horizon is a null surface, which in suitable coordinates can be viewed as the union of two half infinite cylinders originating on a common sphere called the bifurcate sphere. These two cylinders separate the exterior region from a white hole in the past and the black hole in the future. Null geodesics can travel from the white hole into the exterior region and from the exterior region into the black hole, but not the other way around.

Another common geometric feature is the existence of a family of trapped null geodesics within the exterior region. In the Schwarzschild case this family consists of all geodesics tangent to a stationary sphere called the *photon sphere*. Thus, the family of trapped null geodesics has dimension 3. In the Kerr case one still has a three-dimensional family of trapped null geodesics, but this family is no longer located on a single sphere. However, if the angular momentum is small, $|a| \ll M$, then the trapped set stays close to a sphere. Its geometry is more complicated but, due to the complete integrability of the null geodesic flow (see [10]), it is still analytic and can be explicitly described.

The last geometric feature of these space–times is their geometry at spatial infinity. Both Schwarzschild and Kerr are asymptotically flat, so in many respects they behave like small perturbations of the Minkowski space–time. However, in the de Sitter and anti-de Sitter cases one encounters a different geometry, which is beyond the scope of the present paper.

For general information about the geometry of the Schwarzschild/Kerr type space–times, we refer the reader to [11] and [21].

The decay properties of solutions to the linear wave equation in Schwarzschild or Kerr space–times are interesting both intrinsically and as a stepping stone toward the understanding of the stability of the Schwarzschild/Kerr space–time as solutions to the vacuum Einstein equation. There are several useful ways of measuring the decay. Perhaps the most robust one is the local energy decay, that is, the averaged decay of the energy in a compact set in space. Another type of averaged decay is provided by the Strichartz type estimates. Both the local energy decay and the Strichartz estimates are translation invariant in time, and only require finite energy initial data. On the other hand, if one assumes that the initial data have some stronger decay at infinity then one can look for uniform pointwise decay of the solutions in a compact set.

Local energy decay estimates in the Minkowski space–time originate in the work of Morawetz [33]. There are many similar results obtained in the case of perturbations of the Minkowski space–time; see, for example, [33], [24], [25], [37],[39], [40], [1], and [31]. Relevant to us is the case of small long range perturbations of the Minkowski space–time, considered in [32].

Strichartz estimates are well understood in the Minkowski space–time; we refer the reader to the survey paper [20]. However, the case of small perturbations of the Minkowski space–time turned out to be considerably more difficult, and was settled only recently in [32].

Pointwise decay estimates in the Minkowski space–time mirror the decay properties of the fundamental solution. For certain small perturbations of Minkowski, a very useful idea turned out to be the vector field method, introduced by Klainerman and collaborators (see, e.g., [26]). For further references we refer the reader to the monographs [38] and [22].

Let us turn our attention now to the Schwarzschild and Kerr space–times. Near infinity they are small perturbations of the Minkowski space–time, therefore one would expect a similar behavior. However, within a compact spatial region one needs to deal with a new phenomenon, namely trapping; the definition of "compact" here is related to the choice of coordinates. The trapping occurs in two separate ways:

i) Near the event horizon. Indeed, since the event horizon is a null surface, it follows that null geodesics originating on the event horizon and traveling in a tangent direction will remain on the event horizon. This is a two-dimensional family of trapped rays.
ii) Near the photon sphere. This is a four-dimensional family of trapped rays that stay away from the event horizon.

In general, trapping can completely destroy the decay properties for solutions to the wave equation. However, heuristically this is not expected to occur in the Schwarzschild and Kerr cases. This happens for different reasons for the two trapped families of null geodesics. On the one hand, near the event horizon waves are subject to what in the physics literature is called the *red shift effect*, and decay exponentially in the high-frequency approximation. On the other hand, the trapped null geodesics near the photon sphere are hyperbolic, therefore in the high-frequency limit the energy can stay trapped there only up to logarithmic time scales (the so-called Ehrenfest time). Understanding how exactly to take advantage of these features took a bit of time.

The first results regarding the solution of the wave equation on Schwarzschild backgrounds were obtained in [44] and [23] which proved uniform boundedness in the exterior region. The first pointwise decay result (without, however, a rate of decay) was obtained in [43]. Heuristics from [35] suggest that solutions to the wave equation in the Schwarzschild case should decay like v^{-3} in a compact spatial region. Here v, defined later in (3.2), is the so-called advanced time and plays the role of a time variable. For spherically symmetric data, a $v^{-3+\epsilon}$ decay rate was obtained in [12], and under the additional assumption of the initial data vanishing near the event horizon, the t^{-3} decay rate was proved away from the event horizon in [27].

Local energy estimates were first proved in [28] for radially symmetric Schrödinger equations on Schwarzschild backgrounds. In [3–5], those estimates are extended to allow for general data for the wave equation. The same authors, in [6, 7], have provided studies that give certain improved estimates near the photon sphere $r = 3M$.

Variants of these bounds have played an important role in [8] and [16] which use a spherical harmonics decomposition and Morawetz type estimates to prove certain local energy decay estimates as well as analogs of the Morawetz conformal estimates on Schwarzschild backgrounds. This allows one to deduce a v^{-2} uniform decay rate for the local energy away from the event horizon. An additional insight in [16] is that, by energy estimates with respect to a suitably chosen vector field, one can extend these decay estimates in a uniform way up to the event horizon. This provides a quantitative way of exploiting the red shift effect.

The article [29] of the authors and collaborators further contributes to the understanding of the local energy decay estimates in the Schwarzschild space–time in several ways: (1) by providing a simpler global formulation of the decay estimates, (2) by proving the result directly, using a single Morawetz type multiplier without a spherical harmonics decomposition (see also the independent work [13]), (3) by observing that it is possible and indeed natural to extend the decay estimates to a neighborhood of the event horizon inside the black hole, and (4) by providing a refined analysis near the photon sphere, leading to a stronger estimate with only log losses (see also the related articles [6,7]). It is also observed in [29] that the frequency decays exponentially along the null geodesics which are trapped on the event horizon, which leads to a high-frequency microlocal energy decay near these rays; this seems to be a well-known fact in the physics literature, but less so in the mathematics literature. This can be viewed as a microlocal interpretation of the red shift effect and improves the understanding of the earlier ideas in [16].

Further work has been carried out toward establishing Price's law in various contexts, see for instance [27] and [36]; this is outside the scope of the present paper. Finally, we note that stronger decay (both for the energy and pointwise) have been established on de Sitter–Schwarzschild space; see, for example, [9], [17], and [30].

We now turn our attention to the wave equation on the Kerr space–time, which is the focus of the present article. Until recently, even the question of obtaining uniform energy bounds was open. Away from the event horizon this was partially addressed in [19] for individual azimuthal modes; however, these bounds cannot be summed to yield bounds for the general solution. More recently, uniform energy bounds up to the event horizon were established in [14] not only for the Kerr space–time but also for a larger family of perturbations of the Schwarzschild space–time. Some uniform local decay estimates were also obtained in [18].

The aim of this article, see Theorem 4.1, is to establish local energy decay estimates in the Kerr space–time. As an easy corollary, we provide an alternate proof for the uniform energy bounds in [14]. These estimates apply in the full region outside the event horizon, as well as in a small neighborhood on the inside of the event horizon. We remark that some similar local energy decay estimates (but with weights that vanish on a neighborhood of $r = 3M$) were independently proved in [15] using related (but different) methods.

The starting point in our analysis is the earlier work [29] of the authors and collaborators, which establishes similar bounds for the wave equation in the Schwarzschild space–time. The idea is to treat the Kerr geometry as a small perturbation of the Schwarzschild geometry, and then adapt the methods in [29]. Consequently in this article, we are only considering Kerr black hole backgrounds with small angular momentum, which are close to the Schwarzschild space–time. Nevertheless, we are confident that our methods will carry over also to the case of large angular momentum.

Another goal of the earlier article [29] was to establish Strichartz estimates in the Schwarzschild space–time. A further goal of this work is to open the way toward similar bounds in Kerr. Indeed, this has been accomplished in the second author's Ph.D. Thesis [42] and will be published as a separate article.

The local energy estimate in [29] is proved using the multiplier method; the delicate issue there is to show that a suitable multiplier can be found. This method is quite robust under small perturbations of the metric, and for the most part it easily carries over to the Kerr backgrounds with small angular momentum. There is however one region where this does not apply, precisely near the photon sphere $r = 3M$ (which contains all the trapped spatially periodic geodesics in the Schwarzschild space–time, except for the rays along the event horizon, which are not relevant to this discussion). Hence, most of the new analysis here is devoted to understanding what happens there. In effect, as proved in [2], there is no differential multiplier that will achieve the desired goal. We are able to bypass this difficulty by constructing a suitable pseudodifferential multiplier.

The paper is organized as follows. In the next section, we discuss the classical local energy decay estimates in the Minkowski space–time and small perturbation thereof. Then we provide a brief overview of the local energy estimates proved in [29] for the Schwarzschild space–time, along with a discussion of the relevant geometrical issues. Finally, the last section contains the description of the Kerr space–time and all the new results. Our main local energy estimate is contained in Theorem 4.1. This is complemented by higher order bounds in Theorem 4.5.

2 Local Energy Decay in the Minkowski Space–Time

In the Minkowski space–time $\mathbb{R}^{3+1}$, consider the wave equation with constant coefficients

$$\Box u = f, \qquad u(0) = u_0, \qquad \partial_t u(0) = u_1. \tag{2.1}$$

Here $\Box = -\partial_t^2 + \Delta$. More generally, let

$$\Box_g = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j)$$

be the usual d'Alembertian associated to a Lorentzian metric g.

The seminal estimate of Morawetz [33] asserts that for solutions to the homogeneous equation $\Box u = 0$ we have the estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{|x|} |\not\nabla u|^2 (t, x)\, dx\, dt + \int_{\mathbb{R}} |u(t,0)|^2 dt \lesssim \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 \tag{2.2}$$

where $\not\nabla$ denotes the angular derivative. This is obtained combining energy estimates with the multiplier method. The radial multiplier $Qu = (\partial_r + 1/r) u$ is used, where r denotes the radial variable.

Within dyadic spatial regions, one can control the full gradient ∇u, but the square summability with respect to dyadic scales is lost. Precisely, we define the

Minkowski local energy norm $LE_{\mathbf{M}}$ by

$$\|u\|_{LE_{\mathbf{M}}} = \sup_{j\in\mathbb{Z}} 2^{-\frac{j}{2}}\|u\|_{L^2(\mathbb{R}\times\{|x|\in[2^{j-1},2^j]\})} \tag{2.3}$$

and its H^1 counterpart

$$\|u\|_{LE^1_{\mathbf{M}}} = \|\nabla_{x,t}u\|_{LE_{\mathbf{M}}} + \||x|^{-1}u\|_{LE_{\mathbf{M}}}. \tag{2.4}$$

For the inhomogeneous term, we use the dual norm

$$\|f\|_{LE^*_{\mathbf{M}}} = \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}}\|f\|_{L^2(\mathbb{R}\times\{|x|\in[2^{k-1},2^k]\})}. \tag{2.5}$$

Then we have the following scale invariant local energy estimate for solutions u to the inhomogeneous equation (2.1):

$$\|\nabla u\|_{L^\infty_t L^2_x} + \|u\|_{LE^1_{\mathbf{M}}} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} + \|f\|_{LE^*_{\mathbf{M}}+L^1_tL^2_x}. \tag{2.6}$$

This is proved using a small variation of Morawetz's method, with multipliers of the form $a(r)\partial_r + b(r)$ where a is positive, bounded and increasing.

The case of small long range perturbations of the Minkowski space–time was considered in [32]. The metrics g in $\mathbb{R}^{3+1}$ considered there satisfy

$$\sum_{k\in\mathbb{Z}} \sup_{|x|\in[2^{k-1},2^k]} |g(t,x)-g_{\mathbf{M}}| + |x||\nabla_{x,t}g(t,x)| + |x|^2|\nabla^2_{x,t}g(t,x)| \leq \epsilon \tag{2.7}$$

where $g_{\mathbf{M}}$ stands for the Minkowski metric. Then as a special case of the results in [32] we have

Theorem 2.1. [32] Let g be a Lorentzian metric in $\mathbb{R}^{3+1}$ which satisfies (2.7) with ϵ small enough. Then the solution u to the inhomogeneous problem

$$\Box u = f, \qquad u(0) = u_0, \qquad \partial_t u(0) = u_1 \tag{2.8}$$

satisfies the estimate (2.6). □

No general such results are known for large perturbations, where on one hand trapping for large frequencies and on the other hand eigenvalues and resonances for small frequencies create major difficulties. The Schwarzschild and Kerr metrics are such large perturbations where trapping plays a major role.

3 Local Energy Decay in the Schwarzschild Space–Time

In the original coordinates, the Schwarzschild space–time is given as a metric whose line element is (for $I = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2$)

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2 d\omega^2 \tag{3.1}$$

where $d\omega^2$ is the measure on the sphere $\mathbb{S}^2$, and t and r are the time and the radius of the $\mathbb{S}^2$ spheres, respectively. This metric is well defined in two regions,

$$I = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2, \qquad II = \mathbb{R} \times (0, 2M) \times \mathbb{S}^2$$

Let $\Box_{\mathbf{S}}$ denote the associated d'Alembertian.

The singularity at $r = 0$ is a true metric singularity. However, the singularity at the event horizon $r = 2M$ is an apparent singularity that can be removed by a different choice of coordinates. Following [21], let

$$r^* = r + 2M \log(r - 2M) - 3M - 2M \log M, \quad v = t + r^*. \tag{3.2}$$

In the new coordinates (v, r, ω), the metric becomes

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dv^2 + 2dvdr + r^2 d\omega^2$$

and can be extended to a larger manifold $\mathcal{N} = \mathbb{R} \times (0, \infty) \times \mathbb{S}^2$ that consists of copies of the regions I and II and the part of the event horizon separating them. Moreover, if $w = t - r^*$, one can introduce global nonsingular coordinates by rewriting the metric in the Kruskal–Szekeres coordinate system,

$$v' = e^{\frac{v}{4M}}, \qquad w' = -e^{-\frac{w}{4M}}.$$

Returning to the r, v coordinates, we define the notion of trapped null geodesics as follows:

Definition 3.1. We say that a null geodesic in the subset $\mathcal{N}$ of the Schwarzschild space–time is trapped if its maximal extension remains in a bounded region

$$0 < r_{\min} < r(v) < r_{\max} < \infty. \qquad \square$$

There are two families of trapped null geodesics on the Schwarzschild manifold. The first is at the event horizon $r = 2M$, where the trapped geodesics are the vertical ones in the (r, v, ω) coordinates. However, this family of trapped rays turns out to cause no difficulty in the decay estimates. This is due to the so-called red shift effect, whose microlocal interpretation is that the frequency decays exponentially along these trapped rays as $v \to \infty$, causing microlocal exponential decay in the high-frequency limit (see also the discussion following the formula (4.51)). The second family of trapped rays consists of null geodesics approaching the surface $r = 3M$ (called the photon sphere) asymptotically; in particular, null geodesics which are initially tangent to the photon sphere will remain on the surface for all times. Unlike the previous case, the energy is conserved for waves localized along such rays. However, what makes local energy decay estimates at all possible is the fact that the trapped rays on the photon sphere are hyperbolic within the family of null geodesics in the Schwarzschild space–time.

We remark that the family of geodesics described by this definition differs slightly from the one in the introduction in the sense that here we include not only the null geodesics lying on photon sphere, but also the ones approaching it as their affine time goes to ∞.

The (r, v, ω) coordinates are nonsingular on the event horizon, but have the disadvantage that the level sets of v are null surfaces. This is why it is more convenient to introduce

$$\tilde{v} = v - \mu(r)$$

where μ is a smooth function of r. In the $(\tilde{v}, r, \omega)$ coordinates, the metric has the form

$$\begin{aligned} ds^2 = & -\Big(1 - \frac{2M}{r}\Big)d\tilde{v}^2 + 2\left(1 - \Big(1 - \frac{2M}{r}\Big)\mu'(r)\right)d\tilde{v}dr \\ & + \Big(2\mu'(r) - \Big(1 - \frac{2M}{r}\Big)(\mu'(r))^2\Big)dr^2 + r^2 d\omega^2. \end{aligned}$$

On the function μ, we impose the following two conditions:

(i) $\mu(r) \geq r^*$ for $r > 2M$, with equality for $r > 5M/2$.

(ii) The surfaces $\tilde{v} = \text{const}$ are space-like, that is,

$$\mu'(r) > 0, \qquad 2 - \Big(1 - \frac{2M}{r}\Big)\mu'(r) > 0.$$

The first condition (i) ensures that the $(r, \tilde{v}, \omega)$ coordinates coincide with the (r, t, ω) coordinates in $r > 5M/2$. This is convenient but not required for any of our results. What is important is that in these coordinates the metric is asymptotically flat as $r \to \infty$ according to (2.7).

Given $0 < r_e < 2M$, we consider the wave equation

$$\Box_{\mathbf{S}} u = f \tag{3.3}$$

in the cylindrical region

$$\mathcal{M}_R = \{\tilde{v} \geq 0,\ r \geq r_e\} \tag{3.4}$$

with initial data on the space-like surface

$$\Sigma_R^- = \mathcal{M}_R \cap \{\tilde{v} = 0\}. \tag{3.5}$$

The lateral boundary of $\mathcal{M}_R$,

$$\Sigma_R^+ = \mathcal{M}_R \cap \{r = r_e\} \tag{3.6}$$

is also space-like, and can be thought of as the exit surface for all waves which cross the event horizon.

We define the initial (incoming) energy on Σ_R^- as

$$E[u](\Sigma_R^-) = \int_{\Sigma_R^-} \left(|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla\!\!\!\!/\, u|^2\right) r^2 \mathrm{d}r \mathrm{d}\omega \tag{3.7}$$

the outgoing energy on Σ_R^+ as

$$E[u](\Sigma_R^+) = \int_{\Sigma_R^+} \left(|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla\!\!\!\!/\, u|^2\right) r_e^2 \mathrm{d}\tilde{v} \mathrm{d}\omega \tag{3.8}$$

and the energy on an arbitrary $\tilde{v}$ slice as

$$E[u](\tilde{v}_0) = \int_{\mathcal{M}_R \cap \{\tilde{v} = \tilde{v}_0\}} \left(|\partial_r u|^2 + |\partial_{\tilde{v}} u|^2 + |\nabla\!\!\!\!/\, u|^2\right) r^2 \mathrm{d}r \mathrm{d}\omega. \tag{3.9}$$

The choice of the local energy norm $LE_{\rm S}$ is inspired from (2.3). However, there is a loss along the trapped geodesics on the photon sphere. Consequently, we introduce a modified[1] L^2 local energy space

$$\|u\|_{LE_{\rm S}} = \left\| \left(1 - \frac{3M}{r}\right) u \right\|_{LE_{\rm M}}. \tag{3.10}$$

We remark that notations are slightly changed compared to [29] in order to ensure some uniformity across the three models (Minkowski, Schwarzchild, and Kerr) described in the present paper. Here and below, we implicitly assume that all norms are restricted to the set $\mathcal{M}_R$ where we study the wave equation (3.3). Correspondingly, we define the H^1 local energy space

$$\|u\|_{LE^1_{\rm S}} = \|\partial_r u\|_{LE_{\rm M}} + \|\partial_{\tilde{v}} u\|_{LE_{\rm S}} + \|\not\nabla u\|_{LE_{\rm S}} + \|r^{-1} u\|_{LE_{\rm M}}. \tag{3.11}$$

For the inhomogeneous term, we use the norm

$$\|f\|_{LE^*_{\rm S}} = \left\| \left(1 - \frac{3M}{r}\right)^{-1} u \right\|_{LE_{\rm M}}. \tag{3.12}$$

Then we have the following result:

Theorem 3.2. [29] Let u be so that $\Box_{\rm S} u = f$. Then we have

$$E[u](\Sigma_R^+) + \sup_{\tilde{v}} E[u](\tilde{v}) + \|u\|^2_{LE^1_{\rm S}} \lesssim E[u](\Sigma_R^-) + \|f\|^2_{LE^*_{\rm S}}. \tag{3.13}$$

□

Note that, compared to the norms $LE_{\rm M}$, $LE^*_{\rm M}$, the weights have an additional polynomial singularity at $r = 3M$, but there are no additional losses at the event horizon or near ∞. Furthermore, by more refined results in [29], this polynomial loss can be relaxed to a logarithmic loss, that is, the factor $1 - (3M/r)$ can be improved to $|\ln(r - 3M)|^{-1}$ near $r = 3M$. This is related to the fact that the (spatially periodic) trapped rays on the photon sphere are hyperbolic.

We also remark that in the expression of $LE^1_{\rm S}$ it was sufficient to measure $\partial_r u$. This is due to the implicit cancelation caused by the fact that the symbol of the operator ∂_r vanishes on the trapped set.

The choice of $r_e \in (0, 2M)$ is unimportant since the r slices $r = \text{const} \in (0, 2M)$ are space-like. Hence, moving from one such r slice to another is equivalent to solving a local hyperbolic problem. The existence of the Killing vector field ∂_t ensures uniformity for this local problem, therefore no global considerations are involved. Thus, in the proof of the theorem one can assume without any restriction in generality that r_e is close to $2M$.

4 Local Energy Decay in the Kerr Space–Time

The Kerr geometry in Boyer–Lindquist coordinates is given by

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{rr}\mathrm{d}r^2 + g_{\phi\phi}\mathrm{d}\phi^2 + g_{\theta\theta}\mathrm{d}\theta^2$$

where $t \in \mathbb{R}$, $r > 0$, (ϕ, θ) are the spherical coordinates on $\mathbb{S}^2$ and

$$g_{tt} = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}, \qquad g_{t\phi} = -a\frac{2Mr\sin^2\theta}{\rho^2}, \qquad g_{rr} = \frac{\rho^2}{\Delta}$$

$$g_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta, \qquad g_{\theta\theta} = \rho^2$$

with

$$\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

A straightforward computation gives us the inverse of the metric:

$$g^{tt} = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta}, \qquad g^{t\phi} = -a\frac{2Mr}{\rho^2\Delta}, \qquad g^{rr} = \frac{\Delta}{\rho^2},$$

$$g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}, \qquad g^{\theta\theta} = \frac{1}{\rho^2}.$$

The case $a = 0$ corresponds to the Schwarzschild space–time. We shall subsequently assume that a is small $a \ll M$, so that the Kerr metric is a small perturbation of the Schwarzschild metric. We let $\Box_{\mathbf{K}}$ denote the d'Alembertian associated to the Kerr metric.

In the above coordinates, the Kerr metric has singularities at $r = 0$ on the equator $\theta = \pi/2$ and at the roots of Δ, namely $r_\pm = M \pm \sqrt{M^2 - a^2}$. As in the case of the Schwarzschild space, the singularity at $r = r_+$ is just a coordinate singularity, and corresponds to the event horizon. The singularity at $r = r_-$ is also a coordinate singularity; for a further discussion of its nature, which is not relevant for our results, we refer the reader to [11, 21]. To remove the singularities at $r = r_\pm$, we introduce functions r^*, v_+, and ϕ_+ so that (see [21])

$$\mathrm{d}r^* = (r^2 + a^2)\Delta^{-1}\mathrm{d}r, \qquad \mathrm{d}v_+ = \mathrm{d}t + \mathrm{d}r^*, \qquad \mathrm{d}\phi_+ = \mathrm{d}\phi + a\Delta^{-1}\mathrm{d}r.$$

The metric then becomes

$$\begin{aligned}\mathrm{d}s^2 = &-(1-\frac{2Mr}{\rho^2})\mathrm{d}v_+^2 + 2\mathrm{d}r\mathrm{d}v_+ - 4a\rho^{-2}Mr\sin^2\theta\,\mathrm{d}v_+\mathrm{d}\phi_+ - 2a\sin^2\theta\,\mathrm{d}r\mathrm{d}\phi_+ + \rho^2\mathrm{d}\theta^2\\ &+\rho^{-2}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta\,\mathrm{d}\phi_+^2\end{aligned}$$

which is smooth and nondegenerate across the event horizon up to but not including $r = 0$. Just like in [29], we introduce the function

$$\tilde{v} = v_+ - \mu(r)$$

where μ is a smooth function of r. In the $(\tilde{v}, r, \phi_+, \theta)$ coordinates, the metric has the form

$$\begin{aligned}\mathrm{d}s^2 = &\;(1-\frac{2Mr}{\rho^2})\mathrm{d}\tilde{v}^2 + 2\left(1-(1-\frac{2Mr}{\rho^2})\mu'(r)\right)\mathrm{d}\tilde{v}\mathrm{d}r\\ &-4a\rho^{-2}Mr\sin^2\theta\,\mathrm{d}\tilde{v}\mathrm{d}\phi_+ + \left(2\mu'(r) - (1-\frac{2Mr}{\rho^2})(\mu'(r))^2\right)\mathrm{d}r^2\\ &-2a\theta(1+2\rho^{-2}Mr\mu'(r))\sin^2\mathrm{d}r\mathrm{d}\phi_+ + \rho^2\mathrm{d}\theta^2\\ &+\rho^{-2}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta\,\mathrm{d}\phi_+^2.\end{aligned}$$

On the function μ, we impose the following two conditions:

(i) $\mu(r) \geq r^*$ for $r > 2M$, with equality for $r > 5M/2$.

(ii) The surfaces $\tilde{v} = \text{const}$ are space-like, that is,

$$\mu'(r) > 0, \qquad 2 - (1-\frac{2Mr}{\rho^2})\mu'(r) > 0.$$

As long as a is small, we can work with the same function μ as in the case of the Schwarzschild space–time.

For convenience, we also introduce

$$\tilde{\phi} = \zeta(r)\phi_+ + (1-\zeta(r))\phi$$

where ζ is a cutoff function supported near the event horizon and work in the $(\tilde{v}, r, \tilde{\phi}, \theta)$ coordinates which are identical to (t, r, ϕ, θ) outside of a small neighborhood of the event horizon.

Carter [10] showed that the Hamiltonian flow is completely integrable by finding a fourth constant of motion K that is preserved along geodesics. If E and L are the two constants of motion associated with the Killing vector fields ∂_t and ∂_ϕ, the equations for the null geodesics can be reduced to the following (see, e.g., [11] or [34])

$$\begin{aligned}
\rho^2 \dot{t} &= a(L - Ea\sin^2\theta) + \frac{(r^2+a^2)((r^2+a^2)E - aL)}{\Delta}, \\
\rho^2 \dot{\phi} &= \frac{L - Ea\sin^2\theta}{\sin^2\theta} + \frac{(r^2+a^2)aE - a^2L}{\Delta}, \\
\rho^4 \dot{\theta}^2 &= K - \frac{(L - Ea\sin^2\theta)^2}{\sin^2\theta}, \\
\rho^4 \dot{r}^2 &= -K\Delta + ((r^2+a^2)E - aL)^2
\end{aligned} \tag{4.1}$$

where the overdot denotes differentiation with respect to an affine parameter s. This parametrization of the null geodesics is nondegenerate away from the surfaces $r = r_\pm$.

Next we discuss the geometry of the trapped null geodesics. Here by trapped null geodesic we mean a null geodesic which remains for all times in a bounded r region, say $r_- < r_e < \inf r(t) \leq \sup r(t) < \infty$. The level sets $r = r_0$ of r are time-like for $r_0 > r_+$, null for $r = r_+$ and space-like for $r_- < r_0 < r_+$. The latter implies that there are no trapped null geodesics inside the region $\{r_- < r < r_+\}$. On the null surfaces $r = r_\pm$, through each point there is a unique null vector which is tangent and which generates a trapped null geodesic.

To find the trapped null geodesics in the region $r > r_+$, it suffices to consider the behavior of the fourth degree polynomial

$$P(r) = -K\Delta + ((r^2+a^2)E - aL)^2$$

in the last equation in (4.1). At least one of the parameters E, K, and L should be nonzero, and the third equation shows that $K \geq 0$ and that we cannot simultaneously have $E = K = 0$. Thus, P is always nondegenerate. The key observation is that the simple zeroes of P correspond to turning points in the last equation, and only the double zeroes are steady states. There are several cases to consider.

a) If $E = 0$, then $K > 0$. Thus, P has at most one positive root, where it changes sign from $+$ to $-$. This root is a right turning point for the ode, and there are no trapped null geodesics.

b) $E \neq 0$. Then P has degree 4 and $P \geq 0$ in $[r_-, r_+]$. If P has any zero in $[r_-, r_+]$, then the square expression must vanish, and this zero must be a double zero. We claim that in (r_+, ∞) P has either no root or two roots (counted with multiplicity); this is easily seen, as P must have either (at least) two complex conjugate roots or a negative root (the sum of the roots equals 0) and (at least) another one smaller than r_- (since $P(r_-) \geq 0$). There are three subcases:

  b1) P has no roots larger than r_+. Then r is monotone along null geodesics, and there are no trapped null geodesics.

  b2) P has two distinct positive roots $r_+ < r_1 < r_2$. There it must change sign from $+$ to $-$, respectively from $-$ to $+$. Hence, r_1 is a right turning point and r_2 is a left turning point for the ode. Thus, no trapped null geodesics exist.

  b3) P has a double positive real root r_0. Then this root is a steady state, and all other solutions converge to the steady state at one end, and escape to 0 or infinity at the other end.

This analysis shows that the trapped null geodesics in the region $r > r_+$ must converge to the family of geodesics along which r is constant (the equivalent of the photon sphere from the Schwarzschild space–time), which we now proceed to characterize in greater detail. The polynomial P has a double root if the following two relations hold,

$$((r^2 + a^2)E - aL)^2 = K\Delta, \qquad 2rE((r^2 + a^2)E - aL) = K(r - M)$$

which we rewrite in the form

$$K = \frac{r^2 E^2 \Delta}{(r - M)^2}, \qquad aL = E\left(r^2 + a^2 - \frac{2r\Delta}{r - M}\right).$$

The right-hand side in the $\dot{\theta}$ equation must be nonnegative. Substituting in the above two relations, we obtain a necessary condition for the existence of trapped geodesics, namely the inequality

$$(2r\Delta - (r - M)\rho^2)^2 \leq 4a^2 r^2 \Delta \sin^2\theta. \tag{4.2}$$

One can show that this condition is also sufficient. The expression on the left has the form

$$2r\Delta - (r-M)\rho^2 = r^2(r-3M) + 2ra^2 - (r-M)a^2\cos^2\theta.$$

If $a=0$, then it has a single positive nondegenerate zero at $r=3M$, which is the photon sphere in the Schwarzschild metric. Hence, if $0 < a \ll M$ it will still have a single zero which is close to $3M$. A rough computation leads to a bound of the form

$$|r-3M| \leq 2a, \qquad a \ll 2M. \tag{4.3}$$

Thus, all trapped null geodesics lie within $O(a)$ of the $r=3M$ sphere.

We would like a characterization of the aforementioned trapped geodesics in the phase space. Let τ, ξ, Φ, and Θ be the Fourier variables corresponding to t, r, ϕ, and θ, and

$$p_{\mathbf{K}}(r,\theta,\tau,\xi,\Phi,\Theta) = g^{tt}\tau^2 + 2g^{t\phi}\tau\Phi + g^{\phi\phi}\Phi^2 + g^{rr}\xi^2 + g^{\theta\theta}\Theta^2$$

be the principal symbol of $\Box_{\mathbf{K}}$. On any null geodesic, one has

$$p_{\mathbf{K}}(r,\theta,\tau,\xi,\Phi,\Theta) = 0. \tag{4.4}$$

Moreover, the Hamilton flow equations give us

$$\dot{r} = -\frac{\partial p_{\mathbf{K}}}{\partial \xi} = -\frac{2\Delta}{\rho^2}\xi \tag{4.5}$$

$$\dot{\xi} = \frac{\partial p_{\mathbf{K}}}{\partial r} = g^{tt}_{,r}\tau^2 + 2g^{t\phi}_{,r}\tau\Phi + g^{\phi\phi}_{,r}\Phi^2 + g^{rr}_{,r}\xi^2 + g^{\theta\theta}_{,r}\Theta^2. \tag{4.6}$$

We rewrite the latter in the form

$$\rho^2\dot{\xi} = \rho^2\frac{\partial p_{\mathbf{K}}}{\partial r} = -2R_a(r,\tau,\Phi)\Delta^{-2} + \rho^2\partial_r(\rho^{-2})\,p_{\mathbf{K}} + 2(r-M)\xi^2 \tag{4.7}$$

where

$$R_a(r,\tau,\Phi) = (r^2+a^2)(r^3-3Mr^2+a^2r+a^2M)\tau^2 - 2aM(r^2-a^2)\tau\Phi - a^2(r-M)\Phi^2.$$

For geodesics with constant r, one needs to impose the additional condition $\dot{r} = 0$. Hence, from (4.5) either $r = r_\pm$, which corresponds to the geodesics at $r = 2M$ in the Schwarzschild case, or $\xi = 0$. In the latter case from (4.7), we obtain a polynomial equation for r, namely

$$R_a(r, \tau, \Phi) = 0. \tag{4.8}$$

Furthermore, due to (4.4) we must also have the inequality

$$-((r^2 + a^2)^2 - a^2\Delta \sin^2\theta)\tau^2 - 2aMr\tau\Phi + \frac{\Delta - a^2\sin^2\theta}{\sin^2\theta}\Phi^2 \leq 0.$$

If a is small and r is as in (4.3), this allows us to bound Φ in terms of τ,

$$|\Phi| \leq 4M|\tau|. \tag{4.9}$$

For Φ in this range and small a, the polynomial $\tau^{-2}R_a(r, \tau, \Phi)$ can be viewed as a small perturbation of

$$\tau^{-2}R_0(r, \tau, \Phi) = r^4(r - 3M)$$

which has a simple root at $r = 3M$. Hence, for small a the polynomial R_a has a simple root close to $3M$, which we denote by $r_a(\tau, \Phi)$. By homogeneity considerations and the implicit function theorem, we can further express r_a in the form

$$r_a(\tau, \Phi) = 3M\tilde{r}\left(\frac{a}{M}, \frac{\Phi}{M\tau}\right), \qquad \tilde{r} \in C^\infty([-\epsilon, \epsilon] \times [-4, 4]).$$

Since $r_0(\tau, \Phi) = 3M$, it follows that we can write $r_a(\tau, \Phi)$ in the form

$$r_a(\tau, \Phi) = 3M + aF\left(\frac{a}{M}, \frac{\Phi}{M\tau}\right), \qquad F \in C^\infty([-\epsilon, \epsilon] \times [-4, 4]).$$

The above analysis shows that the trapped null geodesics corresponding to frequencies (τ, Φ) are located at radial frequency $\xi = 0$ and position $r = r_a(\tau, \Phi)$. One would be naively led to define the local smoothing spaces associated to the Kerr space–time by replacing the factor $r - 3M$ in (3.10) and (3.12) with the modified factor $r - r_a(\tau, \Phi)$. Unfortunately, this is no longer a scalar function, but a symbol of a pseudodifferential operator. In addition, this operator depends on the time Fourier variable τ, which is inconvenient for energy estimates on time $(\tilde{v})$ slabs.

Consequently, we replace the $r - r_a(\tau, \Phi)$ weight with a polynomial in τ which has the same symbol on the characteristic set $p_K = 0$. More precisely, for r close to $3M$ we factor

$$p_K(r, \theta, \tau, \xi, \Phi, \Theta) = g^{tt}(\tau - \tau_1(r, \theta, \xi, \Phi, \Theta))(\tau - \tau_2(r, \theta, \xi, \Phi, \Theta))$$

where τ_1, τ_2 are real distinct smooth 1-homogeneous symbols. On the cone $\tau = \tau_i$, the symbol $r - r_a(\tau, \phi)$ equals

$$c_i(r, \theta, \xi, \Phi, \Theta) = r - r_a(\tau_i, \Phi) = r - 3M - aF\left(\frac{a}{M}, \frac{\Phi}{M\tau_i}\right), \qquad i = 1, 2.$$

If r is close to $3M$ and $|a| \ll M$, then on the characteristic set of p we have $|\phi| < 4M|\tau|$, therefore the symbols c_i are well defined, smooth, and homogeneous. They are also nonzero outside an $O(a)$ neighborhood of $3M$.

In order to use c_i as symbols of pseudodifferential operators, we need to remove their singularity at frequency 0. Thus, we redefine

$$c_i(r, \theta, \xi, \Phi, \Theta) = r - 3M - a\chi_{\geq 1}F\left(\frac{a}{M}, \frac{\Phi}{M\tau_i}\right)$$

where $\chi_{\geq 1}$ is a smooth symbol which equals 1 for frequencies $\gg 1$ and 0 for frequencies $\ll 1$.

We use the symbols c_i to define associated microlocally weighted function spaces $L^2_{c_i}$ in a neighborhood $V \times \mathbb{S}^2$ of $3M \times \mathbb{S}^2$ which does not depend on a for small a. For functions u supported in $V \times \mathbb{S}^2$, we set

$$\|u\|^2_{L^2_{c_i}} = \|c_i(D, x)u\|^2_{L^2} + \|u\|^2_{H^{-1}}.$$

There is an ambiguity in this notation as we have not specified the coordinate frame in which we view c_i as a pseudodifferential operator. However, it is easy to see that different frames lead to equivalent norms. The quantization that we use for c_i becomes unimportant as well. We also define a dual norm $c_i L^2$ for functions g supported in $V \times \mathbb{S}^2$, namely

$$\|g\|^2_{c_i L^2} = \inf_{c_i(x,D)g_1 + g_2 = g} (\|g_1\|^2_{L^2} + \|g_2\|^2_{H^1}).$$

Since the symbols c_i are nonzero outside an $O(a)$ neighborhood of $3M$, it follows that both norms $L^2_{c_i}$ and $c_i L^2$ are equivalent to L^2 outside a similar neighborhood.

Now we can define local energy norms associated to the Kerr space–time. Let $\chi(r)$ be a smooth cutoff function which is supported in the above neighborhood V of $3M$ and which equals 1 near $3M$. Then we set

$$\|u\|_{LE^1_{\mathrm{K}}} = \|\chi(D_t - \tau_2(D,x))\chi u\|_{L^2_{c_1}} + \|\chi(D_t - \tau_1(D,x))\chi u\|_{L^2_{c_2}} + \|(1-\chi^2)\partial_t u\|_{LE_{\mathrm{M}}} \\ + \|(1-\chi^2)\nabla\!\!\!/ u\|_{LE_{\mathrm{M}}} + \|\partial_r u\|_{LE_{\mathrm{M}}} + \|r^{-1}u\|_{LE_{\mathrm{M}}}. \tag{4.10}$$

We remark that this norm is degenerate on the trapped set and is equivalent to the Schwarzschild norm LE^1_{S} when $a=0$. Indeed, when $a=0$ we have at the symbol level

$$c_1 = c_2 = r - 3M, \qquad \tau_1 = -\tau_2 \approx \sqrt{\xi^2+\lambda^2}$$

and

$$(\tau-\tau_1)^2 + (\tau-\tau_2)^2 \approx \tau^2 + \xi^2 + \lambda^2$$

while the errors are controlled by the L^2 norm of u.

For the nonhomogeneous term in the equation, we define a dual structure,

$$\|f\|_{LE^*_{\mathrm{K}}} = \|(1-\chi)f\|_{LE^*_{\mathrm{M}}} + \|\chi f\|_{c_1L^2 + c_2L^2}.$$

To state the main result of this paper, we use the notations in (3.4)–(3.9), with the parameter r_e chosen so that $r_- < r_e < r_+$:

Theorem 4.1. Let u solve $\Box_{\mathrm{K}} u = f$ in $\mathcal{M}_R$. Then

$$\|u\|^2_{LE^1_{\mathrm{K}}} + \sup_{\tilde{v}\geq 0} E[u](\tilde{v}) + E[u](\Sigma^+_R) \lesssim E[u](\Sigma^-_R) + \|f\|^2_{LE^*_{\mathrm{K}}}. \tag{4.11}$$

in the sense that the left-hand side is finite and the inequality holds whenever the right-hand side is finite. □

The proof of the result uses the multiplier method. Part of the difficulty is caused by the fact that, as shown in [2], there is no differential multiplier that provides us with a positive local energy norm. What we do instead is find a suitable pseudodifferential operator that does the job. This is chosen so that its symbol vanishes on trapped rays, which leads to a local energy norm which is degenerate there.

As in the Schwarzschild case, the choice of $r_e \in (r_-, r_+)$ is unimportant since the r slices $r = \text{const} \in (r_-, r_+)$ are space-like, therefore one can pass from a slice to another by solving a uniform family of local hyperbolic problems. Hence, in the proof of the theorem one can assume without any restriction in generality that r_e is close to r_+. However, the implicit constant in (4.11) may explode as $r_e \to r_-$, as the uniformity is lost in the local problems mentioned above.

Proof. The theorem is proved using a modification of the arguments in [29]. Let us first quickly recall the key steps in the proof of Theorem 3.2 as in [29]. We begin with the energy–momentum tensor

$$Q_{\alpha\beta}[u] = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial^\gamma u \partial_\gamma u.$$

Its contraction with respect to a vector field X is denoted by

$$P_\alpha[u, X] = Q_{\alpha\beta}[u] X^\beta$$

and its divergence is

$$\nabla^\alpha P_\alpha[u, X] = \Box_g u \cdot Xu + \frac{1}{2} Q_{\alpha\beta}[u] \pi^{\alpha\beta}$$

where $\pi^{\alpha\beta}$ is the deformation tensor of X, given by

$$\pi_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha.$$

A special role is played by the Killing vector field

$$K = \partial_{\tilde{v}}$$

whose deformation tensor is zero.

Integrating the above divergence relation for a suitable choice of X does not suffice in order to prove the local energy estimates, as in general the deformation tensor can only be made positive modulo a Lagrangian term of the form $q\partial^\alpha u \partial_\alpha u$. Hence, some lower order corrections are required. For a vector field X, a scalar function q and a 1-form m we define

$$P_\alpha[u, X, q, m] = P_\alpha[u, X] + qu\partial_\alpha u - \frac{1}{2} \partial_\alpha q u^2 + \frac{1}{2} m_\alpha u^2.$$

The divergence formula gives

$$\nabla^{\alpha} P_{\alpha}[u, X, q, m] = \Box_g u\Big(Xu + qu\Big) + Q[u, X, q, m], \tag{4.12}$$

where

$$Q[u, X, q, m] = \frac{1}{2} Q_{\alpha\beta}[u]\pi^{\alpha\beta} + q\partial^{\alpha} u\partial_{\alpha} u + m_{\alpha} u\partial^{\alpha} u + (\nabla^{\alpha} m_{\alpha} - \frac{1}{2}\nabla^{\alpha}\partial_{\alpha} q)\, u^2.$$

So far, these computations apply both for the Schwarzschild and the Kerr metrics. From here on, we will use the bold sub(super)scripts **S**, respectively **K** to indicate when a computation is performed with respect to one metric or another.

To prove the local energy decay in the Schwarzschild space–time, X, q, and m are chosen as in the following lemma:

Lemma 4.2. In the exterior region $r \geq 2M$, there exist a smooth vector field

$$X = b(r)(1 - \frac{3M}{r})\partial_r + c(r)K$$

with c supported near the event horizon and $b > 0$ bounded so that

$$|\partial_r^{\alpha} b| \leq c_{\alpha} r^{-\alpha},$$

a smooth function $q(r)$ with

$$|\partial_r^{\alpha} q| \leq c_{\alpha} r^{-1-\alpha},$$

and a smooth 1-form m supported near the event horizon $r = 2M$ so that

(i) The quadratic form $Q^S[u, X, q, m]$ is positive definite,

$$Q^S[u, X, q, m] \gtrsim r^{-2}|\partial_r u|^2 + \left(1 - \frac{3M}{r}\right)^2 (r^{-2}|\partial_{\tilde{v}} u|^2 + r^{-1}|\not\nabla u|^2) + r^{-4}u^2. \tag{4.13}$$

(ii) $X(2M)$ points toward the black hole, $X(dr)(2M) < 0$, and $\langle m, dr\rangle(2M) > 0$. ■

We remark that by extending smoothly the functions b, c, r, q, and m to $r < 2M$ one can ensure that (i) above holds for $r > r_e$ and (ii) holds at $r = r_e$, provided $r_e < 2M$ is close enough to $2M$. We fix r_e with this property. Then for small enough a, namely $|a| \ll M$, we still have $r_e < r_+$, therefore this r_e is still suitable for the proof of Theorem 4.1.

The local energy estimate is obtained by integrating the divergence relation (4.12) with $X + CK$ instead of X, where C is a large constant, on the domain

$$\mathcal{M}_{[0,\tilde{v}_0]} = \{0 < \tilde{v} < \tilde{v}_0,\ r > r_e\}$$

with respect to the measure induced by the metric, $\mathrm{d}V_{\mathbf{S}} = r^2 \mathrm{d}r\mathrm{d}\tilde{v}\mathrm{d}\omega$. This yields

$$\int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^{\mathbf{S}}[u, X, q, m]\mathrm{d}V_{\mathbf{S}} = -\int_{\mathcal{M}_{[0,\tilde{v}_0]}} \Box_{\mathbf{S}} u\Big((X + CK)u + qu\Big)\mathrm{d}V_{\mathbf{S}} + BDR^{\mathbf{S}}[u] \tag{4.14}$$

where $BDR^{\mathbf{S}}[u]$ denotes the boundary terms

$$BDR^{\mathbf{S}}[u] = \int \langle \mathrm{d}\tilde{v}, P[u, X + CK, q, m]\rangle r^2 \mathrm{d}r\mathrm{d}\omega\Big|_{\tilde{v}=0}^{\tilde{v}=\tilde{v}_0} - \int \langle \mathrm{d}r, P[u, X + CK, q, m]\rangle r_e^2 \mathrm{d}\tilde{v}\mathrm{d}\omega.$$

Using the condition (ii) in the lemma and Hardy type inequalities, it is shown in [29] that for large C and r_e close to $2M$ the boundary terms have the correct sign,

$$BDR^{\mathbf{S}}[u] \le c_1 E[u](\Sigma_R^-) - c_2(E[u](\tilde{v}_0) + E[u](\Sigma_R^+)), \qquad c_1, c_2 > 0. \tag{4.15}$$

Consequently, by applying the Cauchy–Schwarz inequality for the first term on the right of (4.14) we obtain a slightly weaker form of the local energy estimate (3.13), namely

$$E[u](\Sigma_R^+) + \sup_{\tilde{v}} E[u](\tilde{v}) + \|u\|^2_{LEW^1_{\mathbf{S}}} \lesssim E[u](\Sigma_R^-) + \|f\|^2_{LEW^*_{\mathbf{S}}}. \tag{4.16}$$

where the weaker norm $LEW^1_{\mathbf{S}}$ and the stronger norm $LEW^*_{\mathbf{S}}$ are defined by

$$\|u\|^2_{LEW^1_{\mathbf{S}}} = \int_{\mathcal{M}_R} \left(r^{-2}|\partial_r u|^2 + \left(1 - \frac{3M}{r}\right)^2 (r^{-2}|\partial_{\tilde{v}} u|^2 + r^{-1}|\nabla\!\!\!/ u|^2) + r^{-4}u^2 \right) r^2 \mathrm{d}r\mathrm{d}\tilde{v}\mathrm{d}\omega,$$

respectively

$$\|f\|^2_{LEW^*_{\mathbf{S}}} = \int_{\mathcal{M}_R} r^2 \left(1 - \frac{3M}{r}\right)^{-2} f^2 r^2 \mathrm{d}r\mathrm{d}\tilde{v}\mathrm{d}\omega.$$

These norms are equivalent with the stronger norms $LE^1_{\mathbf{S}}$, respectively $LE^*_{\mathbf{S}}$ for r in a bounded set. On the other hand for large r, the Schwarzschild space can be viewed as a small perturbation of the Minkowski space. Thus, the transition from (4.16) to (3.13) is achieved in [29] by cutting away a bounded region and then using a perturbation of a Minkowski space estimate. This part of the proof translates without any changes to the case of the Kerr space–time. Our goal in what follows will be to establish the Kerr counterpart of (4.16), namely

$$E[u](\Sigma_R^+) + \sup_{\tilde{v}} E[u](\tilde{v}) + \|u\|^2_{LEW^1_{\mathbf{K}}} \lesssim E[u](\Sigma_R^-) + \|f\|^2_{LEW^*_{\mathbf{K}}}. \tag{4.17}$$

where the norms $LEW^1_{\mathbf{K}}$, respectively $LEW^*_{\mathbf{K}}$ coincide with $LE^1_{\mathbf{K}}$, respectively $LE^*_{\mathbf{K}}$ for bounded r, and with $LEW^1_{\mathbf{S}}$, respectively $LEW^*_{\mathbf{S}}$ for large r. More precisely, if $\chi(r)$ is a smooth compactly supported cutoff function which equals 1 say for $r < 4M$ then we set

$$\|u\|^2_{LEW^1_{\mathbf{K}}} = \|\chi u\|^2_{LE^1_{\mathbf{K}}} + \|(1-\chi)u\|^2_{LEW^1_{\mathbf{S}}},$$

respectively

$$\|u\|^2_{LEW^*_{\mathbf{K}}} = \|\chi u\|^2_{LE^*_{\mathbf{K}}} + \|(1-\chi)u\|^2_{LEW^*_{\mathbf{S}}}.$$

Different choices for χ lead to different but equivalent norms.

It is useful to first consider the effect of the same multiplier in the Kerr metric. The two metrics are close when measured in the same Euclidean frame $x = r\omega$ with $r \geq r_e$. Precisely, with ∂ standing for ∂_t and ∂_x, $x = r\omega$,

$$|\partial^\alpha[(g_{\mathbf{K}})_{ij} - (g_{\mathbf{S}})_{ij}]| \leq c_\alpha \frac{a}{r^{2+|\alpha|}}, \qquad |\partial^\alpha[(g_{\mathbf{K}})^{ij} - (g_{\mathbf{S}})^{ij}]| \leq c_\alpha \frac{a}{r^{2+|\alpha|}}. \tag{4.18}$$

From this and the size and regularity properties of X, q, and m, it follows that

$$|P^{\mathbf{S}}_\alpha[u, X, q, m] - P^{\mathbf{K}}_\alpha[u, X, q, m]| \lesssim \frac{a}{r^2}|\nabla u|^2, \tag{4.19}$$

respectively

$$|Q^{\mathbf{S}}[u, X, q, m] - Q^{\mathbf{K}}[u, X, q, m]| \lesssim a\left(\frac{1}{r^2}|\nabla u|^2 + \frac{1}{r^4}|u|^2\right). \tag{4.20}$$

Hence, integrating the divergence relation (4.12) in the Kerr space–time over the same domain $\mathcal{M}_{[0,\tilde{v}_0]}$ but with respect to the Kerr induced measure $\mathrm{d}V_{\mathbf{K}} = \rho^2 \mathrm{d}r\mathrm{d}\tilde{v}\mathrm{d}\omega$ we obtain

$$\int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^{\mathbf{K}}[u, X, q, m]\mathrm{d}V_{\mathbf{K}} = -\int_{\mathcal{M}_{[0,\tilde{v}_0]}} \Box_{\mathbf{K}} u\Big((X + CK)u + qu\Big)\mathrm{d}V_{\mathbf{K}} + BDR^{\mathbf{K}}[u]. \tag{4.21}$$

The bound (4.19) shows that for small a the boundary terms retain their positivity properties in (4.15), namely

$$BDR^{\mathrm{K}}[u] \leq c_1 E[u](\Sigma_R^-) - c_2(E[u](\tilde{v}_0) + E[u](\Sigma_R^+)), \qquad c_1, c_2 > 0. \tag{4.22}$$

However, (4.20) merely shows that

$$Q_{\mathrm{K}}[u, X, q, m] \gtrsim r^{-2}|\partial_r u|^2 + \left[\left(1 - \frac{3M}{r}\right)^2 - Ca\right](r^{-2}|\partial_{\tilde{v}} u|^2 + r^{-1}|\nabla\!\!\!/ u|^2) + r^{-4}u^2 \tag{4.23}$$

and the right-hand side is no longer positive definite near $r = 3M$. Thus, we cannot close the argument as in the Schwarzschild case. As shown in [2], changing the vector field X near $r = 3M$ would not help.

To remedy this, we need to use a pseudodifferential modification S of the vector field X. We will choose S so that its kernel is supported in a small neighborhood of $(3M, 3M)$; this ensures that there will be no additional contributions at $r = r_e$. Furthermore, in order to be able to carry out the computations near the initial and final surfaces $\tilde{v} = 0$, $\tilde{v}_0$ we take S to be a first order differential operator with respect to $\tilde{v}$. Similarly, we modify the Lagrangian factor q using a pseudodifferential correction E, which is also a first order differential operator with respect to $\tilde{v}$.

We also need to choose a quantization which is consistent with the Kerr measure. Here we have a few choices which have equivalent results. For our selection, we use Euclidean-like coordinates $x = \omega r$. Given a real symbol s, its Euclidean Weyl quantization s^w is self-adjoint with respect to the Euclidean measure $\mathrm{d}V = r^2 \mathrm{d}r\mathrm{d}\omega$. However, in our case we need to work with the Kerr induced measure $\mathrm{d}V_{\mathrm{K}} = \rho^2 \mathrm{d}r\mathrm{d}\omega$. Hence, we slightly abuse the notation and redefine the Weyl quantization as

$$s^w := \frac{r}{\rho}\, s^w\, \frac{\rho}{r}.$$

If s is a real symbol, then s^w (re)defined above is a self-adjoint operator in $L^2(\mathrm{d}V_{\mathrm{K}})$.

Another issue which does not affect our analysis but needs to be addressed is that we are using pseudodifferential operators in an exterior domain $\{r > r_e\}$ and some care must be given to what happens near $r = r_e$. To keep things simple, in what follows all pseudodifferential operators we work with are compactly supported in the sense that their kernels are supported away from r_e and infinity; even better, supported in a small neighborhood of $3M$.

In what follows, we consider a skew-adjoint pseudodifferential operator S and a self-adjoint pseudodifferential operator E of the form (since we are away from the event horizon, the variable tv coincides with t, and we make this substitution here and later)

$$S = is_1^w + s_0^w \partial_t, \qquad E = e_0^w + \frac{1}{i} e_{-1} \partial_t$$

where $s_1 \in S^1$, $s_0, e_0 \in S^0$, and $e_{-1} \in S^{-1}$ are real symbols, homogeneous outside a neighborhood of 0. We further assume that their kernels are supported close to $3M$.

For a function u with compact support in $\mathcal{M}_{[0,\tilde{v}_0]}$, we use the fact that $\Box_{\mathbf{K}}$ and E are self-adjoint while S is skew-adjoint to compute

$$\Re \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \Box_{\mathbf{K}} u \cdot (S+E) u \, \mathrm{d}V_{\mathbf{K}} = \int_{\mathcal{M}_{[0,\tilde{v}_0]}} Qu \cdot u \, \mathrm{d}V_{\mathbf{K}} \tag{4.24}$$

where

$$Q = \frac{1}{2}([\Box_{\mathbf{K}}, S] + \Box_{\mathbf{K}} E + E \Box_{\mathbf{K}}).$$

In general, the operator Q is a third order differential operator in t, which by the pseudodifferential calculus has the form

$$Q = Q_2^w + 2Q_1^w D_t + Q_0^w D_t^2 + Q_{-1}^w D_t^3$$

where $Q_j^w \in OPS^j$ are self-adjoint pseudodifferential operators. In our case, the operators S and E will be chosen later so that the coefficient D_t^3 vanishes,

$$Q_{-1}^w = 0. \tag{4.25}$$

By analogy with (4.21), this leads us to define the bilinear form

$$IQ^{\mathbf{K}}[u, S, E] = \int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q_2^w u \cdot \overline{u} + 2\Re Q_1^w u \cdot \overline{D_t u} + Q_0^w D_t u \overline{D_t u} \, \mathrm{d}V_{\mathbf{K}}. \tag{4.26}$$

In view of (4.24) and (4.25), for u with compact support in $\mathcal{M}_{[0,\tilde{v}_0]}$ we have the relation

$$IQ^{\mathbf{K}}[u, S, E] = \Re \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \Box_{\mathbf{K}} u \cdot (S+E) u \mathrm{d}V_{\mathbf{K}}. \tag{4.27}$$

We define the principal symbol of the quadratic form $IQ^{\mathrm{K}}[u, S, E]$ as

$$q^{\mathrm{K}}[S, E] = q_2 + 2q_1\tau + q_0\tau^2.$$

The previous relation shows that it satisfies

$$q^{\mathrm{K}}[S, E] = \frac{1}{2i}\{p_{\mathrm{K}}, s\} + p_{\mathrm{K}} e \qquad \mathrm{mod}\ S^0 + \tau S^{-1} + \tau^2 S^{-2}.$$

Suppose we now remove the compact support condition on u. Due to the kernel localization near $3M$ for both operators S and E, there are no contributions on the lateral boundary $r = r_e$. Since both S and E are differential operators in t, the computation leading to (4.27) can be carried out using only integration by parts with respect to the time variable. This leads us to the full counterpart of (4.21), namely

$$IQ^{\mathrm{K}}[u, S, E] = \Re \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \Box_{\mathrm{K}} u \cdot (S + E) u \mathrm{d}V_{\mathrm{K}} + BDR^{\mathrm{K}}[u, S, E] \tag{4.28}$$

where $BDR^{\mathrm{K}}[u, S, E]$ represents the boundary terms at times 0 and $\tilde{v}_0$ obtained in the integration by parts with respect to t. It has the form

$$BDR^{\mathrm{K}}[u, S, E] = \Re \int_{\Sigma_t} B_0^w D_t u \cdot D_t u + B_1^w u \cdot D_t u + B_2^w u \cdot u \mathrm{d}A_K \Big|_{t=0}^{t=\tilde{v}_0}$$

where $B_j^w \in OPS^j$ and $\mathrm{d}A_K$ is the induced volume element on time sections. The exact expressions of the operators B_j^w are not important, as all we need to use here is the bound

$$|BDR^{\mathrm{K}}[u, S, E]| \lesssim E[u](0) + E[u](\tilde{v}_0). \tag{4.29}$$

We add (4.21) with a times (4.28). The boundary terms are estimated by (4.22) and (4.29). Using the duality between the spaces $c_i L^2$ and $L^2_{c_i}$, we can also estimate

$$\left| \int_{\mathcal{M}_{[0,\tilde{v}_0]}} f \cdot (X + CK + q + a(S + E)) u \, \mathrm{d}V_{\mathrm{K}} \right| \lesssim \|f\|_{LEW_{\mathrm{K}}^*} \|u\|_{LEW_{\mathrm{K}}^1}.$$

Hence, in order to prove (4.17) it would suffice to show that the symbols s and e can be chosen so that

$$\int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^{\mathrm{K}}[u, X, q, m] \mathrm{d}V_{\mathrm{K}} + aIQ^{\mathrm{K}}[u, S, E] \gtrsim \|u\|^2_{LEW_{\mathrm{K}}^1}. \tag{4.30}$$

Here we aim to choose S and E uniformly with respect to small a. In effect, our construction below yields symbols s and e which are analytic with respect to a. We remark that the choice of S and E is only important in the region where r is close to $3M$. Outside this region, $Q^{\mathrm{K}}[u, X, q, m]$ is already positive definite and the contribution of $aI\,Q^{\mathrm{K}}[u, S, E]$ is negligible.

We consider first the expression $Q^{\mathrm{K}}[u, X, q, m]$. Near $r = 3M$, it has the form

$$Q^{\mathrm{K}}[u, X, q, m] = \sum q^{\mathrm{K},\alpha\beta}\partial_\alpha u \partial_\beta u + q^{\mathrm{K},0}u^2 \tag{4.31}$$

where its principal symbol $q^{\mathrm{K}} = q^{\mathrm{K},\alpha\beta}\eta_\alpha\eta_\beta$ and the lower order coefficient $q^{\mathrm{K},0}$ are given by the relation

$$q^{\mathrm{K}} = \frac{1}{2i}\{p_{\mathrm{K}}, X\} + qp_{\mathrm{K}}, \qquad q^{K,0} = -\frac{1}{2}\nabla^\alpha\partial_\alpha q.$$

We do not need to exactly compute the above expression in the Kerr case, but it is useful to perform the computation in the simpler case of the Schwarzschild space. There we have

$$p_{\mathrm{S}} = -\left(1 - \frac{2M}{r}\right)^{-1}\tau^2 + \left(1 - \frac{2M}{r}\right)\xi^2 + \frac{1}{r^2}\lambda^2, \qquad X = ib(r)\left(1 - \frac{3M}{r}\right)\xi$$

where λ stands for the spherical Fourier variable. Hence, we obtain

$$\begin{aligned} r^2 q^{\mathrm{S}} &= \frac{1}{2i}\{r^2 p_{\mathrm{S}}, X\} + (q - r^{-1}b(r)(r-3M))(r^2 p_{\mathrm{S}}) \\ &= \alpha_{\mathrm{S}}^2(r)\tau^2 + \beta_{\mathrm{S}}^2(r)\xi^2 + \tilde{q}_{\mathrm{S}}(r)(r^2 p) \end{aligned} \tag{4.32}$$

where, near $r = 3M$,

$$\alpha_{\mathrm{S}}^2(r) = \frac{rb(r)(r-3M)^2}{(r-2M)^2},$$

$$\beta_{\mathrm{S}}^2(r) = \frac{3M}{r^2}b(r^2 - 2Mr) + \left(1 - \frac{3M}{r}\right)(b'(r^2 - 2Mr) - b(r - M)),$$

respectively

$$\tilde{q}_{\mathrm{S}}(r) = q - r^{-1}b(r)(r - 3M).$$

Here we have used the fact that $b > 0$ to write the first two coefficients as squares.

For our choice of q and r, we know that the relation (4.13) holds. This implies that the following two inequalities must hold:

$$q^{S} \gtrsim \xi^2 + (r-3M)^2(\tau^2 + \lambda^2), \qquad q^{S,0} > 0. \tag{4.33}$$

Given the form of q^{S}, the first relation implies that $\tilde{q}_{S}$ is a multiple of $(r-3M)^2$, and that in addition there is a smooth function $\nu(r)$ so that

$$\frac{r^3}{r-2M}\tilde{q}_{S} = \nu(r)\alpha_{S}^2(r), \qquad 0 < \nu < 1.$$

This allows us to obtain the following sum of squares representation for q^{S}:

$$r^2 q^{S} = (1-\nu(r))\alpha_{S}^2(r)\tau^2 + \beta_{S}^2(r)\xi^2 + \nu_1(r)\alpha_{S}^2(r)(\lambda^2 + (r^2 - 2rM)\xi^2), \quad \nu_1 = \frac{r-2M}{r^3}\nu. \tag{4.34}$$

The symbol λ^2 of the spherical Laplacian can also be written as sums of squares of differential symbols,

$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

where in Euclidean coordinates we can write

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{x_i\eta_j - x_j\eta_i, \ i \neq j\}$$

$$r^2 q^{S} = (1-\nu(r))\alpha_{S}^2(r)\tau^2 + \beta_{S}^2(r)\xi^2 + \nu_1(r)\alpha_{S}^2(r)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + (r^2 - 2rM)\xi^2). \tag{4.35}$$

We return now to the question of finding symbols s and e so that the bound (4.30) holds. Near $r = 3M$, the principal symbol of the quadratic form on the left in (4.30) is

$$\frac{1}{2i}\{p_K, X + as\} + p_K(q + ae).$$

In order to prove (4.30) at the very least, we would like the above symbol to be nonnegative, and to satisfy the bound

$$\frac{1}{2i}\{p_K, X + as\} + p_K(q + ae) \gtrsim c_2^2(\tau - \tau_1)^2 + c_1^2(\tau - \tau_2)^2 + \xi^2.$$

However, such a bound would not a priori suffice since translating it to operator bounds would require using the Fefferman–Phong inequality, which does not hold in general for systems. Hence, we prove a more precise result, and show that the symbols s and e can be chosen so that we have a favorable sum of squares representation for the above expression, which extends the sum of squares (4.35) to $a \neq 0$.

Lemma 4.3. Let a be sufficiently small. Then there exist smooth homogeneous symbols $s \in S^1_{\text{hom}} + \tau S^0_{\text{hom}}$, $e \in S^0_{\text{hom}} + \tau S^{-1}_{\text{hom}}$, also depending smoothly on a, so that for r close to $3M$ we have the sum of squares representation (here X and q remain the ones given by Lemma 4.2)

$$\rho^2 \left(\frac{1}{2i} \{p_K, X + as\} + p_K(q + ae) \right) = \sum_{j=1}^{8} \mu_j^2 \tag{4.36}$$

where $\mu_j \in S^1_{\text{hom}} + \tau S^0_{\text{hom}}$ satisfies the following properties:

(i) The decomposition (4.36) extends the decomposition (4.35) in the sense that

$$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = ((1-\nu)^{\frac{1}{2}} \alpha_{\mathbf{S}} \tau, \beta_{\mathbf{S}} \xi, \nu_1^{\frac{1}{2}} \alpha_{\mathbf{S}} \lambda_1, \nu_1^{\frac{1}{2}} \alpha_{\mathbf{S}} \lambda_2, \nu_1^{\frac{1}{2}} \alpha_{\mathbf{S}} \lambda_3, \nu_1^{\frac{1}{2}} \alpha_{\mathbf{S}} \xi) \mod a(S^1_{\text{hom}} + \tau S^0_{\text{hom}})$$

and

$$(\mu_7, \mu_8) \in \sqrt{a}(S^1_{\text{hom}} + \tau S^0_{\text{hom}}).$$

(ii) The family of symbols $\{\mu_j\}_{j=1,6}$ is elliptically equivalent with the family of symbols $(c_2(\tau - \tau_1), c_1(\tau - \tau_2), \xi)$ in the sense that we have a representation of the form

$$\mu = Mv, \qquad v = \begin{pmatrix} c_2(\tau - \tau_1) \\ c_1(\tau - \tau_2) \\ \xi \end{pmatrix}$$

where the symbol valued matrix $M \in M^{8\times 3}(S^0_{\text{hom}})$ has maximum rank 3 everywhere. □

Proof. Setting $\tilde{q}_K = q - 2\{\ln \rho, X\}$, respectively $\tilde{e} = e - 2\{\ln \rho, s\}$ we compute

$$\rho^2 \left(\frac{1}{2i} \{p_K, X + as\} + (q + ae) p_K \right) = \frac{1}{2i} \{\rho^2 p_K, X + as\} + (\tilde{q}_K + a\tilde{e})(\rho^2 p_K).$$

We first choose the symbol s so that the Poisson bracket $\{\rho^2 p_K, X + as\}$ has the correct behavior on the characteristic set $p_K = 0$. Recall that the symbol of X is $ir^{-1}b(r)(r - 3M)\xi$, where the vanishing coefficient at $3M$ corresponds exactly to the location of the trapped rays. Its natural counterpart in the Kerr space–time is the symbol

$$\tilde{s}(r, \tau, \xi, \Phi) = ir^{-1}b(r)(r - r_a(\tau, \Phi))\xi.$$

This coincides with X in the Schwarzschild case $a = 0$, and it is well defined and smooth in a for r near $3M$ and $|\Phi| < 4|\tau|$. In particular, it is well defined in a neighborhood of the characteristic set $p_K = 0$, which is all we use in the sequel.

We use (4.7) to compute the Poisson bracket on the characteristic set $\{p_K = 0\}$:

$$\begin{aligned} \frac{1}{i}\{\rho^2 p_K, \tilde{s}\} = & -(\rho^2 p_K)_r r^{-1}b(r)(r - r_a(\tau, \Phi)) + \xi(\rho^2 p_K)_\xi \partial_r \left(r^{-1}b(r)(r - r_a(\tau, \Phi))\right) \\ = & \ 2r^{-1}b(r)R(r, \tau, \Phi)\Delta^{-2}(r - r_a(\tau, \Phi)) \\ & + \left[2\Delta\partial_r \left(r^{-1}b(r)(r - r_a(\tau, \Phi))\right) - 2(r - M)r^{-1}b(r)(r - r_a(\tau, \Phi))\right]\xi^2. \end{aligned}$$

Since $r_a(\tau, \Phi)$ is the unique zero of $R(r, \tau, \Phi)$ near $r = 3M$ and is close to $3M$, it follows that we can write

$$\frac{1}{2i}\{\rho^2 p_K, \tilde{s}\} = \alpha^2(r, \tau, \Phi)\tau^2(r - r_a(\tau, \Phi))^2 + \beta^2(r, \tau, \Phi)\xi^2 \qquad \text{on } \{p_K = 0\} \tag{4.37}$$

where $\alpha, \beta \in S^0_{\text{hom}}$ are positive symbols. We note that in the Schwarzschild case the symbols α and β are simply functions of r, see the first two terms in (4.32).

Unfortunately $\tilde{s}$ is not a polynomial in τ, which limits its direct usefulness. To remedy that we first note that

$$\tilde{s} - (ir^{-1}b(r)(r - 3M)\xi) \in aS^1_{\text{hom}}.$$

Hence, by (the simplest form of) the Malgrange preparation theorem we can write

$$\frac{1}{i}\tilde{s} = r^{-1}(r - 3M)b(r)\xi + a(s_1(r, \xi, \theta, \Theta, \Phi) + s_0(r, \xi, \theta, \Theta, \Phi)\tau) + ah(\tau, r, \xi, \theta, \Theta, \Phi)p_K$$

with $s_1 \in S^1_{\text{hom}}$, $s_0 \in S^0_{\text{hom}}$, and $h \in S^{-1}_{\text{hom}}$. Then we define the desired symbol s by

$$s = i(s_1 + s_0\tau)$$

thus ensuring that

$$\tilde{s} = X + as \qquad \text{on } \{p_{\mathrm{K}} = 0\}.$$

The Poisson bracket $\frac{1}{i}\{\rho^2 p_{\mathrm{K}}, s\}$ is a third degree polynomial in τ. Hence, after division by $p_{\mathrm{K}} = -g^{tt}(\tau - \tau_1)(\tau - \tau_2)$, taking also (4.35) into account, we can write

$$\frac{1}{2i}\{\rho^2 p_{\mathrm{K}}, X + as\} + \tilde{q}_{\mathrm{K}}(\rho^2 p_{\mathrm{K}}) = \gamma_2 + \gamma_1\tau + [e_{\mathrm{S}} + a(f_0 + f_{-1}\tau)](\tau - \tau_1)(\tau - \tau_2) \tag{4.38}$$

where $f_i \in S^i_{\mathrm{hom}}$ and, by (4.35), the coefficient e_{S} corresponding to the Schwarzschild case is given by

$$e_{\mathrm{S}} = (1 - \nu(r))\alpha^2(r).$$

It remains to show that the principal part $\gamma_2 + \gamma_1\tau + e_{\mathrm{S}}(\tau - \tau_1)(\tau - \tau_2)$ in the right-hand side of (4.38) can be expressed as a sum of squares as in the lemma modulo an error $a(S^0_{\mathrm{hom}} + \tau S^{-1}_{\mathrm{hom}})p_{\mathrm{K}}$,

$$\gamma_2 + \gamma_1\tau + e_{\mathrm{S}}(\tau - \tau_1)(\tau - \tau_2) = \sum \mu_j^2 + a(g_0 + g_{-1}\tau)(\tau - \tau_1)(\tau - \tau_2).$$

Then the symbol e is chosen so that both $a(S^0_{\mathrm{hom}} + \tau S^{-1}_{\mathrm{hom}})p_{\mathrm{K}}$ terms are canceled,

$$\tilde{e} = -[f_0 + g_0 + (f_{-1} + g_{-1})\tau].$$

The coefficients γ_1 and γ_2 can be computed using the relation (4.37) and the fact that $\{\rho^2 p_{\mathrm{K}}, X + as\} = \{\rho^2 p_{\mathrm{K}}, \tilde{s}\}$ on $p_{\mathrm{K}} = 0$ (i.e., when $\tau = \tau_i$). This implies that

$$\gamma_2 + \gamma_1\tau = \alpha^2(r, \tau, \Phi)\tau^2(r - r_a(\tau, \Phi))^2 + \beta^2(r, \tau, \Phi)\xi^2 \qquad \text{when } \tau = \tau_i.$$

We denote

$$\alpha_i = \frac{2|\tau_i|}{\tau_1 - \tau_2}\alpha(r, \tau_i, \Phi)(r - r_a(\tau_i, \Phi)) \in S^0_{\mathrm{hom}}, \qquad \beta_i = \beta(r, \tau_i, \Phi) \in S^0_{\mathrm{hom}},$$

observing that α_i can be used as substitutes for the c_i's in the lemma since they are elliptic multiples of c_i. Then we have the two-dimensional system

$$\gamma_2 + \gamma_1\tau_i = \frac{1}{4}\alpha_i^2(\tau_1 - \tau_2)^2 + \beta_i^2\xi^2$$

which gives the following expressions for γ_1, γ_2:

$$\begin{aligned}\gamma_2 &= \frac{1}{4}(\tau_1-\tau_2)(\alpha_2^2\tau_1-\alpha_1^2\tau_2)+\frac{\tau_1\beta_2^2-\tau_2\beta_1^2}{\tau_1-\tau_2}\xi^2,\\ \gamma_1 &= \frac{1}{4}(\tau_1-\tau_2)(\alpha_1^2-\alpha_2^2)+\frac{\beta_1^2-\beta_2^2}{\tau_1-\tau_2}\xi^2.\end{aligned} \tag{4.39}$$

We use the first components of γ_1 and γ_2 to obtain a sum of squares as follows:

$$\begin{aligned}(\tau_1-\tau_2)(\alpha_2^2\tau_1-\alpha_1^2\tau_2)+\tau(\tau_1-\tau_2)(\alpha_1^2-\alpha_2^2) &= \nu(\alpha_1(\tau-\tau_2)-\alpha_2(\tau-\tau_1))^2\\ &\quad+(1-\nu)(\alpha_1(\tau-\tau_2)+\alpha_2(\tau-\tau_1))^2\\ &\quad-4e_K(\tau-\tau_1)(\tau-\tau_2)\end{aligned} \tag{4.40}$$

where

$$e_K = \frac{(\alpha_1-\alpha_2)^2}{4}+(1-\nu)\alpha_1\alpha_2.$$

We remark that in the Schwarzschild case we have $\tau_2=-\tau_1$ and also $\alpha_1=\alpha_2=\alpha_S$ and $\beta_1=\beta_2=\beta_S$. In particular, this shows that

$$e_K - e_S \in a(S^0_{\text{hom}}+\tau S^{-1}_{\text{hom}})$$

which accounts for the e_S factor in (4.38). It remains to consider the ξ^2 terms in (4.39). This is easier since the coefficients β_1, β_2 are positive and have a small difference $\beta_1-\beta_2\in aS^0_{\text{hom}}$. Precisely, for a large C we can write

$$\begin{aligned}\frac{\tau_1\beta_2^2-\tau_2\beta_1^2}{\tau_1-\tau_2}+\tau\frac{\beta_1^2-\beta_2^2}{\tau_1-\tau_2} &= \frac{1}{2}(\beta_1^2+\beta_2^2-Ca)+\frac{(Ca-\beta_2^2+\beta_1^2)(\tau-\tau_2)^2}{2(\tau_1-\tau_2)^2}\\ &\quad+\frac{(Ca-\beta_1^2+\beta_2^2)(\tau-\tau_1)^2}{2(\tau_1-\tau_2)^2}+O(a)p.\end{aligned}$$

Summing this with (4.40), we obtain the desired sums of squares representation,

$$\begin{aligned}\frac{1}{2i}\{\rho^2 p_K, X+as\}+(\rho^2 p_K)\tilde{q}_K \in\ & \frac{\nu}{4}(\alpha_1(\tau-\tau_2)-\alpha_2(\tau-\tau_1))^2\\ &+\frac{1-\nu}{4}(\alpha_1(\tau-\tau_2)+\alpha_2(\tau-\tau_1))^2+\frac{1}{2}(\beta_1^2+\beta_2^2-Ca)\xi^2\\ &+\frac{(Ca-\beta_2^2+\beta_1^2)(\tau-\tau_2)^2}{2(\tau_1-\tau_2)^2}\xi^2+\frac{(Ca-\beta_1^2+\beta_2^2)(\tau-\tau_1)^2}{2(\tau_1-\tau_2)^2}\xi^2\\ &+a(S^0_{\text{hom}}+S^{-1}_{\text{hom}}\tau)(\tau-\tau_1)(\tau-\tau_2).\end{aligned}$$

Then e is chosen so that the last term accounts for the contribution of $\tilde{e}$.

Part (ii) of the lemma directly follows. For part (i), we still need to specify which are the symbols μ_j. Precisely, we set

$$\mu_1^2 = \frac{1-\nu}{4}(\alpha_1(\tau-\tau_2)+\alpha_2(\tau-\tau_1))^2, \qquad \mu_2^2 = \frac{1}{2}(\beta_1^2+\beta_2^2-Ca)\xi^2$$

$$\mu_7^2 = \frac{(Ca-\beta_2^2+\beta_1^2)(\tau-\tau_2)^2}{2(\tau_1-\tau_2)^2}\xi^2, \qquad \mu_8^2 = \frac{(Ca-\beta_1^2+\beta_2^2)(\tau-\tau_1)^2}{2(\tau_1-\tau_2)^2}\xi^2.$$

Finally for $\mu_{3,4,5}$ and μ_6, we set

$$\mu_{3,4,5}^2 = \frac{\lambda_{1,2,3}^2}{\lambda^2+(r^2-2rM)\xi^2}\frac{\nu}{4}(\alpha_1(\tau-\tau_2)-\alpha_2(\tau-\tau_1))^2,$$

respectively

$$\mu_6^2 = \frac{(r^2-2rM)\xi^2}{\lambda^2+(r^2-2rM)\xi^2}\frac{\nu}{4}(\alpha_1(\tau-\tau_2)-\alpha_2(\tau-\tau_1))^2.$$

It is easy to see that in the case $a=0$ all these symbols coincide with the corresponding Schwarzschild symbols. The proof of the lemma is concluded. ■

In what follows, we use the above lemma to prove the bound (4.30) and conclude the proof of the theorem. We begin with symbols s and e as in the lemma. These are homogeneous symbols, and in order to use the pseudodifferential calculus we need to remove the singularity at frequency 0. This is easily achieved by redefining

$$s := \chi_{>1}s, \qquad e := \chi_{>1}e$$

where $\chi_{>1}$ is a smooth symbol which equals 1 at frequencies $\gg 1$ and vanishes at frequencies $\ll 1$. Since both s and e are only defined near $r=3M$, some spatial truncation is also necessary. Let χ be a smooth cutoff function supported near $3M$ which equals 1 in a neighborhood of $3M$, chosen so that we have a smooth partition of unity in r,

$$1 = \chi^2(r) + \chi_o^2(r).$$

At first we define the truncated operators

$$\tilde{S} = \chi s^w \chi, \qquad \tilde{E} = \chi e^w \chi.$$

This choice would yield an expression $Q^{\mathbf{K}}[u, \tilde{S}, \tilde{E}]$ with a principal symbol

$$q^{\mathbf{K}}_{\mathrm{princ}}[\tilde{S}, \tilde{E}] = \chi^2 \left(\frac{1}{2i} \{p_{\mathbf{K}}, s\} + p_{\mathbf{K}} e \right) + \frac{1}{i} \chi s \{p_{\mathbf{K}}, \chi\}.$$

For these choices of $\tilde{S}$ and $\tilde{E}$ we consider the expression $IQ^{\mathbf{K}}[u, \tilde{S}, \tilde{E}]$ which is given by (4.26). For this we need to insure that (4.25) holds. By the Weyl calculus we can write

$$\frac{1}{2}([\Box_{\mathbf{K}}, \tilde{S}] + \Box_{\mathbf{K}} \tilde{E} + \tilde{E} \Box_{\mathbf{K}}) - (q^{\mathbf{K}}_{\mathrm{princ}}[\tilde{S}, \tilde{E}])^w \in \sum_{j=0}^{3} OPS^{-j} D_t^j.$$

We note that the principal symbol $q^{\mathbf{K}}_{\mathrm{princ}}[\tilde{S}, \tilde{E}]$ for $(1/2)([\Box_{\mathbf{K}}, \tilde{S}] + \Box_{\mathbf{K}} \tilde{E} + \tilde{E} \Box_{\mathbf{K}})$ is only a second order polynomial in τ. This shows that $Q^w_{-1} \in OPS^{-3}$. To eliminate this term we slightly adjust our choice of $\tilde{E}$ by adding a lower order term to it,

$$\tilde{E} = \chi e^w \chi - e^w_{\mathrm{aux}} D_t$$

where the operator e^w_{aux} is chosen so that

$$g^{tt} e^w_{\mathrm{aux}} + e^w_{\mathrm{aux}} g^{tt} = Q^w_{-1}.$$

This is possible since the coefficient g^{tt} of τ^2 in $p_{\mathbf{K}}$ is a scalar function which is nonzero near $r = 3M$. Also as defined $e^w_{\mathrm{aux}} \in OPS^{-3}$ and has kernel supported near $r = 3M$. Note that Lemma 1 still holds, since the principal symbols do not change.

Having ensured that the D_t^3 term does not appear, we divide $IQ^{\mathbf{K}}[u, \tilde{S}, \tilde{E}]$ into two parts,

$$IQ^{\mathbf{K}}[u, \tilde{S}, \tilde{E}] = IQ^{\mathbf{K}}_{\mathrm{princ}}[u, \tilde{S}, \tilde{E}] + IQ^{\mathbf{K}}_{\mathrm{aux}}[u, \tilde{S}, \tilde{E}]$$

where the main component is given by

$$IQ^{\mathbf{K}}_{\mathrm{princ}}[u, \tilde{S}, \tilde{E}] = \int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^w_{2,p} u \cdot \overline{u} + 2\Re Q^w_{1,p} u \cdot \overline{D_t u} + Q^w_{0,p} D_t u \overline{D_t u} \, dV_{\mathbf{K}} \tag{4.41}$$

with operators $Q^w_{2,p}$, $Q^w_{1,p}$ and $Q^w_{0,p}$ defined by

$$Q^w_{2,p} + 2Q^w_{1,p}D_t + Q^w_{0,p}D_t^2 = \chi\left(\frac{1}{2i}\{p_{\mathrm{K}}, s\} + p_{\mathrm{K}}e\right)^w \chi$$

while the remainder is given by a similar expression with operators $Q^w_{2,a}$, $Q^w_{1,a}$, and $Q^w_{0,a}$ whose principal symbols are supported away from $r = 3M$. More precisely, we have

$$Q^w_{2,a} + 2Q^w_{1,a}D_t + Q^w_{0,a}D_t^2 - \left(\frac{1}{i}\chi s\{p_{\mathrm{K}}, \chi\}\right)^w \in OPS^0 + OPS^{-1}D_t + OPS^{-2}D_t^2.$$

Hence, using the fact that the LEW^1_{K} norm is nondegenerate outside an $O(a)$ neighborhood of $3M$ we can bound in an elliptic fashion

$$|IQ^{\mathrm{K}}_{\mathrm{aux}}[u, \tilde{S}, \tilde{E}]| \lesssim \|u\|^2_{LEW^1_{\mathrm{K}}} + \|D_t u\|^2_{H^{-1}_{\mathrm{comp}}} \tag{4.42}$$

where the last term on the right represents the H^{-1} norm of $D_t\, u$ in a compact region in r (precisely, a neighborhood of $3M$).

In order to conclude the proof of the theorem, we turn our attention to the bound (4.30), which we seek to establish with S and E replaced with $\tilde{S}$ and $\tilde{E}$, respectively. We will show that

$$\int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^{\mathrm{K}}[u, X, q, m]\mathrm{d}V_{\mathrm{K}} + aIQ^{\mathrm{K}}_{\mathrm{princ}}[u, \tilde{S}, \tilde{E}] \gtrsim \|u\|^2_{LEW^1_{\mathrm{K}}} - O(a)\|D_t u\|^2_{H^{-1}_{\mathrm{comp}}}. \tag{4.43}$$

We decompose the left-hand side of (4.43) into an outer part and an inner part,

$$\mathrm{LHS}(4.43) = \mathrm{LHS}(4.43)_{\mathrm{out}} + \mathrm{LHS}(4.43)_{\mathrm{in}}$$

where

$$\begin{aligned} \mathrm{LHS}(4.43)_{\mathrm{out}} &= \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \chi_o^2 Q^{\mathrm{K}}[u, X, q, m]\mathrm{d}V_k \\ \mathrm{LHS}(4.43)_{\mathrm{in}} &= \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \chi^2 Q^{\mathrm{K}}[u, X, q, m]\mathrm{d}V_k + aIQ^{\mathrm{K}}_{\mathrm{princ}}[u, \tilde{S}, \tilde{E}]. \end{aligned}$$

For the first part, we use the pointwise positivity of Q^{K} away from $3M$ (see (4.23)) to conclude that

$$\mathrm{LHS}(4.43)_{\mathrm{out}} \gtrsim \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \chi_o^2(r^{-2}|\nabla u|^2 + r^{-4}|u|^2)\mathrm{d}V_{\mathrm{K}}. \tag{4.44}$$

The second part is a quadratic form which for convenience we fully recall here (see (4.31) and (4.41)):

$$\mathrm{LHS}(4.43)_{\mathrm{in}} = \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \chi^2 (q^{K,\alpha\beta}\partial_\alpha u \partial_\beta u + q^{K,0}u^2) \mathrm{d}V^{\mathbf{K}}$$
$$+a\int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^w_{2,p}u\cdot\overline{u} + 2\Re Q^w_{1,p}u\cdot\overline{D_t u} + Q^w_{0,p}D_t u\overline{D_t u}\,\mathrm{d}V_{\mathbf{K}}$$

where the coefficients $q^{K,\alpha\beta}$, $q^{K,0}$, and operators $Q^w_{j,p}$ satisfy

$$q^{K,\alpha\beta}\eta_\alpha\eta_\beta = \frac{1}{2i}\{p_{\mathbf{K}}, X\} + qp_{\mathbf{K}}, \qquad q^{K,0} > 0,$$

respectively

$$Q^w_{2,p} + 2Q^w_{1,p}D_t + Q^w_{0,p}D_t^2 = \chi\left(\frac{1}{2i}\{p_{\mathbf{K}}, s\} + p_{\mathbf{K}}e\right)^w \chi.$$

We carefully observe that in the two parts of the expression for $\mathrm{LHS}(4.43)_{\mathrm{in}}$ the cutoff function χ appears in different places. In the first part, it is applied after the differentiation, while in the second part it is applied before the pseudodifferential operator. It does not make much sense to commute at this point. In the first part, we would produce lower order terms which may significantly alter $q^{K,0}$. In the second part, we would lose the compact support of the kernels for the operators $Q^w_{j,p}$.

Since s and e are chosen as in Lemma 1, it follows that the principal symbol for $\mathrm{LHS}(4.43)_{\mathrm{in}}$ admits the sum of squares representation (4.36). We want to translate this into a sum of squares decomposition for $\mathrm{LHS}(4.43)_{\mathrm{in}}$. However, we want all the lower order terms in the pseudodifferential calculus to have size $O(a)$ (as opposed to $O(1)$), therefore some care is required due to the different positions of the cutoff χ, as explained above. The symbols $\mu_k = \mu_k(a)$ are in general of pseudodifferential type. However, part (i) of the Lemma guarantees that in the Schwarzschild case they are of differential type. Consequently, we write

$$\mu_k(a) = \mu_k(0) + \mu_k(a) - \mu_k(0)$$

and use this decomposition to define the pseudodifferential operators

$$M_k = \chi\mu_k(0)(x, D) + (\mu_k(a) - \mu_k(0))^w\chi.$$

Then using the Weyl calculus, it follows that for LHS(4.43)$_{\mathrm{in}}$ we have the representation

$$\mathrm{LHS}(4.43)_{\mathrm{in}} = \int_{\mathcal{M}_{[0,\tilde{v}_0]}} \sum_k |M_k u|^2 + q^{K,0}\chi^2 u^2 \mathrm{d}V_k + \int_{\mathcal{M}_{[0,\tilde{v}_0]}} R_2^w u \cdot \bar{u} + 2\Re R_1^w D_t u \cdot \bar{u} + R_0^w D_t u \cdot \overline{D_t u} \mathrm{d}V_{\mathrm{K}}.$$

To start with, the symbols for the remainder terms satisfy $r_j \in S^{j-2}$ and depend smoothly on a. In addition, our choice of the operators M_k guarantees that when $a = 0$ the remainder is zero. Hence, we obtain the better relation $r_j \in aS^{j-2}$.

Combining the last relation with (4.44), we obtain the bound

$$\int_{\mathcal{M}_{[0,\tilde{v}_0]}} \chi_o^2 r^{-2}|\nabla u|^2 + r^{-4}|u|^2 + \sum_k |M_k u|^2 \mathrm{d}V_{\mathrm{K}} \lesssim \mathrm{LHS}(4.43) + a(\|u\|^2_{L^2_{\mathrm{comp}}} + \|D_t u\|^2_{H^{-1}_{\mathrm{comp}}})$$

where the last two terms on the right account for the remainder terms involving the operators R_j^w, which can be bounded using norms of u and $D_t u$ in a compact region in r, away from $r = 0$ and $r = \infty$.

It is easy to see that the above left-hand side dominates $\|u\|_{LEW^1_{\mathrm{K}}}$. For r away from $3M$, one uses only the first two terms. On the other hand, for r close to $3M$ we use part (ii) of the lemma, which guarantees that the symbols $c_1(\tau - \tau_2)$, $c_2(\tau - \tau_1)$, and ξ can be recovered in an elliptic fashion from the principal symbols μ_k of M_k. We can now use [41, Corollary II.8], which says

Corollary 4.4. Let $a_j, b \in C^{1,1}S^1$ be real symbols with $|b| \le \sum |a_j|$. Then

$$\|B(x, D)u\|_{L^2} \lesssim \sum \|A_j(x, D)u\|_{L^2} + \|u\|_{L^2}.$$

□

Thus, (4.43) is proved. Together with (4.42), this shows that

$$\|u\|^2_{LEW^1_{\mathrm{K}}} \lesssim \int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^{\mathrm{K}}[u, X, q, m]\mathrm{d}V_{\mathrm{K}} + aIQ^{\mathrm{K}}[u, \tilde{S}, \tilde{E}] + O(a)\|D_t u\|^2_{H^{-1}_{\mathrm{comp}}}.$$

The final step in the proof of (4.30) is to establish that the last error term above is negligible. We can account for it in an elliptic manner. Precisely, for any compactly supported self-adjoint operator $Q \in OPS^{-1}$ we can use Q^2 in a Lagrangian term and integrate by parts (commute) to obtain

$$\begin{aligned}\Re \int_{\mathcal{M}_{[0,\tilde{v}_0]}} (g^{tt})^{-1}\Box_{\mathbf{K}} u \cdot \overline{Q^2 u}\, dV_{\mathbf{K}} &= \int_{\mathcal{M}_{[0,\tilde{v}_0]}} -\partial_t^2 Qu \cdot \overline{Qu} + Q_0 \partial_t u \cdot \overline{Qu}\, dV_{\mathbf{K}} + O(\|u\|^2_{L^2_{comp}}) \\ &= \int_{\mathcal{M}_{[0,\tilde{v}_0]}} |Q\partial_t u|^2 - Q_0 u \cdot Q\partial_t u\, dV_{\mathbf{K}} + \int_{\Sigma_t} -\partial_t Qu \cdot \overline{Qu} + Q_0 u \cdot Qu\, dA_{\mathbf{K}} \Big|_{t=0}^{t=\tilde{v}_0} + O(\|u\|^2_{L^2_{comp}}) \\ &= \|QD_x u\|^2_{L^2} + O(\|QD_x u\|_{L^2}\|u\|_{L^2_{comp}} + \|u\|^2_{L^2_{comp}} + E[u](0) + E[u](\tilde{v}_0))\end{aligned}$$

where $Q_0 \in OPS^0$ and the $(g^{tt})^{-1}$ factor is inserted in order to cancel the coefficient of ∂_t^2 in $\Box_{\mathbf{K}}$. This leads to the elliptic bound

$$\|QD_t u\|^2_{L^2} \lesssim \|u\|^2_{L^2_{comp}} + \|\Box_{\mathbf{K}} u\|^2_{H^{-1}_{comp}} + E[u](0) + E[u](\tilde{v}_0)$$

and further to

$$\|D_t u\|^2_{H^{-1}_{comp}} \lesssim \|u\|^2_{L^2_{comp}} + \|\Box_{\mathbf{K}} u\|^2_{H^{-1}_{comp}} + E[u](0) + E[u](\tilde{v}_0).$$

Thus, (4.30) follows, and the proof of the theorem is concluded. ■

Note that Theorem 4.1 tells us, in particular, that if we start with an initial data $(u_0, u_1) \in H^1 \times L^2$ then $u(\tilde{v}) \in H^1$ is uniformly bounded for all $\tilde{v} > 0$. A natural question to ask is if this is also true for higher H^n norms. For $n \geq 1$, we define

$$\|u\|_{LE^{n+1}_{\mathbf{K}}} = \sum_{|\alpha| \leq n} \|\partial^\alpha u\|_{LE^1_{\mathbf{K}}},$$

respectively

$$\|f\|_{LE^{n*}_{\mathbf{K}}} = \sum_{|\alpha| \leq n} \|\partial^\alpha f\|_{LE^*_{\mathbf{K}}}.$$

The higher order energies are similarly defined,

$$E^{n+1}[u](\Sigma_R^\pm) = \sum_{|\alpha| \leq n} E[\partial^\alpha u](\Sigma_R^\pm), \qquad E^{n+1}[u](\tilde{v}_0) = \sum_{|\alpha| \leq n} E[\partial^\alpha u](\tilde{v}_0).$$

We then have the following

Theorem 4.5. Let n be a positive integer and u satisfy $\Box_{\mathbf{K}} u = f$ with initial data $(u_0, u_1) \in H^{n+1} \times H^n$ on Σ_R^- and $f \in LE_{\mathbf{K}}^{n*}(\mathcal{M}_R)$. Then

$$E^{n+1}[u](\Sigma_R^+) + \sup_{\tilde{v}>0} E^{n+1}[u](\tilde{v}_0) + \|u\|^2_{LE_{\mathbf{K}}^{n+1}} \lesssim \|u_0\|^2_{H^{n+1}} + \|u_1\|^2_{H^n} + \|f\|^2_{LE_{\mathbf{K}}^{n*}}. \qquad \Box$$

Proof. We remark that by trace regularity results, we have

$$\sum_{|\alpha|\leq n-1} \|\partial^\alpha f\|_{L^2(\Sigma_R^-)} \lesssim \|f\|_{LE_{\mathbf{K}}^{n*}}.$$

Since the initial surface Σ_R^- is space-like, we can use the equation to derive all higher $\tilde{v}$ derivatives of u in terms of the Cauchy data (u_0, u_1) and f,

$$E^{n+1}[u](\Sigma_R^-) \lesssim \|u_0\|^2_{H^{n+1}} + \|u_1\|^2_{H^n} + \|f\|^2_{LE_{\mathbf{K}}^{n*}}.$$

Thus, it suffices to prove that for $\tilde{v}_0 > 0$ we have

$$E^{n+1}[u](\Sigma_R^+) + E^{n+1}[u](\tilde{v}_0) + \|u\|^2_{LE_{\mathbf{K}}^{n+1}} \lesssim E^{n+1}[u](\Sigma_R^-) + \|f\|^2_{LE_{\mathbf{K}}^{n*}}. \tag{4.45}$$

We will prove this for $n = 1$, and the proof for the other cases will follow in a similar manner by induction.

Since $\partial_{\tilde{v}}$ is a Killing vector field, we have $\Box_{\mathbf{K}}(\partial_{\tilde{v}} u) = \partial_{\tilde{v}} f$. Then by Theorem 4.1, we obtain

$$E[\partial_{\tilde{v}} u](\Sigma_R^+) + E[\partial_{\tilde{v}} u](\tilde{v}_0) + \|\partial_{\tilde{v}} u\|^2_{LE_{\mathbf{K}}^1} \lesssim E^2[u](\Sigma_R^-) + \|f\|^2_{LE_{\mathbf{K}}^{1*}}. \tag{4.46}$$

In order to control the rest of the second order derivatives, we take advantage of the equation, which takes the form

$$(g^{\tilde{v}\tilde{v}} \partial_{\tilde{v}\tilde{v}} + 2g^{\tilde{v}\tilde{\phi}} \partial_{\tilde{v}\tilde{\phi}} + L) u = f \tag{4.47}$$

where L is a spatial partial differential operator of order 2. This is most useful in the region where $\partial_{\tilde{v}}$ is time-like. Given $\epsilon > 0$, this happens in the region of the form $r > 2M + \epsilon$ provided that a is sufficiently small. The fact that $\partial_{\tilde{v}}$ is time-like is equivalent to the ellipticity of the spatial part L of $\Box_{\mathbf{K}}$. From (4.46), we obtain at $\tilde v = \tilde v_0$

$$\|Lu\|^2_{L^2(\Sigma_{\tilde v_0})} \lesssim E[u](v_0) + E[\partial_{\tilde v} u](\tilde v_0) + \|f\|^2_{L^2(\Sigma_{\tilde v_0})}.$$

The operator L on the left is elliptic in $r \geq 2M + \epsilon$, therefore by a standard elliptic estimate we obtain

$$E[\nabla u](\Sigma_{\tilde v_0} \cap \{r > 2M + \epsilon\}) \lesssim E[u](\tilde v_0) + E[\partial_{\tilde v} u](\tilde v_0) + \|f\|^2_{L^2(\Sigma_{\tilde v_0})}.$$

A similar elliptic analysis leads to the corresponding local energy bound,

$$\|\nabla u\|^2_{LE^1_{\mathbf{K}}(\mathcal{M}_R \cap \{r > 2M + 2\epsilon\})} \lesssim \|u\|^2_{LE^1_{\mathbf{K}}(\mathcal{M}_R)} + \|\partial_{\tilde v} u\|^2_{LE^1_{\mathbf{K}}(\mathcal{M}_R)} + \|f\|^2_{L^2(\mathcal{M}_R)}.$$

We are left to deal with the case $r < 2M + 2\epsilon$, where g^{rr} is small and simply using Equation (4.47) does not suffice. Let $\zeta(r)$ be a smooth cutoff function such that $\zeta = 1$ on $[r_e, r_+ + 2\epsilon]$ and $\zeta = 0$ when $r > r_+ + 3\epsilon$. Then we need bounds for the function $w = \zeta u$, which solves

$$\Box_{\mathbf{K}} w = \zeta f + [\Box_{\mathbf{K}}, \zeta] u := g.$$

The commutator above is supported in the region $\{2M + 2\epsilon \leq r \leq 2M + 3\epsilon\}$ where we already have good estimates for u. Recall that in the region $\{r < 2M + 3\epsilon\}$ the $LE^1_{\mathbf{K}}$ and $LE^*_{\mathbf{K}}$ norms are equivalent with the H^1, respectively L^2 norm. Hence, it remains to prove that for all functions w with support in $\{r < 2M + 3\epsilon\}$ which solve $\Box_{\mathbf{K}} w = g$ we have

$$E[\nabla w](\Sigma_R^+) + E[\nabla w](\tilde v_0) + \|\nabla w\|^2_{H^1(\mathcal{M}_R)} \lesssim E^2[u](\Sigma_R^-) + \|g\|^2_{H^1(\mathcal{M}_R)}. \tag{4.48}$$

This is an estimate which is localized near the event horizon, and we will prove it taking advantage of the red shift effect.

Since $\partial_{\tilde v}$ is a Killing vector field, this bound follows directly from Theorem 4.1 for the $\partial_{\tilde v} w$ component of ∇w,

$$E[\partial_{\tilde v} w](\Sigma_R^+) + E[\partial_{\tilde v} w](\tilde v_0) + \|\partial_{\tilde v} w\|^2_{H^1(\mathcal{M}_{[0,\tilde v_0]})} \lesssim E^2[w](\Sigma_R^-) + \|g\|^2_{H^1(\mathcal{M}_{[0,\tilde v_0]})}. \tag{4.49}$$

Consider now the angular derivatives of w, $\partial_\omega w$. We know that

$$[\Box_{\mathbf{S}}, \partial_\omega] = 0$$

since the Schwarzschild metric is spherically symmetric. Hence, by (4.18) it follows that $[\Box_{\mathbf{K}}, \partial_\omega]$ is a second order operator whose coefficients have size $O(a)$. Hence, by Theorem 4.1 we obtain

$$E[\partial_\omega w](\Sigma_R^+) \quad +E[\partial_\omega w](\tilde{v}_0) + \|\partial_\omega w\|^2_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})} \lesssim E^2[w](\Sigma_R^-) + \|g\|^2_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})}$$
$$+a\|\partial_\omega w\|^2_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})}. \tag{4.50}$$

We still need to bound $\partial_r w$. For that, we compute the commutator

$$[\Box_{\mathbf{K}}, \partial_r]w = -(\partial_r g^{rr})\partial_{rr} w + Tw \tag{4.51}$$

where T stands for a second order operator with no ∂_r^2 terms.

The key observation, which is equivalent to the red shift effect, is that the coefficient $\gamma = \partial_r g^{rr} > 0$ near $r = 2M$; a similar argument based on this observation was previously made in [14]. This can be interpreted geometrically in terms of the Hamilton flow for the Schwarzschild space–time, which on the trapped set on the event horizon $\{r = 2M, \tau = 0, \lambda = 0\}$ has the form

$$\dot{\xi} = -\frac{1}{2}\gamma\xi.$$

This shows that on this trapped set the frequency decreases exponentially, which heuristically implies microlocal exponential energy decay near these geodesics in the high-frequency limit. This property is stable with respect to small perturbations of the metric, so it transfers to the Kerr space–time with small angular momentum.

Thus, for X, C, and q as in Lemma 4.2 we can write the equation for $\partial_r w$ in the form

$$(\Box_{\mathbf{K}} - \gamma[(X + CK) + q])\partial_r w = \partial_r g + Tw$$

with T as above and most importantly, a positive coefficient γ. Because of this, the operator

$$B = \Box_{\mathbf{K}} - \gamma[(X + CK) + q]$$

satisfies the same estimate in Theorem 4.1 as $\Box_g$ for functions supported near the event horizon. Indeed, the same proof goes through as in Theorem 3.2. Writing the integral identity (4.21) for w, we see that the contribution of the term proportional to γ is negative, therefore we obtain the inequality

$$\int_{\mathcal{M}_{[0,\tilde{v}_0]}} Q^{\mathrm{K}}[w,X,q,m]\mathrm{d}V_{\mathrm{K}} \leq -\int_{\mathcal{M}_{[0,\tilde{v}_0]}} (\partial_r g + Tw)\Big((X+CK)w + qw\Big)\mathrm{d}V_{\mathrm{K}} + BDR^{\mathrm{K}}[w].$$

By (4.23), the left-hand side is positive definite for $r < 2M + 3\epsilon$. Using Cauchy–Schwarz for the first term on the right and (4.22) for the second, we obtain

$$E[\partial_r w](\Sigma_R^+) + E[\partial_r w](\tilde{v}_0) + \|\partial_r w\|^2_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})} \lesssim E^2[w](\Sigma_R^-) + \|\partial_r g + Tw\|^2_{L^2(\mathcal{M}_{[0,\tilde{v}_0]})}.$$

Since T contains no second order r derivatives, this leads to

$$E[\partial_r w](\Sigma_R^+) + E[\partial_r w](\tilde{v}_0) + \|\partial_r w\|^2_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})} \lesssim E^2[w](\Sigma_R^-) + \|g\|_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})} + \|\nabla_{\omega,\tilde{v}} w\|^2_{H^1(\mathcal{M}_{[0,\tilde{v}_0]})}. \tag{4.52}$$

Then the desired bound (4.48) follows by combining (4.49), (4.50), and (4.52) with appropriate coefficients. ■

As an easy corollary, one obtains from Sobolev embeddings the pointwise boundedness result,

Corollary 4.5. If u satisfies $\Box_{\mathrm{K}} u = 0$ in $\mathcal{M}_R$ with initial data $(u_0, u_1) \in H^2 \times H^1$ in Σ_R^-, then

$$\|u\|_{L^\infty} \lesssim \|u_0\|_{H^2} + \|u_1\|_{H^1}.$$ □

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