Linear Waves on higher dimensional Schwarzschild black holes and Schwarzschild de Sitter spacetimes

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Declaration of Originality

The research presented in this thesis was conducted at the Department of Pure Mathematics and Mathematical Statistics, University of Cambridge in the period between October 2008 and April 2012. The work contained in this thesis is original and was entirely performed by myself. The content of Chapter 1 has been submitted in part to the *Smith-Rayleigh-Knight Prize* in January 2010 and in full for publication in a journal; a shortened version is also available online at `http://arxiv.org/abs/1012.5963`. The content of Chapter 2 is unpublished in any form at the time of submission. This dissertation has not been submitted for any degree or other qualification.

Summary

I study linear waves on higher dimensional Schwarzschild black holes and Schwarzschild de Sitter spacetimes.

In the first part of this thesis two decay results are proven for general finite energy solutions to the linear wave equation on higher dimensional Schwarzschild black holes. I establish uniform energy decay and improved interior first order energy decay in all dimensions with rates in accordance with the $3+1$-dimensional case. The method of proof departs from earlier work on this problem. I apply and extend the new physical space approach to decay of Dafermos and Rodnianski. An integrated local energy decay estimate for the wave equation on higher dimensional Schwarzschild black holes is proven.

In the second part of this thesis the global study of solutions to the linear wave equation on expanding de Sitter and Schwarzschild de Sitter spacetimes is initiated. I show that finite energy solutions to the initial value problem are globally bounded and have a limit on the future boundary that can be viewed as a function on the standard cylinder.

Both problems are related to the Cauchy problem in General Relativity.

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Contents

Preface

In the dynamical formulation of General Relativity Theory we are presented with a Cauchy problem describing the evolution of an initial geometric data set under the laws of gravitation, namely Einstein's field equations; (an interesting introduction to the Cauchy problem in General Relativity is for example to be found in [12]). In the absence of symmetry this problem has only been fully understood for initial data sufficiently close to Minkowski space in the seminal work of Christodoulou and Klainerman [11], and for initial data close to de Sitter space by Friedrich [28].

The resolution of the Cauchy problem with initial data close to a given black hole spacetime has remained elusive for many years. Our expectations for the global behaviour of these solutions are the content of the *nonlinear black hole stability conjecture* and its proof is one of the major open problems in the field; (for a rough formulation of this conjecture see e.g. [18]).

The motivation for the problems treated in this thesis is largely drawn from their relation to the nonlinear stability problem and my results can be viewed as *linear stability* statements. Instead of studying the dynamics of the spacetime itself (which is governed by equations that are *hyperbolic* in nature) we fix a spacetime manifold and study solutions to the *linear wave equation* on this background. (For an introduction to the linear theory in the context of black hole spacetimes see [17].) We are interested in uniform boundedness and decay properties of general finite energy solutions and aim at proving them in a suitably robust way.

It is important to recognize here the role of the method used in the proof. While in this thesis we always take an explicitly known solution of the Einstein equations as the background spacetime manifold the methods we use and develop manifestly lend themselves to the application of our proofs to the study of the wave equation on *small perturbations* of the background geometry. It is mainly due to this fact that our work has relevance to the question of black hole stability.

We shall be concerned with the study of linear waves on two explicitly known black hole spacetimes: *(i)* higher dimensional Schwarzschild black holes and *(ii)* Schwarzschild de Sitter spacetimes. The treatment of linear waves on black hole spacetimes in $3+1$ dimensions has been completed in the much larger class of rotating Kerr black holes; (for a review of the black hole stability problem for linear scalar perturbations see [19]). In view of recent advances in the analysis of hyperbolic partial differential equations (see e.g. [17] and references therein) which have lead to more robust methods (in particular a new physical space approach to proving decay [22]) it is natural to apply these methods to the study of linear waves on higher dimensional black holes. (Black hole spacetimes in high dimensions are also of independent interest for high energy physics; more on the relevance for theories in high energy physics can be found in [25].) Moreover I have advanced these methods which has lead to more robust proofs of already known results in 3+1 dimensions. Our interest in *(ii)* stems from the fact that the expansion of a spacetime is believed to introduce another decay mechanism into the problem which may bring the resolution of the nonlinear stability problem in the context of expanding black hole spacetimes within reach. At any rate, our present understanding of cosmology justifies the choice made in this thesis of expanding spacetimes as background manifolds corresponding to a positive cosmological constant; (for a historical account of the discovery of the expanding universe see [5]).

Let us give an informal statement of the results in this thesis.

Result 1 (Decay of linear waves on higher dimensional Schwarzschild black holes)**.** *In Chapter 1 we consider solutions to the linear wave equation on higher dimensional Schwarzschild black hole spacetimes and prove robust nondegenerate energy decay estimates that are in principle required in a nonlinear stability problem. More precisely, it is shown that for solutions ϕ to the wave equation on the domain of outer communications of the Schwarzschild spacetime manifold $(\mathcal{M}^n_m, g)$ (where $n \geq 3$ is the spatial dimension, and $m > 0$ is the mass of the black hole) the associated energy flux $E[\phi](\Sigma_\tau)$ through a foliation of hypersurfaces Σ_τ (terminating at future null infinity and crossing the event horizon to the future of the bifurcation sphere, obtained by timelike translations along the Killing vectorfield $T = \partial_t$ where $\partial_t\tau = 1$) decays,*

$$E[\phi](\Sigma_\tau) \leq \frac{CD}{\tau^2}\,,$$

where C is a constant depending on n and m, and $D < \infty$ is a suitable higher order initial energy on Σ_0 (i.e. an energy involving commutations with the Killing vectorfields of $\mathcal{M}^n_m$, in particular T); moreover we improve the decay rate for the first order energy to

$$E[\partial_t\phi](\Sigma^R_\tau) \leq \frac{CD_\delta}{\tau^{4-2\delta}}$$

for any $\delta > 0$ where Σ^R_τ denotes the hypersurface Σ_τ truncated at an arbitrarily large fixed radius $R < \infty$ provided the higher order energy D_δ on Σ_0 is finite. We conclude our treatment by interpolating between these two results to obtain the pointwise estimate

$$|\phi|_{\Sigma^R_\tau} \leq \frac{CD'_\delta}{\tau^{\frac{3}{2}-\delta}}\,.$$

A precise statement of Result 1 is given in Section 1.1.1. The decay argument developed in Section 1.5 is sufficiently robust to imply similar decay results for solutions to the wave equation on a wide class of asymptotically flat spacetimes. We elaborate on our argument for Minkowski space in $3+1$ dimensions in Appendix A.

Result 2 (Decay of linear waves on asymptotically flat spacetimes)**.** *Improved interior first order energy decay as stated in Result 1 holds for solutions to the linear wave equation in all dimensions on a wide class of asymptotically flat black hole exteriors, whenever an integrated local energy decay estimate is available. In particular, it holds on small perturbations of Minkowski space.*

Note that our decay results are obtained in the domain of outer communications or in other words the exterior of the black hole up to and including the event horizon. This is precisely the region of spacetime which is expected to be *stable.* In my recent work I turned to cosmological black hole spacetimes and investigated what is expected to be the stable region of Schwarzschild de Sitter spacetimes. The exterior of the black hole does here not only contain a static region but also what shall be referred to as the *expanding region* lying to the future of the static domain beyond a cosmological horizon. While the linear theory for the static region (namely the domain enclosed by the event and cosmological horizons) has already been addressed in [16, 4, 38], we make use of an additional stability mechanism in the expanding region.

Result 3 (Global boundedness of linear waves on Schwarzschild de Sitter spacetimes)**.** *In Chapter 2 I describe the global study of linear waves on de Sitter and Schwarzschild de Sitter spacetimes. We prove that solutions to the Cauchy problem for the linear wave equation posed in the past of the future boundary Σ^+ of the expanding region of (subextremal) Schwarzschild de Sitter spacetimes $(\mathcal{M}_\Lambda^{(m)}, g)$ are globally bounded and have a limit on Σ^+ which can be viewed as a function on the standard cylinder $\mathbb{R}\times\mathbb{S}^2$ provided their energy is initally finite. In fact, if Σ is a spacelike hypersurface in the past of Σ^+ (such that Σ^+ is in the domain of dependence of Σ) and the energy $D[\phi]$ of a solution ϕ to the linear wave equation on Σ is finite, then*

$$\int_{\Sigma^+} \big|\overset{\circ}{\nabla}\phi\big|^2 \,\mathrm{d}\mu_{\overset{\circ}{g}} \leq C\, D[\phi]$$

where C is a constant that only depends on Σ, m and Λ, and $\overset{\circ}{g}$ and $\overset{\circ}{\nabla}$ denote the standard metric and derivatives on the cylinder. Moreover, we show that on de Sitter spacetimes this limit is identically zero for solutions to the Klein-Gordon equation $\Box_g\phi = m_\Lambda\phi$ (provided $m_\Lambda \geq 2\frac{\Lambda}{3}$).

The precise formulation of Result 3 is given in Section 2.2 for de Sitter spacetimes and in Section 2.3 for Schwarzschild de Sitter spacetimes. The fact that a global solution to the linear wave equation on expanding black hole spacetimes has a limit on the future boundary as a function on the cylinder for which we have an explicit global integral bound is a nontrivial statement which is in agreement with our expectations for the nonlinear stability problem.

Chapter 1

Decay of linear waves on higher dimensional Schwarzschild black holes

1.1 Overview

The study of the wave equation on black hole spacetimes has generated considerable interest in recent years. As discussed in the preface this stems mainly from its role as a model problem for the nonlinear black hole stability problem [18, 19], and more recent advances in the analysis of linear waves [17].

In this Chapter we study the linear wave equation on higher dimensional Schwarzschild black holes. The motivation for this problem lies — apart from the above mentioned relation to the nonlinear stability problem (which is expected to be simpler in the higher dimensional case [9]; for work on the 5-dimensional case under symmetry see also [15, 30]) — on one hand in the purely mathematical curiosity of dealing with higher dimensions and on the other hand in its interest for theories of high energy physics [25].

In the philosophy of [11] it is understood that the resolution of the nonlinear stability problem requires an understanding of the linear equations in a sufficiently robust setting. In particular, we require a proof of the uniform boundedness and decay of solutions to the linear wave equation based on the method of energy currents which (ideally) only uses properties of the spacetime that are stable under perturbations, and does not rely heavily on the specifics of the unperturbed metric; (for an introduction in the context of black hole spacetimes see [17]). Correspondingly we establish on higher dimensional Schwarzschild spacetime backgrounds boundedness and decay results analogous to the current state of the art in the $3+1$-dimensional case [36].

The decay argument presented here departs from earlier work that either makes use of multipliers with weights in the temporal variable (notably [10, 2, 7, 20, 36]) which in one

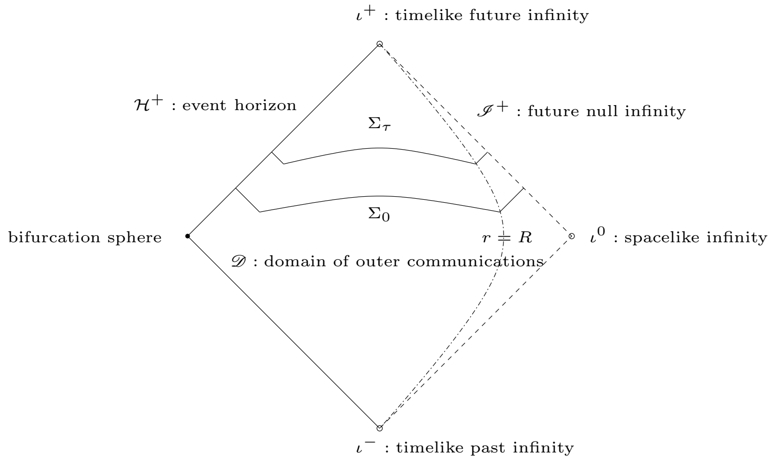

Figure 1.1: The hypersurface Σ_0 in the domain of outer communications $\mathcal{D}$.

form or the other are due to Morawetz [39], or that relies on the exact stationarity of the spacetime (such as [8, 46, 24] based on Fourier analytic methods). Here we follow the new physical-space approach to decay of Dafermos and Rodnianski [22], which only uses multipliers with weights in the radial variable. Thus my work — especially the improvement of Section 1.5.3 — is of independent interest for the 3 + 1-dimensional Schwarzschild and Minkowski case and also for a wider class of spacetimes including Kerr black hole exteriors.

1.1.1 Statement of the Theorems

We consider solutions to the wave equation

$$\Box_g \phi = 0 \tag{1.1.1}$$

on higher dimensional Schwarzschild black hole spacetimes; these backgrounds are a family of $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^n_m, g)$ parametrized by the mass of the black hole $m > 0$, $(n \geq 3)$. They arise as spherically symmetric solutions of the vacuum Einstein equations, the governing equations of General Relativity, and are discussed as such in Section 1.2; for the relevant concepts see also [17, 29].

More precisely, we consider solutions to (1.1.1) on the domain of outer communications $\mathcal{D}$ of $\mathcal{M}$ — which comprises the exterior up to and including the event horizons of the black hole — with initial data prescribed on a hypersurface Σ_0 consisting of an incoming null segment crossing the event horizon to the future of the bifurcation sphere, a spacelike segment and an outgoing null segment emerging from a larger sphere of radius R terminating at future null infinity; see figure 1.1 (the exact parametrization – which is chosen merely for technical reasons – is given in Section 1.4).

In the exterior of the black hole the metric g takes the classical form in (t, r)-coordinates

[45],

$$g = -\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}t^2 + \big(1 - \frac{2m}{r^{n-2}}\big)^{-1}\,\mathrm{d}r^2 + r^2\,\overset{\circ}{\gamma}_{n-1}\,, \tag{1.1.2}$$

where $r > \sqrt[n-2]{2m}$, $t \in (-\infty, \infty)$, and $\overset{\circ}{\gamma}_{n-1}$ denotes the standard metric on the unit $n-1$-sphere; however this coordinate system breaks down on the horizon $r = \sqrt[n-2]{2m}$ and we shall for that reason introduce in Section 1.2 the global geometry of $(\mathcal{M}^n_m, g)$ using a double null foliation, from which we derive an alternative double null coordinate system for the exterior of the black-hole,

$$g = -4\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}u^*\,\mathrm{d}v^* + r^2\,\overset{\circ}{\gamma}_{n-1}\,, \tag{1.1.3}$$

so called Eddington-Finkelstein coordinates.

In this work both the conditions on the initial data and the statements on the decay of the solutions are formulated using the concepts of energy and the energy momentum tensor associated to (1.1.1) in particular (see Section 1.1.2 and also Appendix B.2):

$$T_{\mu\nu}[\phi] = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\,g_{\mu\nu}\,\partial^\alpha\phi\partial_\alpha\phi\,. \tag{1.1.4}$$

The corresponding 1-contravariant-1-covariant tensorfield fulfills the physical requirement that the linear transformation $-T : T\mathcal{M} \to T\mathcal{M}$ maps the hyperboloid of future-directed unit timelike vectors into the closure of the open future cone at each point. Physically,

$$-T \cdot u \in T_p\mathcal{M}$$

is the energy-momentum density relative to an observer at $p \in \mathcal{M}$ with 4-velocity $u \in T_p\mathcal{M}$, and it is for this reason that we refer to

$$\varepsilon = g(T \cdot u, u) = T(u, u) \geq 0$$

as the energy density at $p \in \mathcal{M}$ relative to the observer with 4-velocity $u \in T_p\mathcal{M}$. One may think of a spacelike hypersurface as a collection of locally simultaneous observers with a 4-velocity given by the normal. The hypersurfaces relative to which we establish energy decay are simply defined by $\Sigma_\tau \doteq \varphi_\tau(\Sigma_0 \cap \mathcal{D})$, where φ_τ denotes the 1-parameter group of isometries generated by $\frac{\partial}{\partial t}$. The energy flux through the hypersurface Σ_τ is then given by

$$E[\phi](\Sigma_\tau) \doteq \int_{\Sigma_\tau} \Big(J^N[\phi], n\Big) \tag{1.1.5}$$

where $(J^N[\phi], n) \doteq T[\phi](N, n)$, n is the normal[1] to Σ_τ and N is a timelike φ_τ-invariant future directed vectorfield which is constructed in Section 1.3 for the purpose of turning $\varepsilon^N \doteq T(N, N)$ into a nondegenerate energy up to and including the horizon. Note that the energy $E[\phi](\Sigma_\tau)$ in particular bounds a suitably defined $\dot{\mathrm{H}}^1$-norm on Σ_τ.

[1]On spacelike segments of Σ_τ the vector n is indeed timelike; however, on the null segments of the hypersurfaces Σ_τ the "normal" n is in fact a null vector, but the notation is kept for convenience; see Appendix B.1.

The classes of solutions to (1.1.1) to which our results apply are formulated in terms of finite energy conditions on the initial data, for the purpose of which we list the following quantities:

$$D_2^{(2)}(\tau_0) \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \sum_{k=0}^{1} r^2 \Big(\frac{\partial(r^{\frac{n-1}{2}}\partial_t^k\phi)}{\partial v^*}\Big)^2\Big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \Big(\sum_{k=0}^{2} J^N[\partial_t^k\phi], n\Big) \quad (1.1.6)$$

$$\begin{aligned} D_5^{(4-\delta)}(\tau_0) \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Big\{ \sum_{k=0}^{1} r^{4-\delta} \Big(\frac{\partial^2(r^{\frac{n-1}{2}}\partial_t^k\phi)}{\partial v^{*2}}\Big)^2 \\ + \sum_{k=0}^{4} r^2 \Big(\frac{\partial(r^{\frac{n-1}{2}}\partial_t^k\phi)}{\partial v^*}\Big)^2 + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2 \Big(\frac{\partial r^{\frac{n-1}{2}}\Omega_i\partial_t^k\phi}{\partial v^*}\Big)^2 \Big\}\Big|_{u^*=\tau_0} \\ + \int_{\Sigma_{\tau_0}} \Big(\sum_{k=0}^{5} J^N[\partial_t^k\phi] + \sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N[\Omega_i\partial_t^k\phi], n\Big) \quad (1.1.7) \end{aligned}$$

$$\begin{aligned} D_{7+[\frac{n}{2}]}^{(4-\delta)}(\tau_0) \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Big\{ \sum_{k=0}^{2} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^{4-\delta} \Big(\frac{\partial^2(r^{\frac{n-1}{2}}\Omega^\alpha\partial_t^k\phi)}{\partial v^{*2}}\Big)^2 \\ + \sum_{k=0}^{5} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^2 \Big(\frac{\partial r^{\frac{n-1}{2}}\Omega^\alpha\partial_t^k\phi}{\partial v^*}\Big)^2 + \sum_{k=0}^{4} \sum_{|\alpha|\leq[\frac{n}{2}]+2} r^2 \Big(\frac{\partial r^{\frac{n-1}{2}}\Omega^\alpha\partial_t^k\phi}{\partial v^*}\Big)^2 \Big\}\Big|_{u^*=\tau_0} \\ + \int_{\Sigma_{\tau_0}} \Big(\sum_{k=0}^{6} \sum_{|\alpha|\leq[\frac{n}{2}]+1} J^N[\Omega^\alpha\partial_t^k\phi] + \sum_{k=0}^{5} \sum_{|\alpha|\leq[\frac{n}{2}]+2} J^N[\Omega^\alpha\partial_t^k\phi], n\Big) \quad (1.1.8) \end{aligned}$$

Here $\Omega_i : i = 1, \ldots, \frac{n(n-1)}{2}$ are the generators of the spherical isometries of the spacetime $\mathcal{M}$, α is a multiindex, and for any radius R we denote by R^* the corresponding Regge-Wheeler radius (1.2.22). (See also Section 1.4.2.)

Among the propositions on linear waves on higher dimensional Schwarzschild black hole spacetimes proven in this thesis, we wish to highlight the following conclusions[2].

Theorem 1 (Energy decay). *Let ϕ be a solution of the wave equation $\Box_g\phi = 0$ on $\mathcal{D} \subset \mathcal{M}_m^n$, where $n \geq 3$ and $m > 0$, with initial data prescribed on Σ_{τ_0} ($\tau_0 > 0$).*

- *If $D \doteq D_2^{(2)}(\tau_0) < \infty$ then there exists a constant $C(n,m)$ such that*

$$E[\phi](\Sigma_\tau) \leq \frac{C\,D}{\tau^2} \quad (\tau > \tau_0)\,. \quad (1.1.9)$$

[2]The "redshift" proposition, and the "integrated local energy decay" proposition are to be found on page 28 in Section 1.3 and page 35 in Section 1.4 respectively.

- *Furthermore if for some* $0 < \delta < \frac{1}{2}$ *and* $R > \sqrt[n-2]{8nm/\delta}$ *also* $D' \doteq D_5^{(4-\delta)}(\tau_0) < \infty$ *then there exists a constant* $C(n, m, \delta, R)$ *such that*

$$E[\partial_t\phi](\Sigma'_\tau) \leq \frac{C\,D'}{\tau^{4-2\delta}} \quad (\tau > \tau_0)\,, \tag{1.1.10}$$

  where $\Sigma'_\tau \doteq \Sigma_\tau \cap \{r \leq R\}$.

While each of these energy decay statements lend themselves to prove pointwise estimates for ϕ and $\partial_t\phi$ respectively (see Section 1.6) we would like to emphasize that using the (refined) integrated local energy decay estimates of Section 1.4 an interpolation argument allows us to improve the pointwise bound on ϕ directly in the interior.[3]

Theorem 2 (Pointwise decay). *Let* ϕ *be a solution of the wave equation as in Theorem 1. If for some* $0 < \delta < \frac{1}{4}$, $D \doteq D^{(4-\delta)}_{7+[\frac{n}{2}]}(\tau_0) < \infty \quad (\tau_0 > 1)$ *then there exists a constant* $C(n, m, \delta, R)$ *such that*

$$r^{\frac{n-2}{2}}|\phi|\Big|_{\Sigma'_\tau} \leq \frac{C\,D}{\tau^{\frac{3}{2}-\delta}} \qquad (\sqrt[n-2]{2m} \leq r < R,\ \tau > \tau_0) \tag{1.1.11}$$

where Σ'_τ *and* R *are as in Theorem 1.*

Remark 3 (Decay rates and method of proof). Theorems 1 and 2 extend the presently known decay results for linear waves on $3+1$-dimensional Schwarzschild black holes to higher dimensions $n > 3$; for $3+1$-dimensional Schwarzschild black holes (1.1.9) was first established in [20], and (1.1.10, 1.1.11) more recently in [36]. However, both proofs use multipliers with weights in t, [20] by using the conformal Morawetz vectorfield in the decay argument, and [36] by using in addition the scaling vectorfield. Here we extend (1.1.9) to higher dimensions $n > 3$ in the spirit of [22] only using multipliers with weights in r, and provide a new proof of the improved decay results (1.1.10) and (1.1.11) in the $n = 3$-dimensional case in particular.

1.1.2 Overview of the Proof

In this section we give an overview of the work in this part of my thesis, and present some of the ideas in the proof that lead to Theorem 1; references to previous work is made when useful, but for a more detailed account of previous work on the wave equation on Schwarzschild black hole spacetimes see §1.3 in [23] and references therein.

[3] In this work we use the term "interior" to refer to a region of finite radius, i.e. the term "interior region" is used interchangeably with "a region of compact r (including the horizon)", and is of course *not* meant to refer to the interior of the black hole, which is not considered in my present work.

Energy Identities. Let us recall that the wave equation (1.1.1) arises from an action principle and that the corresponding energy momentum tensor is conserved. Indeed, here we find (1.1.4) and by virtue of the wave equation (1.1.1)

$$\nabla^\mu T_{\mu\nu} = (\Box_g \phi)(\partial_\nu \phi) = 0\,. \tag{1.1.12}$$

Moreover, the energy momentum tensor (1.1.4) satisfies the positivity condition, namely $T(X, Y) \geq 0$ for all future-directed *causal* vectors X, Y at a point.

Now let X be a vectorfield on $\mathcal{M}$. We define the energy current $J^X[\phi]$ associated to the multiplier X by

$$J^X_\mu[\phi] \doteq T_{\mu\nu}[\phi] X^\nu\,. \tag{1.1.13}$$

Then

$$K^X \doteq \nabla^\mu J^X_\mu = {}^{(X)}\pi^{\mu\nu} T_{\mu\nu} \tag{1.1.14}$$

where we have used that $T_{\mu\nu}$ is conserved and symmetric. Here

$${}^{(X)}\pi(Y, Z) \doteq \frac{1}{2}(\mathcal{L}_X g)(Y, Z) = \frac{1}{2} g(\nabla_Y X, Z) + \frac{1}{2} g(Y, \nabla_Z X) \tag{1.1.15}$$

is the *deformation tensor* of X.

Remark 1.1. If X is a Killing field, i.e. X generates an isometry of g, ${}^{(X)}\pi = 0$, then $K^X = 0$, i.e. J^X is conserved.

In the following we shall refer to

$$\int_{\mathcal{R}} K^X \,\mathrm{d}\mu_g = \int_{\partial\mathcal{R}} {}^*J^X \tag{1.1.16}$$

as the *energy identity for J^X (or simply X) on* $\mathcal{R}$, where $\mathcal{R} \subset \mathcal{M}$; (this is of course the content of Stokes' Theorem, and ${}^*J$ denotes the Hodge-dual of J, see also Appendix B.2). Moreover we refer to X in (1.1.16) as the *multiplier vectorfield.* We will largely be concerned with the construction of vectorfields X, associated currents J^X and their modifications, and the application of (1.1.16) and various derived energy inequalities to appropriately chosen domains $\mathcal{R} \subset \mathcal{D}$.

The new approach [22] to obtaining robust decay estimates requires us to first establish (i) uniform boundedness of energy, (ii) an integrated local energy decay estimate and (iii) good asymptotics towards null infinity.

Redshift effect. The reason (i) is nontrivial as compared to Minkowski space is that the energy corresponding to the multiplier ∂_t degenerates on the horizon (the vectorfield ∂_t becomes null on the horizon and no control on the angular derivatives is obtained, c.f. [17]); it was recognized in [20], and formulated more generally in [17], that the redshift property of Killing horizons is the key to obtaining an estimate for the *nondegenerate* energy (i.e. an energy with respect to a strictly timelike vectorfield up to the horizon, which controls all derivatives tangential to the horizons). An explicit construction of a suitable timelike vectorfield N is given in Section 1.3 which allows us to state the redshift property in the language of multipliers and energy currents, and a proof of the uniform boundedness of the nondegenerate energy is given (independently of other calculations in this work) in Section 1.5.1.

Integrated local energy decay. Section 1.4 is devoted to establishing (ii). This is achieved by the use of radial multiplier vectorfields of the form $f(r^*)\partial_{r^*}$ (see Section 1.4.1). In Section 1.4.2 a construction of a positive definite current for the high angular frequency regime is given using a decomposition on the sphere. In Section 1.4.3 a more general construction of a current is given using a commutation with the angular momentum operators. We wish to emphasize that the decay results of Section 1.5 — albeit with a higher loss of differentiability — could be obtained solely on the basis of the latter current, without the recourse in Section 1.4.2 to the Fourier expansion on the sphere. However, the dependence on the initial data is significantly improved by virtue of the integrated local energy decay estimate Prop. 1.11; here (see Section 1.4.4) the results of Section 1.4.2 and Section 1.4.3 are combined in order to replace the commutation with the angular momentum operators by a commutation with the vectorfield ∂_t only. The difficulty in both constructions lies in overcoming the "trapping" obstruction, which is the insight that it is impossible to prove an integrated local energy decay estimate on spacetime regions that contain the photon sphere without losing derivatives (see [17]). In the context of the Schwarzschild spacetime the need for vectorfields whose associated currents give rise to positive definite spacetime integrals was first recognised and used in [6, 20], and such estimates have since then been extended by many authors [37, 1].

p-hierarchy. In Section 1.5.2 we use a multiplier of the form $r^p\partial_{v^*}$ that gives rise to a weighted energy inequality which we consequently exploit in a hierarchy of two steps; this approach — which yields the corresponding quadratic decay rate in (1.1.9) — is pioneered in [22] for a large class of spacetimes, including the $3+1$-dimensional Schwarzschild and Kerr black hole spacetimes. In Section 1.5.3 a further commutation with ∂_{v^*} is carried out, which allows us to extend the hierarchy of commuted weighted energy inequalities to four steps, yielding the correponding decay rate for the first order energy. The argument involves dealing with an (arbitrarily small) degeneracy of the first order energy density at infinity which corresponds to the δ-loss in the decay estimate (1.1.10). In both cases (iii) is ensured by the imposition of higher order finite energy conditions on the initial data.

Interpolation. The pointwise decay of Theorem 2 then follows from Theorem 1 and the (refined) integrated local energy decay estimates of Section 1.4.4 by a simple interpolation argument given in Section 1.6.

Final Comments. The currents in Section 1.4.2 and Section 1.4.3 and the corresponding integrated local energy decay result already appeared in the Smith-Rayleigh-Knight essay [42]. Independently a version of integrated local energy decay was subsequently obtained in [35]. [42] also contained an alternative proof of (1.1.9) of Theorem 1 using the conformal Morawetz vectorfield, which is included in this thesis in Section 1.5.4.

1.2 Global causal geometry of the higher dimensional Schwarzschild solution

In this Section, we give a discussion (in the spirit of §3 of [13]) of the global geometry of the $n+1$-dimensional Schwarzschild black hole spacetime [45], the underlying manifold on which the wave equation is studied in this thesis.

The $n+1$-dimensional Schwarzschild spacetime manifold $\mathcal{M}$ ($n \geq 3,\ n \in \mathbb{N}$) is spherically symmetric, i.e. $\mathrm{SO}(n)$ acts by isometry. The group orbits are $(n-1)$-spheres, and the quotient $\mathcal{Q}$=$\mathcal{M}/\mathrm{SO}(n)$ is a 2-dimensional Lorentzian manifold with boundary. The metric g on $\mathcal{M}$ assumes the form

$$g = \overset{\mathcal{Q}}{g} + \gamma_r = \overset{\mathcal{Q}}{g} + r^2 \overset{\circ}{\gamma}_{n-1} \tag{1.2.1}$$

where $\overset{\mathcal{Q}}{g}$ is the Lorentzian metric on $\mathcal{Q}$ to be discussed below, $\overset{\circ}{\gamma}_{n-1}$ is the standard metric on $\mathbb{S}^{n-1}$, and r is the *area radius*; or more precisely in local coordinates $x^a : a = 1,2$ on $\mathcal{Q}$, and local coordinates $y^A : A = 1, \ldots, n-1$ on $\mathbb{S}^{n-1}$

$$g_{(x,y)} = g_{ab}(x)\,\mathrm{d}x^a\,\mathrm{d}x^b + r^2(x)(\overset{\circ}{\gamma}_{n-1})_{AB}\,\mathrm{d}y^A\,\mathrm{d}y^B\,.$$

Remark 1.2 (Area radius). Since

$$\det \gamma_r = (r^2)^{n-1} \det \overset{\circ}{\gamma}_{n-1}$$

the area radius $r(x)$ is related to the area of the $(n-1)$-sphere at x by

$$\mathrm{Area}(x) = \int_{\mathrm{S}(x)} \mathrm{d}\mu_{\gamma_r} = \int_{\mathbb{S}^{n-1}} r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} = \omega_n r^{n-1}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the area of the unit $(n-1)$-sphere.

The $n+1$-dimensional Schwarzschild spacetime is a solution of the vacuum Einstein equations, which in other words means that its Ricci curvature vanishes identically. This implies in particular (see derivation below) that the area radius function r satisfies the Hessian equations

$$\nabla_a \partial_b r = \frac{(n-2)}{2r}\big[1 - (\partial^c r)(\partial_c r)\big] g_{ab}\ , \tag{1.2.2}$$

as a result of which the *mass function* m on $\mathcal{Q}$ defined[4] by

$$1 - \frac{2m}{r^{n-2}} = g^{ab}\,\partial_a r\,\partial_b r \tag{1.2.3}$$

is constant; we take this parameter $m > 0$ to be positive.

Hessian equations. The non-vanishing connection coefficients of g are Γ^c_{ab}, $\Gamma^C_{AB} = \mathring{\Gamma}^C_{AB}$ and

$$\begin{aligned}\Gamma^a_{AB} &= -r\,g^{ab}\,\partial_b r\,(\mathring{\gamma}_{n-1})_{AB}\\ \Gamma^A_{aB} &= \frac{1}{r}\,\partial_a r\,\delta^A_B\ .\end{aligned}$$

Since the components of the Ricci curvature are given by

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}$$

we obtain the decomposition

$$\begin{aligned}R_{ab} &= \overset{\mathcal{Q}}{R}_{ab} - \partial_b\Gamma^A_{aA} + \Gamma^A_{dA}\Gamma^d_{ab} - \Gamma^A_{Bb}\Gamma^B_{aA}\\ &= K\,g_{ab} - (n-1)\,\frac{1}{r}\,\nabla_a\partial_b r\end{aligned}$$

where we have used that

$$\overset{\mathcal{Q}}{R}_{ab} = K\,g_{ab}\ ,$$

K being the Gauss curvature of the 2-dimensional manifold $\mathcal{Q}$, and similarly

$$R_{aA} = 0\ ,$$

$$\begin{aligned}R_{AB} &= \partial_a\Gamma^a_{AB} + \Gamma^a_{ba}\Gamma^b_{AB} + \Gamma^D_{aD}\Gamma^a_{AB} - \Gamma^a_{DB}\Gamma^D_{Aa} - \Gamma^D_{aB}\Gamma^a_{AD} + \mathring{R}_{AB}\\ &= (\mathring{\gamma}_{n-1})_{AB}\big[(n-2) - (n-3)\,(\partial^a r)(\partial_a r) - \nabla^a(r\partial_a r)\big]\ ,\end{aligned}$$

because the Ricci curvature of $\mathbb{S}^{n-1}$ is simply

$$\mathring{R}_{AB} = (n-2)(\mathring{\gamma}_{n-1})_{AB}\ .$$

The *vacuum* Einstein equations

$$R_{\mu\nu} = 0$$

therefore reduce to the following system on $\mathcal{Q}$:

$$K\,g_{ab} - (n-1)\,\frac{1}{r}\,\nabla_a\partial_b r = 0 \tag{1.2.4a}$$

$$(n-2) - (n-3)\,(\partial^a r)(\partial_a r) - \nabla^a(r\partial_a r) = 0\ . \tag{1.2.4b}$$

---

[4]We choose the normalization of the mass function to be independent of the dimension n; this is motivated by a consideration of the *mass equations* in the presence of matter, see Remark 1.3.

Taking the trace of (1.2.4a) gives

$$2\,K - (n-1)\,\frac{1}{r}\,\nabla^a \partial_a r = 0$$

and substituting for $\nabla^a \partial_a r$ from (1.2.4b) yields

$$\begin{aligned} K &= \frac{n-1}{2r}\,\nabla^a \partial_a r \\ &= \frac{n-1}{2r^2}\,\bigl[\nabla^a (r\partial_a r) - (\partial^a r)(\partial_a r)\bigr] \\ &= \frac{(n-1)(n-2)}{2r^2}\bigl[1 - (\partial^a r)(\partial_a r)\bigr]\ . \end{aligned}$$

Then returning to (1.2.4a), we arrive at the *Hessian* equations (1.2.2). The mass function defined by (1.2.3),

$$m = \frac{r^{n-2}}{2}\bigl[1 - (\partial^a r)(\partial_a r)\bigr]$$

is thus constant on $\mathcal{Q}$:

$$\begin{aligned} \partial_a m &= \frac{n-2}{2}\,r^{n-3}\bigl[1 - (\partial^b r)(\partial_b r)\bigr]\partial_a r - \frac{1}{2}\,r^{n-2}\nabla_a\bigl[(\partial^b r)(\partial_b r)\bigr] \\ &= 0\ . \end{aligned}$$

We take $m > 0$. Moreover

$$K = \frac{(n-1)(n-2)\,m}{r^n}\ .$$

Remark 1.3 (Normalization of the mass). One may prefer a normalization of the mass function that depends on the dimension n, i.e. favour the definition

$$1 - \frac{a_n\,m}{r^{n-2}} = g^{ab}\,\partial_a r\,\partial_b r$$

where a_n is a constant depending on n. However, in the presence of matter the Einstein equations in $(n+1)$ dimensions necessarily take the form

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = (n-1)\,\omega_n\,T_{\mu\nu}$$

(in units where Newton's constant $G = 1$ and the speed of light $c = 1$) and we fix the a_n by taking the prefactor in the *mass equations*

$$\partial_a m = \omega_n r^{n-1}\,(T_{ab} - g_{ab}\,\mathrm{tr}\,T)\partial^b r$$

to be the area of the $(n-1)$-sphere. The analogous calculation of the Hessian equations in $(n+1)$ dimensions in the presence of matter shows that this is precisely satisfied for

$$a_n = 2\ ,$$

independently of n.

On $\mathcal{Q}$ we choose functions u, v whose level sets are outgoing and incoming null curves respectively, which are increasing towards the future. These functions define a null system of coordinates, in which the metric $\overset{\mathcal{Q}}{g}$ takes the form

$$\overset{\mathcal{Q}}{g} = -\Omega^2 \,\mathrm{d}u\,\mathrm{d}v \;. \tag{1.2.5}$$

Note. The only non-vanishing connection coefficients are

$$\Gamma^u_{uu} = \frac{2}{\Omega}\frac{\partial\Omega}{\partial u} \qquad \Gamma^v_{vv} = \frac{2}{\Omega}\frac{\partial\Omega}{\partial v} \;.$$

The Gauss curvature is given by

$$K = -\frac{2}{\Omega^2}\overset{\mathcal{Q}}{R}_{uv} = \frac{4}{\Omega^2}\frac{\partial^2 \log\Omega}{\partial u\,\partial v} \;.$$

The Hessian equations (1.2.2) in null coordinates read

$$\frac{\partial^2 r}{\partial u^2} - \frac{2}{\Omega}\frac{\partial\Omega}{\partial u}\frac{\partial r}{\partial u} = 0 \tag{1.2.6a}$$

$$\frac{\partial^2 r}{\partial u\,\partial v} + \frac{n-2}{r}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} = -\frac{n-2}{4r}\Omega^2 \tag{1.2.6b}$$

$$\frac{\partial^2 r}{\partial v^2} - \frac{2}{\Omega}\frac{\partial\Omega}{\partial v}\frac{\partial r}{\partial v} = 0\,, \tag{1.2.6c}$$

and the defining equation for the mass function (1.2.3) is

$$1 - \frac{2m}{r^{n-2}} = -\frac{4}{\Omega^2}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} \;. \tag{1.2.7}$$

The system ((1.2.6b), (1.2.7)) can be rewritten as the partial differential equation

$$\frac{\partial^2 r^*}{\partial u \partial v} = 0 \tag{1.2.8}$$

for a new radial function $r^*(r)$ that is related to r by

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{1}{1-\frac{2m}{r^{n-2}}} \,. \tag{1.2.9}$$

Remark 1.4. Indeed, by substituting for Ω^2 from (1.2.7) in (1.2.6b),

$$r\frac{\partial^2 r}{\partial u\,\partial v} = (n-2)\frac{2m}{r^{n-2}-2m}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v}\,,$$

we see that

$$r^* = \int \frac{1}{1-\frac{2m}{r^{n-2}}}\,\mathrm{d}r \tag{1.2.10}$$

satisfies

$$\begin{aligned}\frac{\partial^2 r^*}{\partial u\,\partial v} &= -\frac{(n-2)\frac{2m}{r^{n-1}}}{\left(1-\frac{2m}{r^{n-2}}\right)^2}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} + \frac{1}{1-\frac{2m}{r^{n-2}}}\frac{\partial^2 r}{\partial u\,\partial v}\\ &= 0\,.\end{aligned}$$

A solution of ((1.2.8),(1.2.9)) is given by

$$r^* = \frac{1}{(n-2)} \sqrt[n-2]{2m} \log |uv| \,, \tag{1.2.11}$$

or

$$\begin{aligned} |uv| &= e^{\frac{(n-2)\, r^*}{\sqrt[n-2]{2m}}} \\ &= e^{\frac{(n-2)\, r}{\sqrt[n-2]{2m}}} \exp\Big[\int \frac{n-2}{x^{n-2}-1}\,\mathrm{d}x\Big|_{x=\frac{r}{\sqrt[n-2]{2m}}}\Big] \,. \end{aligned} \tag{1.2.12}$$

Remark 1.5. The general solution is

$$r^* = f(u) + g(v) \,.$$

Since

$$\begin{aligned} r^* &= r + \int \frac{2m}{r^{n-2} - 2m}\,\mathrm{d}r \\ &= r + \sqrt[n-2]{2m} \int \frac{1}{x^{n-2}-1}\,\mathrm{d}x\Big|_{x=\frac{r}{\sqrt[n-2]{2m}}} \end{aligned}$$

we require the representation in terms of null coordinates to be such that $r^* = -\infty$ is contained in the (u, v) plane *and* the metric to be non-degenerate at $r = \sqrt[n-2]{2m}$ and take

$$\begin{aligned} f(u) &= \sqrt[n-2]{2m} \log |u|^{\frac{1}{n-2}} \\ g(v) &= \sqrt[n-2]{2m} \log |v|^{\frac{1}{n-2}} \end{aligned}$$

so that

$$r^* = (n-2)^{-1} \sqrt[n-2]{2m} \log |uv| \,. \tag{1.2.13}$$

We find more explicitly,

$$uv = \begin{cases} e^{\frac{r}{2m}} \big(1 - \frac{r}{2m}\big) & , n = 3 \\ e^{\frac{2r}{\sqrt{2m}}} \frac{\big(1 - \frac{r}{\sqrt{2m}}\big)}{\big(1 + \frac{r}{\sqrt{2m}}\big)} & , n = 4 \\ e^{\frac{(n-2)\, r}{\sqrt[n-2]{2m}}} \Big(1 - \frac{r}{\sqrt[n-2]{2m}}\Big) \begin{cases} 1 & , n \text{ odd} \\ \big(1 + \frac{r}{\sqrt[n-2]{2m}}\big)^{-1} & , n \text{ even} \end{cases} & \\ \times \prod_{j=1}^{[\frac{n-3}{2}]} \Big(\frac{r^2}{(2m)^{\frac{2}{n-2}}} - 2\cos\big(\frac{2\pi j}{n-2}\big) \frac{r}{(2m)^{\frac{1}{n-2}}} + 1\Big)^{\cos(2\pi j \frac{n-3}{n-2})} & , n \geq 5 \\ \times \prod_{j=1}^{[\frac{n-3}{2}]} \exp\Big[2 \sin\big(2\pi j \frac{n-3}{n-2}\big) \arctan\Big(\frac{\frac{r}{\sqrt[n-2]{2m}} - \cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big)\Big] & \end{cases} \tag{1.2.14}$$

Note, in particular that the $u = 0$ and $v = 0$ lines are the constant $r = \sqrt[n-2]{2m}$ curves, and that all other curves of constant radius are hyperbolas in the (u, v) plane — timelike for $r > \sqrt[n-2]{2m}$, spacelike for $r < \sqrt[n-2]{2m}$. This outlines the well-known *global* causal geometry of the Schwarzschild solution (see figure 1.2).

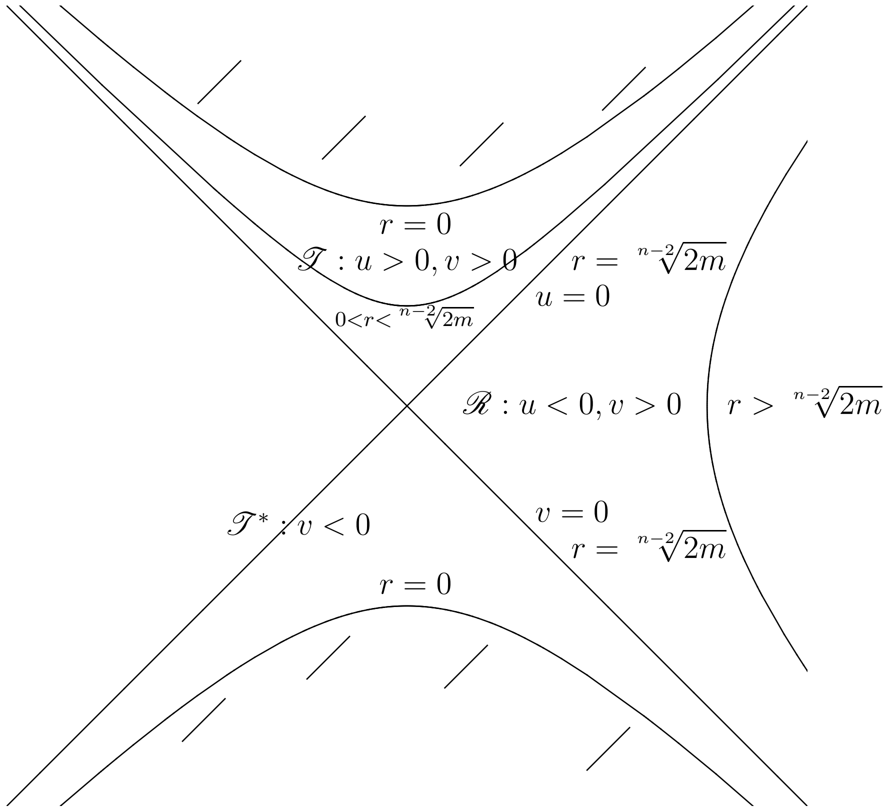

Figure 1.2: Global causal geometry of the Schwarzschild solution.

Remark 1.6. The rational function in (1.2.12) can be integrated using elementary methods (see Appendix B.2.1) to obtain:

$$|uv| = e^{\frac{(n-2)\,r}{\sqrt[n-2]{2m}}} \begin{cases} \left|\dfrac{r}{2m} - 1\right| & , n = 3 \\[2ex] \dfrac{|\frac{r}{\sqrt{2m}} - 1|}{|\frac{r}{\sqrt{2m}} + 1|} & , n = 4 \\[2ex] \left|\dfrac{r}{\sqrt[n-2]{2m}} - 1\right| \begin{cases} 1 & , n \text{ odd} \\ |\frac{r}{\sqrt[n-2]{2m}} + 1|^{-1} & , n \text{ even} \end{cases} & \\ \quad \times \displaystyle\prod_{j=1}^{[\frac{n-3}{2}]} \left|\dfrac{r^2}{(2m)^{\frac{2}{n-2}}} - 2\cos(\dfrac{2\pi j}{n-2})\dfrac{r}{(2m)^{\frac{1}{n-2}}} + 1\right|^{\cos(2\pi j\frac{n-3}{n-2})} & , n \geq 5 \\ \quad \times \displaystyle\prod_{j=1}^{[\frac{n-3}{2}]} \exp\left[2\sin\big(2\pi j\dfrac{n-3}{n-2}\big)\arctan\Big(\dfrac{\frac{r}{\sqrt[n-2]{2m}} - \cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big)\right] & \end{cases}$$

Note that

$$|\frac{r^2}{(2m)^{\frac{2}{n-2}}} - 2\cos(\frac{2\pi j}{n-2})\frac{r}{(2m)^{\frac{1}{n-2}}} + 1| \geq |1 - \cos^2(\frac{2\pi j}{n-2})| > 0 \qquad (j = 1, \ldots, [\frac{n-3}{2}],\ n \geq 5)\,.$$

The condition that u, v are increasing towards the future selects the sign, and we finally obtain (1.2.14). We have

$$uv \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \iff r \begin{cases} > \sqrt[n-2]{2m} \\ = \sqrt[n-2]{2m} \\ < \sqrt[n-2]{2m} \end{cases}$$

Note also that $r = 0$ is the spacelike hyperbola

$$uv = \begin{cases} 1 & , n = 3, 4; n \geq 6 \text{ even} \\ \prod_{j=1}^{\frac{n-3}{2}} \exp\left[-2\sin(2\pi j \frac{n-3}{n-2}) \arctan\cot(\frac{2\pi j}{n-2})\right] & , n \geq 5 \text{ odd} \end{cases}$$
$$\leq 1\,.$$

It is easy to see that for (1.2.11) the trapped region, the apparent horizon, the exterior, and the antitrapped regions respectively are given by

$$\begin{aligned} \mathscr{T} &\doteq \left\{(u,v) \in \mathcal{Q} : \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial v} < 0\right\} = \left\{(u,v) \in \mathcal{Q} : u > 0, v > 0\right\} \\ \mathscr{A} &\doteq \left\{(u,v) \in \mathcal{Q} : \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial v} = 0\right\} = \left\{(u,v) \in \mathcal{Q} : u = 0, v > 0\right\} \\ \mathscr{R} &\doteq \left\{(u,v) \in \mathcal{Q} : \frac{\partial r}{\partial u} < 0, \frac{\partial r}{\partial v} > 0\right\} = \left\{(u,v) \in \mathcal{Q} : u < 0, v > 0\right\} \\ \mathscr{T}^* &\doteq \left\{(u,v) \in \mathcal{Q} : \frac{\partial r}{\partial u} > 0\right\} = \left\{(u,v) \in \mathcal{Q} : v < 0\right\}. \end{aligned}$$

Indeed, using the solution (1.2.11)

$$\frac{\partial r^*}{\partial u} = \frac{1}{n-2}\frac{\sqrt[n-2]{2m}}{u} \qquad \frac{\partial r^*}{\partial v} = \frac{1}{n-2}\frac{\sqrt[n-2]{2m}}{v}$$

and on the other hand with (1.2.9)

$$\frac{\partial r^*}{\partial u} = \frac{1}{1-\frac{2m}{r^{n-2}}}\frac{\partial r}{\partial u} \qquad \frac{\partial r^*}{\partial v} = \frac{1}{1-\frac{2m}{r^{n-2}}}\frac{\partial r}{\partial v}$$

we deduce

$$\frac{\partial r}{\partial u} = \left(1 - \frac{2m}{r^{n-2}}\right)\frac{\sqrt[n-2]{2m}}{(n-2)u} \tag{1.2.15a}$$

$$\frac{\partial r}{\partial v} = \left(1 - \frac{2m}{r^{n-2}}\right)\frac{\sqrt[n-2]{2m}}{(n-2)v} \tag{1.2.15b}$$

Note this forms a partition of $\mathcal{Q} = \overline{\mathscr{T} \cup \mathscr{A} \cup \mathscr{R} \cup \mathscr{T}^*}$, and that in view of (1.2.7) $r < \sqrt[n-2]{2m}$ in $\mathscr{T}$, $r = \sqrt[n-2]{2m}$ in $\mathscr{A}$ and $r > \sqrt[n-2]{2m}$ in $\mathscr{R}$. We shall refer to

$$\mathcal{D} \doteq \overline{\mathcal{R}} = \left\{(u,v) \in \mathcal{Q} : u \leq 0, v \geq 0\right\} \tag{1.2.16}$$

as the *domain of outer communications.*

Finally,

$$\begin{aligned}
\Omega^2 &= -4\big(1-\frac{2m}{r^{n-2}}\big)^{-1}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} && \text{by (1.2.7)} \\
&= -\big(\frac{2}{n-2}\big)^2\big(1-\frac{2m}{r^{n-2}}\big)\frac{(2m)^{\frac{2}{n-2}}}{uv} && \text{by (1.2.15)} \qquad (1.2.17a)
\end{aligned}$$

$$= \begin{cases}
4\frac{(2m)^3}{r}e^{-\frac{r}{2m}} & , n=3 \\
\big(\frac{2m}{r}\big)^2\big(\frac{r}{\sqrt{2m}}+1\big)^2 e^{-\frac{2r}{\sqrt{2m}}} & , n=4 \\
\big(\frac{2}{n-2}\big)^2\frac{(2m)^{\frac{n}{n-2}}}{r^{n-2}}\begin{cases} 1 & \text{, n odd} \\ \big(\frac{r}{\sqrt[n-2]{2m}}+1\big)^2 & \text{, n even}\end{cases} & \\
\quad\times\prod_{j=1}^{[\frac{n-3}{2}]}\Big(\frac{r^2}{(2m)^{\frac{2}{n-2}}}-2\cos\big(\frac{2\pi j}{n-2}\big)\frac{r}{\sqrt[n-2]{2m}}+1\Big)^{1-\cos(2\pi j\frac{n-3}{n-2})} & , n\geq 5 \\
\quad\times\prod_{j=1}^{[\frac{n-3}{2}]}\exp\Big[-2\sin\big(2\pi j\frac{n-3}{n-2}\big)\arctan\Big(\frac{\frac{r}{\sqrt[n-2]{2m}}-\cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big)\Big] & \\
\quad\times e^{-\frac{(n-2)\,r}{\sqrt[n-2]{2m}}} &
\end{cases}$$

(1.2.17b)

One may now think of r as a function of u, v implicitly defined by (1.2.14). In $\mathscr{R}$ where $r > \sqrt[n-2]{2m}$ recall from (1.2.12) that this relation is

$$uv = -\exp\Big[\int\frac{n-2}{x^{n-2}-1}\,\mathrm{d}x\big|_{x=\frac{r}{\sqrt[n-2]{2m}}}\Big]e^{\frac{(n-2)\,r}{\sqrt[n-2]{2m}}} \qquad (1.2.18)$$

and may be complemented in this region $\mathscr{R}$ where $v-u>|u+v|$ by the coordinate t defined by

$$t = \frac{2}{n-2}\sqrt[n-2]{2m}\,\mathrm{artanh}\big(\frac{u+v}{v-u}\big)\,; \qquad (1.2.19)$$

we will denote by $\overline{\Sigma}_t$ the corresponding level sets in $\mathcal{D}$. We find

$$\mathrm{d}t = \frac{1}{n-2}\sqrt[n-2]{2m}\,\big(\frac{1}{v}\,\mathrm{d}v-\frac{1}{u}\,\mathrm{d}u\big) \qquad (1.2.20)$$

and from (1.2.18)

$$v\,\mathrm{d}u+u\,\mathrm{d}v = -\frac{\frac{r^{n-2}}{2m}}{\frac{r^{n-2}}{2m}-1}\exp\Big[\int\frac{n-2}{x^{n-2}-1}\,\mathrm{d}x\big|_{x=\frac{r}{\sqrt[n-2]{2m}}}\Big]e^{\frac{(n-2)\,r}{\sqrt[n-2]{2m}}}\frac{n-2}{\sqrt[n-2]{2m}}\,\mathrm{d}r\ .$$

Alternatively (1.2.20) can be written as

$$\begin{aligned}
v\,\mathrm{d}u-u\,\mathrm{d}v &= -uv\frac{n-2}{\sqrt[n-2]{2m}}\,\mathrm{d}t \\
&= \exp\Big[\int\frac{n-2}{x^{n-2}-1}\,\mathrm{d}x\big|_{x=\frac{r}{\sqrt[n-2]{2m}}}\Big]e^{\frac{(n-2)\,r}{\sqrt[n-2]{2m}}}\frac{n-2}{\sqrt[n-2]{2m}}\,\mathrm{d}t\ .
\end{aligned}$$

Hence

$$-\Omega^2\,\mathrm{d}u\,\mathrm{d}v = \frac{\Omega^2}{4uv}\Big[\big(v\,\mathrm{d}u - u\,\mathrm{d}v\big)^2 - \big(v\,\mathrm{d}u + u\,\mathrm{d}v\big)^2\Big]$$
$$= -\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}t^2 + \frac{1}{\big(1-\frac{2m}{r^{n-2}}\big)}\,\mathrm{d}r^2\ .$$

We have arrived at the classic expression for the Schwarzschild metric in its original coordinates for the *exterior* region:

$$g = -\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}t^2 + \big(1 - \frac{2m}{r^{n-2}}\big)^{-1}\,\mathrm{d}r^2 + r^2\,\overset{\circ}{\gamma}_{n-1}\ . \tag{1.2.21}$$

In Regge-Wheeler coordinates (t, r^*), where r^* is centered at the photon sphere $r = \sqrt[n-2]{nm}$,

$$r^* = \int_{(nm)^{\frac{1}{n-2}}}^{r} \frac{1}{1-\frac{2m}{r'^{n-2}}}\,\mathrm{d}r'\,, \tag{1.2.22}$$

the metric obviously takes the conformally flat form

$$g = \big(1 - \frac{2m}{r^{n-2}}\big)\big(-\,\mathrm{d}t^2 + \,\mathrm{d}r^{*2}\big) + r^2\,\overset{\circ}{\gamma}_{n-1}\ . \tag{1.2.23}$$

We shall also use the Eddington-Finkelstein coordinates

$$u^* = \frac{1}{2}(t - r^*) \qquad v^* = \frac{1}{2}(t + r^*) \tag{1.2.24}$$

which are again double null coordinates:

$$g = -4\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}u^*\,\mathrm{d}v^* + r^2\,\overset{\circ}{\gamma}_{n-1}\ . \tag{1.2.25}$$

The two systems of null coordinates in $\mathscr{R}$ are related by

$$-\frac{1}{n-2}\frac{\sqrt[n-2]{2m}}{u}\,\mathrm{d}u = \,\mathrm{d}u^* \qquad \frac{1}{n-2}\frac{\sqrt[n-2]{2m}}{v}\,\mathrm{d}v = \,\mathrm{d}v^*$$

or

$$u = -e^{-\frac{(n-2)\,u^*}{\sqrt[n-2]{2m}}} \qquad v = e^{\frac{(n-2)\,v^*}{\sqrt[n-2]{2m}}}\ . \tag{1.2.26}$$

1.3 The Red-shift effect

In this section we prove a manifestation of the *local redshift effect* in the Schwarzschild geometry of Section 1.2 in the framework of multiplier vectorfields.

Proposition 1.7 (local redshift effect)**.** *Let ϕ be a solution of the wave equation* (1.1.1)*, then there exists a φ_t-invariant future-directed timelike smooth vectorfield N on $\mathcal{D}$, two radii $\sqrt[n-2]{2m} < r_0^{(N)} < r_1^{(N)}$, and a constant $b > 0$ such that*

$$K^N(\phi) \geq b\,(J^N(\phi), N) \qquad \big(\sqrt[n-2]{2m} \leq r < r_0^{(N)}\big) \tag{1.3.1}$$

and $N = T$ $(r \geq r_1^{(N)})$.

The vectorfield N will be constructed explicitly with the following vectorfields.

T-vectorfield. Here φ_t is the 1-parameter group of diffeomorphisms generated by the vectorfield

$$T = \frac{1}{2}\frac{n-2}{\sqrt[n-2]{2m}}\big(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\big)\,; \tag{1.3.2}$$

note that in $\mathscr{R}$ where $r > \sqrt[n-2]{2m}$ (recall (1.2.20))

$$T = \frac{\partial}{\partial t}\,.$$

T is a Killing vectorfield,

$$^{(T)}\pi = 0\,. \tag{1.3.3}$$

For,

$$\begin{aligned} {}^{(T)}\pi_{uu} = 0 \qquad & {}^{(T)}\pi_{vv} = 0 \\ {}^{(T)}\pi_{uv} = \frac{1}{4}\frac{n-2}{\sqrt[n-2]{2m}} g_{uv}\big(v\frac{\partial r}{\partial v} &- u\frac{\partial r}{\partial u}\big)\frac{\partial \log\Omega^2}{\partial r} = 0 \\ {}^{(T)}\pi_{aA} &= 0 \\ {}^{(T)}\pi_{AB} = \frac{1}{2}\frac{n-2}{\sqrt[n-2]{2m}}\big(v\frac{\partial r}{\partial v} &- u\frac{\partial r}{\partial u}\big)\, r\,(\overset{\circ}{\gamma}_{n-1})_{AB} = 0\,. \end{aligned} \tag{1.3.4}$$

T is timelike in the exterior, spacelike in the interior of the black hole and *null on the horizon*,

$$g(T,T) = \frac{1}{4}\frac{(n-2)^2}{(2m)^{\frac{2}{n-2}}}\,uv\,\Omega^2 = -\big(1-\frac{2m}{r^{n-2}}\big)\begin{cases} <0 & r > \sqrt[n-2]{2m} \\ =0 & r = \sqrt[n-2]{2m} \\ >0 & r < \sqrt[n-2]{2m} \end{cases} \tag{1.3.5}$$

In particular,

$$\begin{aligned} T|_{\mathcal{H}^+} &= \frac{1}{2}\frac{n-2}{\sqrt[n-2]{2m}}\,v\frac{\partial}{\partial v} \\ T|_{\mathcal{H}^+\cap\mathcal{H}^-} &= 0\,. \end{aligned} \tag{1.3.6}$$

Y-vectorfield. Let us also define a vectorfield Y on $\mathcal{H}^+$ conjugate to T (that is to say Y is null and orthogonal to the sections of $\mathcal{H}^+$ and normalized by (1.3.8)):

$$Y|_{\mathcal{H}^+} = -\frac{2}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u} \tag{1.3.7}$$

Indeed,

$$g(T,Y)|_{\mathcal{H}^+} = -2 \tag{1.3.8}$$

because

$$\Omega^2|_{\mathcal{H}^+} = -4\frac{\sqrt[n-2]{2m}}{n-2}\frac{1}{v}\frac{\partial r}{\partial u}\,.$$

Furthermore, as a consequence of (1.2.6b)

$$\frac{\partial^2 r}{\partial u\,\partial v}\bigg|_{\mathcal{H}^+} = -\frac{n-2}{4r}\Omega^2\bigg|_{\mathcal{H}^+} = \frac{1}{v}\frac{\partial r}{\partial u}\bigg|_{\mathcal{H}^+}$$

we have

$$\begin{aligned} [T,Y]|_{\mathcal{H}^+} &= [T,Y]^u \, \frac{\partial}{\partial u}\bigg|_{\mathcal{H}^+} + [T,Y]^v \, \frac{\partial}{\partial v}\bigg|_{\mathcal{H}^+} \\ &= \frac{n-2}{\sqrt[n-2]{2m}} \frac{1}{\frac{\partial r}{\partial u}} \Big[v \frac{1}{\frac{\partial r}{\partial u}} \frac{\partial^2 r}{\partial u \, \partial v} - 1 \Big] \, \frac{\partial}{\partial u}\bigg|_{\mathcal{H}^+} \\ &= 0 \,. \end{aligned} \tag{1.3.9}$$

E_A-vectorfields. We denote by $E_A : A = 1, \dots, n-1$ an orthonormal frame field tangential to the orbits of the spherical isometry,

$$g(E_A, E_B) = \delta_{AB} = \begin{cases} 1\,, & A = B \\ 0\,, & A \neq B \end{cases} \tag{1.3.10a}$$

$$g(E_A, Y)|_{\mathcal{H}^+} = 0\,, \qquad g(E_A, T) = 0|_{\mathcal{H}^+}\,, \qquad (A = 1, \dots, n-1)\,. \tag{1.3.10b}$$

We can now prove that the surface gravity of the event horizon is *positive*; this is essential for the existence of the redshift effect, (see more generally [17], and also [3] for work where this is not the case).

Lemma 1.8 (surface gravity)**.** *On* $\mathcal{H}^+$

$$\nabla_T T = \kappa_n T \tag{1.3.11}$$

with

$$\kappa_n = \frac{1}{2} \frac{n-2}{\sqrt[n-2]{2m}} > 0\,. \tag{1.3.12}$$

κ_n *is called the* surface gravity*.*

Note. $T = \kappa_n (v \frac{\partial}{\partial v} - u \frac{\partial}{\partial u})$

Proof. Since ${}^{(T)}\pi = 0$, we have

$$g(\nabla_X T, Y) = -g(\nabla_Y T, X)$$

for all vectorfields X, Y. Therefore,

$$g(\nabla_T T, T) = -g(\nabla_T T, T) = 0$$

$$g(\nabla_T T, E_A) = -g(\nabla_{E_A} T, T) = -\frac{1}{2} E_A \cdot g(T,T) = 0 \quad : A = 1, \dots, n-1,$$

because $g(T,T) = 0$ on $\mathcal{H}^+$, and similarly

$$g(\nabla_T T, Y) = -\frac{1}{2} Y \cdot g(T,T)\,.$$

Now,

$$g(T,T) = -\frac{n-2}{\sqrt[n-2]{2m}} \, u \frac{\partial r}{\partial u}$$

so

$$Y \cdot g(T,T)|_{\mathcal{H}^+} = 2\frac{n-2}{\sqrt[n-2]{2m}}\,.$$

We obtain

$$\begin{aligned}\nabla_T T &= -\frac{1}{2}g(\nabla_T T, T)Y - \frac{1}{2}g(\nabla_T T, Y)T + \sum_{j=1}^{n-1} g(\nabla_T T, E_A)E_A \\ &= \frac{1}{2}\frac{n-2}{\sqrt[n-2]{2m}}\,T\,.\end{aligned}$$

□

Alternatively, κ_n is characterized by

$$\nabla_T Y = -\kappa_n Y \tag{1.3.13}$$

on $\mathcal{H}^+$. Clearly

$$g(\nabla_T Y, Y) = \frac{1}{2}T \cdot g(Y,Y) = 0$$

since Y is null along $\mathcal{H}^+$, and

$$g(\nabla_T Y, T) \overset{(1.3.9)}{=} g(\nabla_Y T, T) \overset{(1.3.3)}{=} -g(\nabla_T T, Y) = 2\kappa_n\,;$$

also

$$g(\nabla_T Y, E_A) \overset{(1.3.9)}{=} g(\nabla_Y T, E_A) \overset{(1.3.3)}{=} -g(\nabla_{E_A} T, Y) = 0 \quad : A = 1, \ldots, n-1,$$

because $\nabla_{E_A} T = 0$. Note, for later use,

$$\nabla_{E_A} Y = -\frac{2}{\sqrt[n-2]{2m}}\,E_A \tag{1.3.14}$$

on $\mathcal{H}^+$.

We defined Y on $\mathcal{H}^+$ conjugate to T. Next we extend Y to a neighborhood of the horizon by

$$\nabla_Y Y = -\sigma(Y+T)\,, \qquad (\sigma \in \mathbb{R})\,,$$

where in fact we will shall assume

$$\sigma > \frac{16}{n-2}(2m)^{\frac{3}{n-2}}\,,$$

and then we extend Y to $\mathscr{R}$ by Lie-transport along the integral curves of T:

$$[T,Y] = 0\,.$$

Proposition 1.9 (redshift)**.** *For the future-directed timelike vectorfield*

$$N = T + Y \tag{1.3.15}$$

there is a $b > 0$ *such that on* $\mathcal{H}^+$

$$K^N \geq b\,(J^N, N)\,. \tag{1.3.16}$$

Proof. Let us calculate

$$
\begin{aligned}
K^Y =& {}^{(Y)}\pi^{\mu\nu} T_{\mu\nu} \\
=& \frac{1}{4}\Big\{ {}^{(Y)}\pi(T,T)\, T(Y,Y) + 2\,{}^{(Y)}\pi(T,Y)\, T(Y,T) + {}^{(Y)}\pi(Y,Y)\, T(T,T) \Big\} \\
& - \sum_{A=1}^{n-1} \Big\{ {}^{(Y)}\pi(E_A,Y)\, T(E_A,T) + {}^{(Y)}\pi(E_A,T)\, T(E_A,Y) \Big\} \\
& + \sum_{A,B=1}^{n-1} {}^{(Y)}\pi(E_A,E_B)\, T(E_A,E_B)
\end{aligned}
$$

Now, on one hand, on $\mathcal{H}^+$,

$$
\begin{aligned}
{}^{(Y)}\pi(T,T) &= g(\nabla_T Y, T) = 2\kappa_n \\
{}^{(Y)}\pi(T,Y) &= \frac{1}{2} g(\nabla_T Y, Y) + \frac{1}{2} g(T, \nabla_Y Y) = \sigma \\
{}^{(Y)}\pi(Y,Y) &= g(\nabla_Y Y, Y) = 2\sigma \\
{}^{(Y)}\pi(E_A,Y) &= \frac{1}{2} g(\nabla_{E_A} Y, Y) + \frac{1}{2} g(E_A, \nabla_Y Y) = 0 \\
{}^{(Y)}\pi(E_A,T) &= \frac{1}{2} g(\nabla_{E_A} Y, T) + \frac{1}{2} g(E_A, \nabla_T Y) = 0 \\
{}^{(Y)}\pi(E_A,E_B) &= \frac{1}{2} g(\nabla_{E_A} Y, E_B) + \frac{1}{2} g(E_A, \nabla_{E_B} Y) = -\frac{2}{\sqrt[n-2]{2m}} \delta_{AB}\,.
\end{aligned}
$$

Thus

$$
K^Y = \frac{1}{2}\kappa_n\, T(Y,Y) + \frac{1}{2}\sigma\, T(Y+T,T) - \frac{2}{\sqrt[n-2]{2m}} \sum_{A=1}^{n-1} T(E_A,E_A)\,.
$$

On the other hand, on $\mathcal{H}^+$,

$$
\begin{aligned}
T(Y,Y) &= \Big(\frac{2}{\frac{\partial r}{\partial u}} \frac{\partial \phi}{\partial u}\Big)^2 \\
T(Y,T) &= \big|\nabla\!\!\!/ \phi\big|^2_{r^2 \overset{\circ}{\gamma}_{n-1}} \\
T(T,T) &= \big(\kappa_n v \frac{\partial \phi}{\partial v}\big)^2
\end{aligned}
$$

and, on $\mathcal{H}^+$,

$$
\begin{aligned}
T(E_A,E_B) = (E_A \cdot \phi)(E_B \cdot \phi) - \frac{1}{2}(2m)^{\frac{2}{n-2}} \delta_{AB} \big|\nabla\!\!\!/ \phi\big|_{r^2 \overset{\circ}{\gamma}_{n-1}} \\
- \frac{1}{2}(n-2)(2m)^{\frac{1}{n-2}} \frac{v}{\frac{\partial r}{\partial u}} \delta_{AB} \big(\frac{\partial \phi}{\partial u}\big)\big(\frac{\partial \phi}{\partial v}\big)\,.
\end{aligned}
$$

Using Cauchy's inequality, on $\mathcal{H}^+$,

$$\begin{aligned}
-\frac{2}{\sqrt[n-2]{2m}}\sum_{A=1}^{n-1}T(E_A,E_A) = & \\
&=(n-3)(2m)^{\frac{1}{n-2}}\left|\nabla\!\!\!/\phi\right|^2_{r^2\overset{\circ}{\gamma}_{n-1}}+(n-2)(n-1)\frac{v}{\frac{\partial r}{\partial u}}(\frac{\partial\phi}{\partial u})(\frac{\partial\phi}{\partial v})\\
&\geq(n-3)(2m)^{\frac{1}{n-2}}T(Y,T)-\frac{1}{4}\kappa_n T(Y,Y)\\
&\qquad-\frac{1}{\kappa_n}\frac{2(n-1)}{(n-2)}(2m)^{\frac{2}{n-2}}\,T(T,T)\\
&\geq-\frac{1}{4}\kappa_n T(Y,Y)-\frac{n-1}{\kappa_n^2}(2m)^{\frac{1}{n-2}}T(T,T)\,.
\end{aligned}$$

Since we have chosen $\sigma > 2\frac{n-1}{\kappa_n^2}(2m)^{\frac{1}{n-2}}$, K^Y has a sign,

$$K^Y \geq \frac{1}{4}\kappa_n\, T(Y,Y) + \sigma'\, T(Y+T,T)$$

for $0<\sigma'<\frac{\sigma}{2}-\frac{n-1}{\kappa_n^2}(2m)^{\frac{1}{n-2}}$, or

$$K^Y \geq b\,T(Y+T,Y+T)$$

for $0<b<\min\{\frac{\kappa_n}{4},\frac{\sigma'}{2}\}$. This yields the result

$$K^N = K^Y \geq b\,T(N,N) = b\,(J^N,N)\,. \qquad \square$$

Finally, we find an explicit expression for Y. Consider the vectorfield

$$\hat{Y} = -\frac{2}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u}$$

on $\mathscr{R}\cup\mathscr{A}$ formally defined by the expression for Y on $\mathcal{H}^+$. In $\mathscr{R}$

$$\hat{Y} = \frac{2}{1-\frac{2m}{r^{n-2}}}\frac{\partial}{\partial u^*}\,.$$

$\hat{Y}$ generates geodesics, this being a consequence of the Hessian equations (1.2.6a),

$$\nabla_{\hat{Y}}\hat{Y} = \big(\frac{2}{\frac{\partial r}{\partial u}}\big)^2\Big[-\frac{1}{\frac{\partial r}{\partial u}}\frac{\partial^2 r}{\partial u^2}+\frac{2}{\Omega}\frac{\partial\Omega}{\partial u}\Big]\frac{\partial}{\partial u} = 0\,,$$

and is Lie-transported by T:

$$[T,\hat{Y}] = \frac{2}{(\frac{\partial r}{\partial u})^2}\big([T,\frac{\partial}{\partial u}]\cdot r\big)\frac{\partial}{\partial u}-\frac{2}{\frac{\partial r}{\partial u}}[T,\frac{\partial}{\partial u}] = -\kappa_n\hat{Y}+\kappa_n\hat{Y} = 0$$

because $[T,\frac{\partial}{\partial u}]=\kappa_n\frac{\partial}{\partial u}$. Y as constructed above coincides with

$$Y = \alpha(r)\hat{Y}+\beta(r)T \qquad (1.3.17)$$

where

$$\alpha(r) = 1 + \frac{\sigma}{4\kappa_n}\big(1 - \frac{2m}{r^{n-2}}\big)$$
$$\beta(r) = \frac{\sigma}{4\kappa_n}\big(1 - \frac{2m}{r^{n-2}}\big)\,.$$

Indeed, on $\mathcal{H}^+$,

$$Y|_{\mathcal{H}^+} = \hat{Y}|_{\mathcal{H}^+} = -\frac{2}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u}\bigg|_{\mathcal{H}^+}$$

and

$$\begin{aligned}\nabla_Y Y|_{\mathcal{H}^+} = \nabla_{\hat{Y}} Y|_{\mathcal{H}^+} &= (\hat{Y}\cdot\alpha)\hat{Y}|_{\mathcal{H}^+} + \nabla_{\hat{Y}}\hat{Y}\Big|_{\mathcal{H}^+} + (\hat{Y}\cdot\beta)T|_{\mathcal{H}^+}\\ &= -\sigma\,(Y+T)|_{\mathcal{H}^+}\end{aligned}$$

since

$$\hat{Y}\cdot\alpha|_{\mathcal{H}^+} = \frac{\sigma}{4\kappa_n}(n-2)\frac{2m}{r^{n-1}}\hat{Y}\cdot r|_{\mathcal{H}^+} = -\sigma$$
$$\hat{Y}\cdot\beta|_{\mathcal{H}^+} = -\sigma$$

and Y remains Lie-transported by T:

$$[T,Y] = (T\cdot\alpha)\hat{Y} + (T\cdot\beta)T + \alpha\,[T,\hat{Y}] + \beta\,[T,T] = 0$$

since

$$T\cdot\alpha = 0 = T\cdot\beta\,.$$

Thus the vectorfield Y is given explicitly by

$$Y = \begin{cases} -\frac{2}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u} & \text{on } \mathcal{H}^+ \\ \Big[1 + \frac{\sigma}{4\kappa_n}\big(1-\frac{2m}{r^{n-2}}\big)\Big]\frac{2}{1-\frac{2m}{r^{n-2}}}\frac{\partial}{\partial u^*} + \frac{\sigma}{4\kappa_n}\big(1-\frac{2m}{r^{n-2}}\big)\frac{\partial}{\partial t} & \text{in } \mathscr{R}\end{cases} \tag{1.3.18}$$

Clearly, by continuity, we can choose two values $\sqrt[n-2]{2m} < r_0^{(N)} < r_1^{(N)} < \infty$ and set

$$N = \begin{cases} T+Y & \sqrt[n-2]{2m} \leq r \leq r_0^{(N)} \\ T & r \geq r_1^{(N)}\end{cases}$$

with a smooth φ_t-invariant transition of the timelike vectorfield N in $r_0^{(N)} \leq r \leq r_1^{(N)}$, such that (1.3.16) extends to the neighborhood $\sqrt[n-2]{2m} < r < r_0^{(N)}$ of the event horizon.

Remark 1.10 (Interpretation of Prop. 1.9). Consider a small strip along the horizon

$$\mathscr{S} = \bigcup_{v^*\in[v_1^*,v_2^*]} \mathscr{V}(v^*) = \bigcup_{v^*\in[v_1^*,v_2^*]} [u_1^*,\infty]\times\{v^*\}\,.$$

Then the energy identity for N in $\mathscr{S}$ becomes (upon neglecting for u_1^* large enough any difference in the contributions from $\{u_1^*\} \times [v_1^*, v_2^*]$ and the corresponding segment on the horizon) the inequality

$$\int_{\mathscr{V}(v_2^*)} (J^N, n) + b \int_{\mathscr{S}} (J^N, n) \leq \int_{\mathscr{V}(v_1^*)} (J^N, n)$$

where we have replaced N by the normal n to $\mathscr{V}(v^*)$ in Prop. 1.9, which implies the energy decay reminiscent of the redshift:

$$\int_{\mathscr{V}(v_2^*)} (J^N, n) \leq e^{-b(v_2^* - v_1^*)} \int_{\mathscr{V}(v_1^*)} (J^N, n)$$

Note that the decay is determined from the surface gravity.

1.4 Integrated Local Energy Decay

In this Section we prove several *integrated local energy decay* statements, i.e. estimates on the energy density of solutions to (1.1.1) integrated on (bounded) *space-time* regions; this in an essential ingredient for the decay mechanism employed in Section 1.5.

Let $\mathcal{R}_{r_0,r_1}(t_0, t_1, u_1^*, v_1^*)$ be the region composed of a trapezoid and characteristic rectangles as follows, (see figure 1.3):

$$\begin{aligned}\mathcal{R}_{r_0,r_1}(t_0, t_1, u_1^*, v_1^*) \doteq\ & \Big\{(t,r) : t_0 \leq t \leq t_1, r_0 \leq r \leq r_1\Big\} \\ & \cup \Big\{(t,r) : r \leq r_0, \frac{1}{2}(t - r^*) \leq u_1^*, t_0 + r_0^* \leq t + r^* \leq t_1 + r_0^*\Big\} \\ & \cup \Big\{(t,r) : r \geq r_1, \frac{1}{2}(t + r^*) \leq v_1^*, t_0 - r_1^* \leq t - r^* \leq t_1 - r_1^*\Big\}\end{aligned} \tag{1.4.1}$$

We denote by

$$\mathcal{R}^\infty_{r_0,r_1}(t_0) \doteq \bigcup_{t_1 \geq t_0} \ \bigcup_{u_1^* \geq \frac{1}{2}(t_1 - r_0^*)} \ \bigcup_{v_1^* \geq \frac{1}{2}(t_1 + r_1^*)} \mathcal{R}(t_0, t_1, u_1^*, v_1^*) \tag{1.4.2}$$

and its past boundary by

$$\Sigma_{\tau_0} \doteq \partial^- \mathcal{R}^\infty_{r_0,r_1}(t_0) \qquad \tau_0 = \frac{1}{2}(t_0 - r_1^*)\,. \tag{1.4.3}$$

We shall first state the central estimate.

Proposition 1.11 (Integrated local energy decay estimate)**.** *Let ϕ be a solution of the wave equation $\Box_g \phi = 0$. Then there exist $(2m)^{\frac{1}{n-2}} < r_0 < r_1$ and a constant $C(n,m)$ depending on the dimension n and the mass m, such that*

$$\begin{aligned}\int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)} \left\{ \frac{1}{r^n}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2 + \frac{1}{r^{n+1}}\Big(\frac{\partial\phi}{\partial t}\Big)^2 + \frac{1}{r^3}\Big(1 - \frac{2m}{r^{n-2}}\Big)\big|\nabla\!\!\!\!/\,\phi\big|^2_{r^2\mathring{\gamma}_{n-1}} \right\} \mathrm{d}\mu_g \\ \leq C(n,m) \int_{\Sigma_{\tau_0}} \Big(J^T(\phi) + J^T(T\cdot\phi), n\Big)\end{aligned} \tag{1.4.4}$$

for any $t_0 \geq 0$, where $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

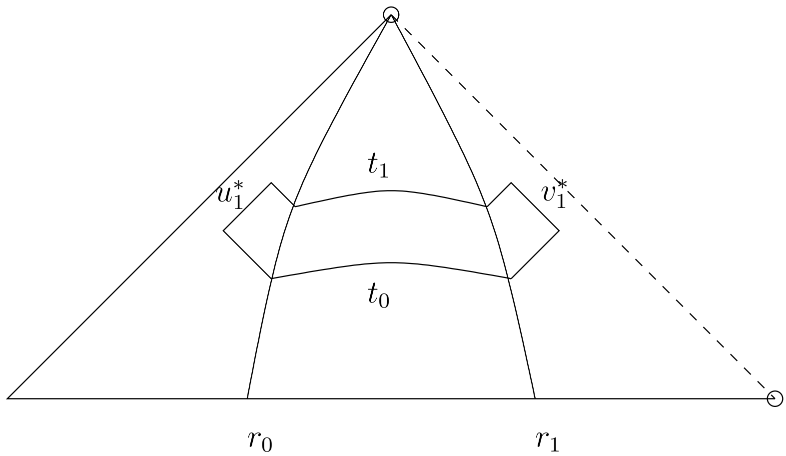

Figure 1.3: The region $\mathcal{R}_{r_0,r_1}(t_0,t_1,u_1^*,v_1^*)$.

The degeneracy at infinity can in fact be improved:

Proposition 1.12 (Improved integrated local energy decay estimate)**.** *Let ϕ be a solution of the wave equation $\Box_g\phi = 0$, then there exists a constant $C(n,m,\delta)$ for each $0<\delta<1$ such that*

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \Big\{ \frac{1}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{1}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial t}\big)^2 + \frac{1}{r}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \Big\}\,\mathrm{d}\mu_g \leq$$

$$\leq C(n,m,\delta)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big) \quad (1.4.5)$$

for any $t_0 \geq 0$, where $r_0<r_1$ are as above, and $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

As a consequence of the redshift effect of Section 1.3, and the uniform boundedness of the nondegenerate energy (which is proven independently in Section 1.5.1), we can infer in a more geometric formulation:

Corollary 1.13 (nondegenerate integrated local energy decay)**.** *Let ϕ be a solution of* (1.1.1)*, then for any $R > \sqrt[n-2]{2m}$ there exists a constant $C(n,m,R)$ such that*

$$\int_{\tau'}^{\tau}\mathrm{d}\overline{\tau}\int_{\Sigma'_{\overline{\tau}}}\Big(J^N(\phi),n\Big) \leq C(n,m,R)\int_{\Sigma_{\tau'}}\Big(J^N(\phi)+J^T(T\cdot\phi),n\Big)\,, \quad (1.4.6)$$

for all $\tau'<\tau$, where $\Sigma'_{\tau} \doteq \Sigma_\tau \cap \{r\leq R\}$.

Proof. Let (we use standard notation for causal sets, see e.g.[29])

$$\mathcal{R}'(\tau',\tau) \doteq \mathrm{J}^-(\Sigma'_\tau)\cap\mathrm{J}^+(\Sigma_{\tau'})\,.$$

In $\mathcal{R}'(\tau',\tau)\cap\{r<r_0^{(N)}\}$ we have by Prop. 1.7

$$\Big(J^N(\phi),n\Big) \leq \frac{1}{b}K^N(\phi)\,,$$

and in $\mathcal{R}'(\tau',\tau)\cap\{r\geq r_1^{(N)}\}$ trivially $(J^N(\phi),n)\leq(J^T(\phi),n)$. Therefore using the energy identity for N on $\mathcal{R}'(\tau',\tau)$ the estimate (1.4.6) follows from Prop. 1.35 and Prop. 1.11. $\square$

In the above, no control is obtained on a spacetime integral of ϕ^2 itself; however, all that is needed for the decay argument of Section 1.5 is an estimate for the integral of ϕ^2 on timelike boundaries.

Proposition 1.14 (zeroth order terms on timelike boundaries)**.** *Let* ϕ *be solution of the wave equation* (1.1.1)*, and* $R > \sqrt[n-2]{8nm}$*. Then there is a constant* $C(n,m,R)$ *such that*

$$\int_{2\tau'+R^*}^{2\tau+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \, \phi^2|_{r=R} \leq \\ \leq C(n,m,R) \int_{2\tau'+R^*}^{2\tau+R^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{ \Big(\frac{\partial \phi}{\partial r^*} \Big)^2 + \big| \nabla\!\!\!/ \phi \big|^2 \Big\} \big|_{r=R} \\ + C(n,m,R) \int_{\Sigma_{\tau'}} \Big(J^T(\phi), n \Big) \quad (1.4.7)$$

for all $\tau' < \tau$*.*

The central result of Prop. 1.11 combines results for two different regimes, that of high angular frequencies and that of low angular frequencies. First we will use radial multiplier vectorfields to construct positive definite currents to deal with the former regime, and then a more general current using a commutation with angular momentum operators for the latter.

Remark 1.15*.* The specific parametrization (1.4.3) has technical advantages, but Σ_τ can in principle be replaced by a foliation of strictly *spacelike* hypersurfaces terminating at future null infinity and crossing the event horizon to the future of the bifurcation sphere.

1.4.1 Radial multiplier vectorfields

A *radial multiplier* is a vectorfield of the form

$$X = f(r^*) \frac{\partial}{\partial r^*} \,. \quad (1.4.8)$$

We would like the associated current to be positive, however we find in general, as it is shown below:

$$K^X = \frac{f'}{1 - \frac{2m}{r^{n-2}}} \Big(\frac{\partial \phi}{\partial r^*} \Big)^2 + \frac{f}{r} \Big(1 - \frac{nm}{r^{n-2}} \Big) \big| \nabla\!\!\!/ \phi \big|^2_{r^2 \overset{\circ}{\gamma}_{n-1}} \\ - \frac{1}{2} \Big[f' + (n-1) \frac{f}{r} \Big(1 - \frac{2m}{r^{n-2}} \Big) \Big] \partial^\alpha \phi \, \partial_\alpha \phi \quad (1.4.9)$$

Note. The prefactor to the angular derivatives vanishes at the *photon sphere* at $r = \sqrt[n-2]{nm}$.

Calculation of the deformation tensor ${}^{(X)}\pi$. It is convenient to work in Eddington-Finkelstein coordinates

$$X = \frac{1}{2} f(r^*) \frac{\partial}{\partial v^*} - \frac{1}{2} f(r^*) \frac{\partial}{\partial u^*} \,. \tag{1.4.10}$$

For the connection coefficients of (1.2.25) one obtains

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial u^*}} \frac{\partial}{\partial u^*} &= -(n-2)\,\frac{2m}{r^{n-1}}\,\frac{\partial}{\partial u^*} \\
\nabla_{\frac{\partial}{\partial v^*}} \frac{\partial}{\partial v^*} &= \quad (n-2)\,\frac{2m}{r^{n-1}}\,\frac{\partial}{\partial v^*} \\
\nabla_{E_A} E_B &= \nabla\!\!\!/_{E_A} E_B + \frac{r}{2}(\mathring{\gamma}_{n-1})_{AB}\frac{\partial}{\partial u^*} - \frac{r}{2}(\mathring{\gamma}_{n-1})_{AB}\frac{\partial}{\partial v^*} \\
\nabla_{\frac{\partial}{\partial u^*}} E_B &= -\frac{1}{r}\big(1-\frac{2m}{r^{n-2}}\big)E_B \\
\nabla_{\frac{\partial}{\partial v^*}} E_B &= \quad \frac{1}{r}\big(1-\frac{2m}{r^{n-2}}\big)E_B \,.
\end{aligned} \tag{1.4.11}$$

Therefore

$$\begin{gathered}
{}^{(X)}\pi_{u^*u^*} = g(\nabla_{\frac{\partial}{\partial u^*}}X, \frac{\partial}{\partial u^*}) = \big(1-\frac{2m}{r^{n-2}}\big)f' \\
{}^{(X)}\pi_{v^*v^*} = g(\nabla_{\frac{\partial}{\partial v^*}}X, \frac{\partial}{\partial v^*}) = \big(1-\frac{2m}{r^{n-2}}\big)f' \\
\begin{aligned}
{}^{(X)}\pi_{u^*v^*} &= \frac{1}{2}g(\nabla_{\frac{\partial}{\partial u^*}}X, \frac{\partial}{\partial v^*}) + \frac{1}{2}g(\frac{\partial}{\partial u^*}, \nabla_{\frac{\partial}{\partial v^*}}X) \\
&= -\big(1-\frac{2m}{r^{n-2}}\big)\big(f' + (n-2)\frac{2m}{r^{n-1}}f\big)
\end{aligned} \\
{}^{(X)}\pi_{aA} = 0 \\
\begin{aligned}
{}^{(X)}\pi_{AB} &= \frac{1}{2}g(\nabla_{E_A}X, E_B) + \frac{1}{2}g(E_A, \nabla_{E_B}X) \\
&= f\,r\,\big(1-\frac{2m}{r^{n-2}}\big)(\mathring{\gamma}_{n-1})_{AB}
\end{aligned}
\end{gathered} \tag{1.4.12}$$

The formula for K^X above (1.4.9) is now obtained by writing out (see also Appendix B.2)

$$K^X = {}^{(X)}\pi^{\alpha\beta}\,T_{\alpha\beta}$$

and rearranging the terms so as to complete $(\frac{\partial\phi}{\partial u^*})^2 + (\frac{\partial\phi}{\partial v^*})^2$ to $(\frac{\partial\phi}{\partial r^*})^2$. This rearrangement is also related to the following modification of currents; for observe that

$$\Box(\phi^2) = 2(\partial^\alpha\phi)(\partial_\alpha\phi) \tag{1.4.13}$$

if $\Box\phi = 0$.

First modified current. Denoting by

$$J^{X,0}_\mu = T_{\mu\nu}X^\nu \tag{1.4.14}$$

define the first modified current by

$$J_\mu^{X,1} = J_\mu^{X,0} + \frac{1}{4}\Big(f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big)\Big)\partial_\mu(\phi^2) - \frac{1}{4}\partial_\mu\Big(f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big)\Big)\phi^2 . \quad (1.4.15)$$

Consequently the divergences are

$$K^{X,0} = \nabla^\mu J_\mu^{X,0} = K^X \quad (1.4.16)$$

$$\begin{aligned} K^{X,1} = \nabla^\mu J_\mu^{X,1} &= K^X + \frac{1}{4}\Big(f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big)\Big)\Box(\phi^2) \\ &\quad - \frac{1}{4}\Box\Big(f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big)\Big)\phi^2 \\ &= \frac{f'}{1 - \frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{f}{r}\big(1 - \frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\mathring{\gamma}_{n-1}} \\ &\quad - \frac{1}{4}\Box\Big(f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big)\Big)\phi^2 \end{aligned} \quad (1.4.17)$$

Since for any function w

$$\begin{aligned} \Box(w) &= (g^{-1})^{\mu\nu}\nabla_\mu\partial_\nu w \\ &= -\frac{1}{1 - \frac{2m}{r^{n-2}}}\partial_{u^*}\partial_{v^*}w - \frac{n-1}{2r}\big(\partial_{u^*}w - \partial_{v^*}w\big) + \Delta\!\!\!\!/\,_{r^2\mathring{\gamma}_{n-1}} w , \end{aligned} \quad (1.4.18)$$

a straight-forward calculation for

$$w = f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big) \quad (1.4.19)$$

shows

$$\begin{aligned} &\Box\Big(f' + (n-1)\frac{f}{r}\big(1 - \frac{2m}{r^{n-2}}\big)\Big) = \\ &\quad = \frac{1}{1 - \frac{2m}{r^{n-2}}}f''' + 2(n-1)\frac{f''}{r} + (n-1)\Big[(n-3) + (n-1)\frac{2m}{r^{n-2}}\Big]\frac{f'}{r^2} \\ &\quad + (n-1)\Big[\Big((n-1)(n-2) - (n-3)\Big)\big(\frac{2m}{r^{n-2}}\big)^2 - n\frac{2m}{r^{n-2}} - (n-3)\Big]\frac{f}{r^3} . \end{aligned} \quad (1.4.20)$$

Thus we finally obtain

$$\begin{aligned} K^{X,1} &= \frac{f'}{1 - \frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{f}{r}\big(1 - \frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\mathring{\gamma}_{n-1}} \\ &\quad - \frac{1}{4}\frac{f'''}{1 - \frac{2m}{r^{n-2}}}\phi^2 - \frac{n-1}{2}\frac{f''}{r}\phi^2 - \frac{n-1}{4}\Big[(n-3) + (n-1)\frac{2m}{r^{n-2}}\Big]\frac{f'}{r^2}\phi^2 \\ &\quad - \frac{n-1}{4}\Big[(n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2 - n\frac{2m}{r^{n-2}} - (n-3)\Big]\frac{f}{r^3}\phi^2 . \end{aligned} \quad (1.4.21)$$

Applications of the first modified current. The proofs of Prop. 1.12 and Prop. 1.14 are applications of this formula, as it appears in the energy identity for $J^{X,1}$ on ${}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$, see Appendix B.2.

Proof of Prop. 1.14. Choose $f = 1$ identically, then

$$K^{X,1} = \frac{1}{r}\big(1 - \frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} + \frac{n-1}{4}\Big[(n-3) + n\frac{2m}{r^{n-2}} - (n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2\Big]\frac{1}{r^3}\phi^2\,. \quad (1.4.22)$$

Since precisely

$$g\big(J^{X,1},\frac{\partial}{\partial r^*}\big) = \frac{1}{4}\big(\frac{\partial\phi}{\partial v^*}\big)^2 + \frac{1}{4}\big(\frac{\partial\phi}{\partial u^*}\big)^2 - \frac{1}{2}\big(1 - \frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} + \frac{n-1}{2r}\big(1-\frac{2m}{r^{n-2}}\big)\phi\frac{\partial\phi}{\partial r^*} + \frac{n-1}{4r^2}\Big[1-(n-1)\frac{2m}{r^{n-2}}\Big]\big(1-\frac{2m}{r^{n-2}}\big)\phi^2 \quad (1.4.23)$$

we deduce from the energy identity for $J^{\frac{\partial}{\partial r^*},1}$ in ${}^{R}\mathcal{D}_{\tau'}^{\tau}$ that

$$\begin{aligned}
&\int_{R^*+2\tau'}^{R^*+2\tau}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-1}\times\\
&\quad\times\Big\{\frac{1}{4}\big(\frac{\partial\phi}{\partial v^*}\big)^2 + \frac{1}{4}\big(\frac{\partial\phi}{\partial u^*}\big)^2 + \frac{n-1}{4R^2}\Big[\frac{1}{2} - (n-1)\frac{2m}{R^{n-2}}\Big]\big(1-\frac{2m}{R^{n-2}}\big)\phi^2\Big\}\Big|_{r=R}\\
&\quad+\int_{{}^{R}\mathcal{D}_{\tau'}^{\tau}}\frac{n-1}{4r}\Big[(n-3)+n\frac{2m}{r^{n-2}} - (n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2\Big]\frac{1}{r^2}\phi^2\,\mathrm{d}\mu_g \leq\\
&\qquad\leq\int_{R^*+2\tau'}^{R^*+2\tau}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-1}\times\\
&\quad\times\Big\{\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} + \frac{n-1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big\}\Big|_{r=R}\\
&\qquad\qquad+C(n,m)\int_{\Sigma_{\tau'}}\Big(J^T(\phi),n\Big)\,, \quad (1.4.24)
\end{aligned}$$

where we have used Prop. B.5 for the boundary terms on $\partial^{R}\mathcal{D}_{\tau'}^{\tau}\setminus\{r=R\}$; note that

$$(n-3) + n\frac{2m}{r^{n-2}} - (n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2 > 0 \qquad (R > \sqrt[n-2]{8nm})\,. \qquad \square$$

Proof of Prop. 1.12. On one hand we need $f' = \mathcal{O}(\frac{1}{r^{1+\delta}})$ in view of (1.4.21) while on the other we already know from the proof of Prop. 1.14 that $f = 1$ generates a positive bulk term for r large enough. We choose

$$f = 1 - \big(\frac{R}{r}\big)^\delta \quad (1.4.25)$$

(where $R > \sqrt[n-2]{2m}$ is chosen suitably in the last step of the proof) and indeed find

$$K^{X,1} = \delta\frac{R^\delta}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{f}{r}\big(1-\frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}$$

$$+\Big\{\frac{n-1}{4}(n-3)\Big[1-\big(\frac{R}{r}\big)^{\delta}(1+\delta)\Big]+\frac{1}{4}\big(\frac{R}{r}\big)^{\delta}\Big[2(n-1)-(2+\delta)\Big]\delta(1+\delta)$$
$$+\Big[\frac{n-1}{4}n\Big[1-\big(\frac{R}{r}\big)^{\delta}\Big]-\frac{\delta}{4}\big(\frac{R}{r}\big)^{\delta}\Big[n(n+\delta)-2(1+\delta)^2\Big]\Big]\frac{2m}{r^{n-2}}$$
$$-\Big[\frac{(n-1)^3}{4}\Big[1-\big(\frac{R}{r}\big)^{\delta}\Big]-\frac{\delta}{4}\big(\frac{R}{r}\big)^{\delta}\Big[\big(n-(1+\delta)\big)(n-1)-\delta^2\Big]\Big]\big(\frac{2m}{r^{n-2}}\big)^2\Big\}\frac{1}{r^3}\phi^2$$
$$\geq 0 \quad (1.4.26)$$

for $r \geq R_1 > R$, $R_1 = R_1(n,m) > \sqrt[n-2]{2m}$ chosen large enough. This gives control on $\frac{\partial\phi}{\partial r^*}$ and the angular derivatives:

$$\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\Big\{\delta\frac{R^{\delta}}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial r^*}\big)^2+\frac{f(R_1)}{r}\big(1-\frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\}\leq\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}K^{X,1}$$

Here and in the following $\tau_2 > \tau_1 > \frac{1}{2}\big(t_0 - R^*\big)$. For $\frac{\partial\phi}{\partial t}$ we use the auxiliary current (see also Appendix B.3)

$$J^{\mathrm{aux}}_{\mu}=\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\delta\frac{R^{\delta}}{r^{1+\delta}}\partial_{\mu}(\phi^2)$$

to find easily

$$\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\delta\frac{R^{\delta}}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial t}\big)^2\leq\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\Big\{\delta(n+\delta)\frac{R^{\delta}}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial r^*}\big)^2$$
$$+\delta\frac{R^{\delta}}{r^{1+\delta}}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}+\delta(n+\delta)\frac{R^{\delta}}{r^{3+\delta}}\phi^2+K^{\mathrm{aux}}\Big\}$$

Note that for $r \geq R_1$ in particular

$$\frac{1}{4}\Big[2(n-1)-(2+\delta)\Big]\delta(1+\delta)\frac{R^{\delta}}{r^{3+\delta}}\phi^2\leq K^{X,1}$$

hence

$$\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\delta\frac{R^{\delta}}{r^{1+\delta}}\Big\{\big(\frac{\partial\phi}{\partial t}\big)^2+\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big\}\leq C(n,m,\delta)\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\Big\{K^{X,1}+K^{\mathrm{aux}}\Big\}\leq$$
$$\leq C(n,m,\delta)\int_{{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}}\Big\{K^{X,1}+K^{\mathrm{aux}}\Big\}$$
$$+C(n,m,\delta)\int_{{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}\cap\{R<r<R_1\}}\Big\{\frac{1}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial t}\big)^2+\frac{1}{r^3}\phi^2\Big\}$$

By Prop. B.5 (also (B.6))

$$\int_{\partial^{R}\mathcal{D}_{\tau_1}^{\tau_2}}{}^*J^{X,1}\leq C(n,m,\delta)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi),n\Big)$$
$$+C(n,m,\delta)\int_{R^*+2\tau_1}^{R^*+2\tau_2}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}r^{n-1}\times$$
$$\times\Big\{\frac{1}{2}\big(\frac{\partial\phi}{\partial v^*}\big)^2+\frac{1}{2}\big(\frac{\partial\phi}{\partial u^*}\big)^2+\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}+\frac{1}{r^2}\phi^2\Big\}\big|_{r=R}$$

and by Prop. B.12

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J^{\mathrm{aux}} \leq C(n,m,\delta)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi),n\Big) \\ + \int_{R^*+2\tau_1}^{R^*+2\tau_2}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\Big\{\frac{\delta}{2}\frac{1}{2}\Big[\big(\frac{\partial\phi}{\partial u^*}\big)^2+\big(\frac{\partial\phi}{\partial v^*}\big)^2\Big]+\frac{\delta}{2}\frac{R^{2\delta}}{r^{2+2\delta}}\phi^2\Big\}_{r=R}.$$

Therefore by the energy identity for $J^{X,1}$ and J^{aux} on ${}^R\mathcal{D}_{\tau_1}^{\tau_2}$:

$$\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}}\Big\{K^{X,1}+K^{\mathrm{aux}}\Big\} \leq C(n,m,\delta)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi),n\Big) \\ + C(n,m,\delta)\int_{R^*+2\tau_1}^{R^*+2\tau_2}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\times \\ \times\Big\{\frac{1}{2}\big(\frac{\partial\phi}{\partial v^*}\big)^2+\frac{1}{2}\big(\frac{\partial\phi}{\partial u^*}\big)^2+\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}+\frac{1}{r^2}\phi^2\Big\}\big|_{r=R}$$

Our earlier (1.4.24) derived from the current $J^{\frac{\partial}{\partial r^*},1}$ now allows us to control the $\frac{\partial\phi}{\partial v^*}$, $\frac{\partial\phi}{\partial u^*}$ derivatives and ϕ^2 on the $r=R$ boundary together with the ϕ^2 term in the region $R\leq r\leq R_1$ in one step:

$$\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\frac{R^\delta}{r^{1+\delta}}\Big\{\big(\frac{\partial\phi}{\partial t}\big)^2+\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big\} \leq C(n,m,\delta)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi),n\Big) \\ + C(n,m,\delta)\int_{R^*+2\tau_1}^{R^*+2\tau_2}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\times \\ \times\Big\{\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}+\frac{n-1}{2}\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big\}\big|_{r=R} \\ + C(n,m,\delta)\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}\cap\{R<r<R_1\}}\frac{1}{r^{1+\delta}}\big(\frac{\partial\phi}{\partial t}\big)^2$$

With t_0 fixed, we can now choose R by Prop. 1.11 such that

$$\int_{{}^{R_1}\mathcal{D}_{\tau_1}^{\tau_2}}\frac{1}{r^{1+\delta}}\Big\{\big(\frac{\partial\phi}{\partial t}\big)^2+\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big\} \leq C(n,m,\delta)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big).$$

□

While it is possible to find simple functions $f\geq 0$ to ensure the positivity of $K^{X,1}$ asymptotically, this is not possible in the entire domain of outer communication; this is a consequence of *trapping*, which more concretely appears as the indefiniteness of sign in (1.4.21) at the photon sphere $r=\sqrt[n-2]{nm}$.

In the following our strategy will be to prove non-negativity of $K^{X,1}$ not pointwise but by using Poincaré inequalities after integration over the spheres (the group orbits of $\mathrm{SO}(n)$). This is achieved in two alternative constructions: in Section 1.4.2 with a decomposition into spherical harmonics, and in Section 1.4.3 by a commutation with angular momentum operators.

1.4.2 High angular frequencies

Here the idea is to control with the second term in (1.4.21) after a decomposition of ϕ into spherical harmonics all other terms of order ϕ^2. For this dominant term to be positive we evidently need

$$f(r^*)\begin{cases}<0 & r<\sqrt[n-2]{nm}\\ =0 & r=\sqrt[n-2]{nm}\\ >0 & r>\sqrt[n-2]{nm}\end{cases}.$$

Since f should also be bounded one may guess that

$$f(r^*)=\arctan\big(\frac{(n-1)\,r^*}{\sqrt[n-2]{nm}}\big)$$

is a good choice; while it can ensure positivity at the photon sphere, it fails to do so away from the photon sphere in the intermediate regions near the horizon and in the asymptotics. After having briefly recalled the decomposition into spherical harmonics, we will therefore give a more refined construction of f, nonetheless guided by the overall characteristics of this function, which will in particular allow us to track the dependence of the lowest spherical harmonic number (for which we can establish non-negativity) on the dimension n.

Fourier expansion on the sphere $\mathbb{S}^{n-1}$. We recall that by considering homogeneous harmonic polynomials on $\mathbb{R}^n$ (see e.g. discussion of spherical harmonics in [44]) we find that all eigenvalues of

$$-\mathring{\Delta}_{n-1}+\big(\frac{n-2}{2}\big)^2$$

on $\mathbb{S}^{n-1}$ are given by

$$\big(l+\frac{n-2}{2}\big)^2\qquad(l\geq 0)\,.$$

Let E_l, $l\geq 0$, be the corresponding eigenspace in $\mathrm{L}^2(\mathbb{S}^{n-1})$. Recall

$$\dim_{\mathbb{C}}E_l=\big(l+\frac{n-2}{2}\big)\frac{2}{l}\binom{n-2+l-1}{l-1}$$

and furthermore

$$\mathrm{L}^2(\mathbb{S}^{n-1})=\bigoplus_{l\geq 0}E_l\,.$$

Denote by π_l the orthogonal projection of $\mathrm{L}^2(\mathbb{S}^{n-1})$ onto E_l, then for $\phi\in\mathrm{L}^2(\mathbb{S}^{n-1})$

$$\phi=\sum_{l\geq 0}\pi_l\phi\,. \tag{1.4.27}$$

This is the Fourier expansion on the sphere $\mathbb{S}^{n-1}$. We find

$$\mathring{\Delta}_{n-1}\,\pi_l\phi=-l(l+n-2)\pi_l\phi\,. \tag{1.4.28}$$

Now,

$$\begin{aligned} l(l+n-2)\int_{\mathbb{S}^{n-1}}(\pi_l\phi)^2\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} &= -\int_{\mathbb{S}^{n-1}}(\pi_l\phi)(\overset{\circ}{\triangle}_{n-1}\,\pi_l\phi)\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ &= \int_{\mathbb{S}^{n-1}}|\overset{\circ}{\nabla}_{n-1}\,\pi_l\phi|^2\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \end{aligned}$$

and assuming that

$$\pi_l\phi = 0 \qquad (0\leq l<L)$$

for some $L>0$,

$$\begin{aligned} L(L+n-2)\frac{1}{r^2}\int_{S_r}\phi^2\,\mathrm{d}\mu_{\gamma_r} &\leq \frac{1}{r^2}\sum_{l\geq L} l(l+n-2)\int_{\mathbb{S}^{n-1}}(\pi_l\phi)^2 r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ &= \frac{1}{r^2}\sum_{l\geq L}\int_{\mathbb{S}^{n-1}}|\overset{\circ}{\nabla}_{n-1}\,\pi_l\phi|^2\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-1} \\ &= \frac{1}{r^2}\sum_{l\geq L}\int_{\mathbb{S}^{n-1}}|\pi_l\,\overset{\circ}{\nabla}_{n-1}\,\phi|^2\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-1} \\ &= \int_{\mathbb{S}^{n-1}}\frac{1}{r^2}|\overset{\circ}{\nabla}_{n-1}\,\phi|^2\,r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ &= \int_{S_r}\big|\nabla\!\!\!/\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,\mathrm{d}\mu_{\gamma_r}\,, \end{aligned}$$

where for the commutation of π_l with $\overset{\circ}{\nabla}_{n-1}$ we have used that the projection is of the form

$$(\pi_l\phi)(r\xi) = \int_{\mathbb{S}^{n-1}}\pi_l(\langle\xi,\xi'\rangle)\phi(r\xi')\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}(\xi')\,. \tag{1.4.29}$$

We have proven the following Poincaré-type inequality:

Lemma 1.16 (Poincaré inequality)**.** *Let* $\phi\in\mathrm{H}^1(S_r)$, $S_r=(\mathbb{S}^{n-1},r^2\,\overset{\circ}{\gamma}_{n-1})$, *have vanishing projection to* E_l, $0\leq l<L$, *for some* $L\in\mathbb{N}$, *i.e.*

$$\pi_l\phi=0 \qquad (0\leq l<L)\,,$$

then

$$\int_{S_r}\big|\nabla\!\!\!/\phi\big|^2\,\mathrm{d}\mu_{\gamma_r}\geq L(L+n-2)\frac{1}{r^2}\int_{S_r}\phi^2\,\mathrm{d}\mu_{\gamma_r}\,.$$

Construction of the multiplier function for high angular frequencies. The idea is to prescribe the 3rd derivative of f and to find its 2nd and 1st derivatives by integration with boundary values and parameters that ensure that f remains bounded. Let

$$\alpha=\frac{n-1}{(nm)^{\frac{1}{n-2}}} \tag{1.4.30}$$

and $\gamma \geq 2$, $\gamma \in \mathbb{N}$. Consider

$$f^{\mathrm{III}}_{\gamma,\alpha}(r^*) = \begin{cases} -1\,, & |r^*| \leq \frac{1}{\gamma\alpha} \\ \ \ 1\,, & \frac{1}{\gamma\alpha} < |r^*| \leq b_{\gamma,\alpha} \\ \big(\frac{b_{\gamma,\alpha}}{r^*}\big)^6\,, & |r^*| \geq b_{\gamma,\alpha} \end{cases} \tag{1.4.31}$$

where

$$b_{\gamma,\alpha} = \frac{5}{6}\frac{2}{\gamma\alpha}\,. \tag{1.4.32}$$

Note that $b_{\gamma,\alpha}$ is chosen so that

$$\int_0^\infty f^{\mathrm{III}}_{\gamma,\alpha}(r^*)\,\mathrm{d}r^* = 0\,. \tag{1.4.33}$$

Now define

$$f^{\mathrm{II}}_{\gamma,\alpha}(r^*) = \int_0^{r^*} f^{\mathrm{III}}_{\gamma,\alpha}(t)\,\mathrm{d}t\,. \tag{1.4.34}$$

Obviously $f^{\mathrm{II}}_{\gamma,\alpha}(-r^*) = -f^{\mathrm{II}}_{\gamma,\alpha}(r^*)$ and in explicit form

$$f^{\mathrm{II}}_{\gamma,\alpha}(r^*) = \begin{cases} -r^* & |r^*| \leq \frac{1}{\gamma\alpha} \\ r^* - \frac{2}{\gamma\alpha} & \frac{1}{\gamma\alpha} < r^* \leq b_{\gamma,\alpha} \\ r^* + \frac{2}{\gamma\alpha} & -b_{\gamma,\alpha} \leq r^* < -\frac{1}{\gamma\alpha} \\ -\frac{b^6_{\gamma,\alpha}}{5r^{*5}} & |r^*| \geq b_{\gamma,\alpha} \end{cases}\,. \tag{1.4.35}$$

The functions $f^{\mathrm{II}}_{\gamma,\alpha}$ and $f^{\mathrm{III}}_{\gamma,\alpha}$ are sketched in figure 1.4. Next define

$$f^{\mathrm{I}}_{\gamma,\alpha} = \int_{-\infty}^{r^*} f^{\mathrm{II}}_{\gamma,\alpha}(t)\,\mathrm{d}t\,. \tag{1.4.36}$$

Here we find

$$f^{\mathrm{I}}_{\gamma,\alpha}(r^*) = \begin{cases} \frac{b^6_{\gamma,\alpha}}{20r^{*4}} & r^* \leq -b_{\gamma,\alpha} \\ \frac{b^2_{\gamma,\alpha}}{20} + \frac{1}{2}(r^{*2} - b^2_{\gamma,\alpha}) + \frac{2}{\gamma\alpha}(r^* + b_{\gamma,\alpha}) & -b_{\gamma,\alpha} \leq r^* \leq -\frac{1}{\gamma\alpha} \\ \frac{13}{12}\frac{1}{(\gamma\alpha)^2} - \frac{r^{*2}}{2} & -\frac{1}{\gamma\alpha} \leq r^* \leq 0 \end{cases} \tag{1.4.37}$$

and $f^{\mathrm{I}}_{\gamma,\alpha}(r^*) = f^{\mathrm{I}}_{\gamma,\alpha}(-r^*)$, as sketched in figure 1.5. Finally define

$$f^0_{\gamma,\alpha}(r^*) = \int_0^{r^*} f^{\mathrm{I}}_{\gamma,\alpha}(t)\,\mathrm{d}t\,. \tag{1.4.38}$$

Here again $f^0_{\gamma,\alpha}(-r^*) = -f^0_{\gamma,\alpha}(r^*)$ and in particular

$$f^0_{\gamma,\alpha}\big(\frac{1}{\gamma\alpha}\big) = \int_0^{\frac{1}{\gamma\alpha}} \big(\frac{13}{12}\frac{1}{(\gamma\alpha)^2} - \frac{t^2}{2}\big)\,\mathrm{d}t = \frac{11}{12}\frac{1}{(\gamma\alpha)^3}\,. \tag{1.4.39}$$

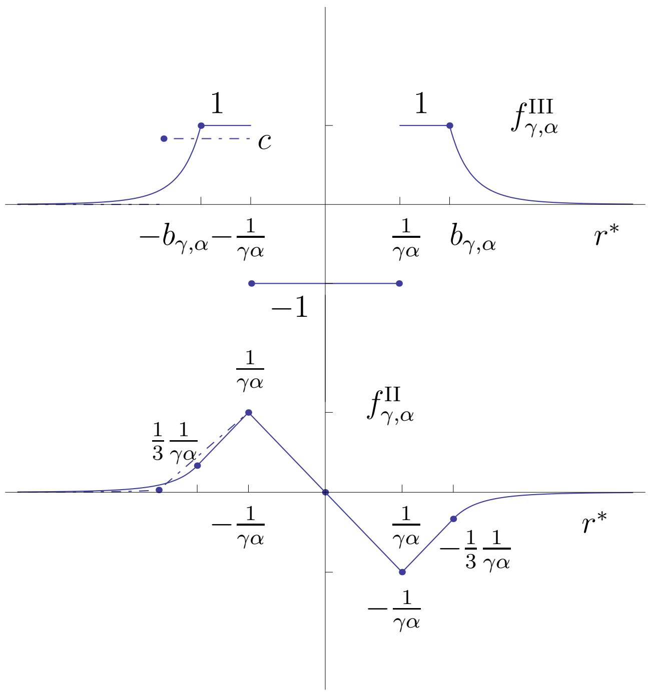

Figure 1.4: Sketch of the functions $f^{\mathrm{II}}_{\gamma,\alpha}$ and $f^{\mathrm{III}}_{\gamma,\alpha}$, and the adjusted functions (dot-dashed) for $r^* \leq 0$.

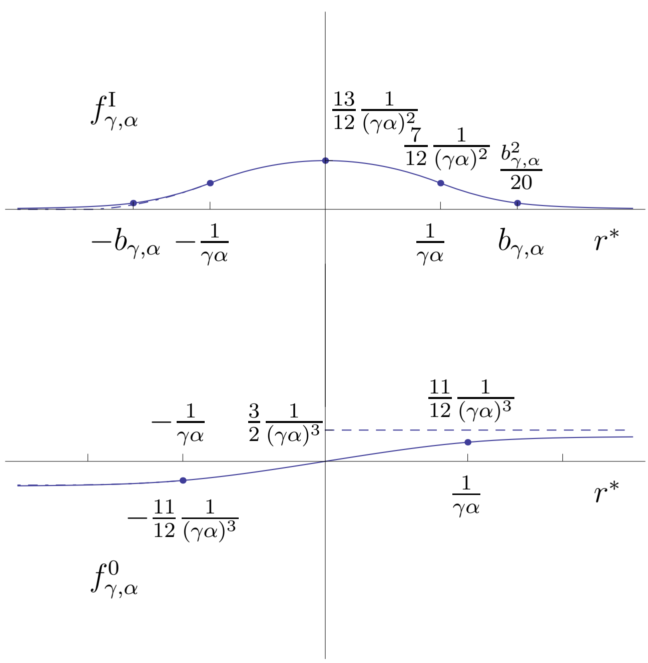

Figure 1.5: Sketch of the functions $f^{\mathrm{I}}_{\gamma,\alpha}$ and $f^{0}_{\gamma,\alpha}$, and the adjusted functions (dot-dashed) for $r^* \leq 0$.

Moreover the calculus yields

$$f(b_{\gamma,\alpha}) > \frac{1}{(\gamma\alpha)^3}$$
$$\lim_{r^*\to\infty} f^0_{\gamma,\alpha}(r^*) < \frac{3}{2}\frac{1}{(\gamma\alpha)^3}\,. \tag{1.4.40}$$

The function $f^0_{\gamma,\alpha}$ is sketched in figure 1.5. While this function would suffice in the region $r^* \geq -\frac{1}{\gamma\alpha}$ it does not fall-off fast enough as $r^* \to -\infty$.

Lemma 1.17. *With r^* defined by (1.2.22) we have for all $n \geq 3$*

$$\lim_{r^*\to-\infty} (1 - \frac{2m}{r^{n-2}})(-r^*) = 0\,.$$

In fact,

$$(1 - \frac{2m}{r^{n-2}}) \leq \frac{(2m)^{\frac{1}{n-2}}}{(-r^*)}$$

for all $r^* < 0$.

Proof. See Appendix B.2.2. ☐

It is easy to convince oneself that one can make an adjustment to f^{III} on $r^* \leq 0$ that introduces faster decay while keeping the area under the graph of f^{III} *and* f^{II} fixed. In other words, there are constants

$$b_{\gamma,\alpha} \leq b \leq \frac{4}{\gamma\alpha} \qquad \frac{1}{4} \leq c \leq 1 \tag{1.4.41}$$

such that if we redefine $f^{\mathrm{III}}_{\gamma,\alpha}$ for $r^* \leq 0$ as

$$f^{\mathrm{III}}_{\gamma,\alpha}(r^*) = \begin{cases} -1 & -\frac{1}{\gamma\alpha} \leq r^* \leq 0 \\ c & -b \leq r^* \leq \frac{1}{\gamma\alpha} \\ (1 - \frac{2m}{r^{n-2}})^6 (\frac{b}{(2m)^{\frac{1}{n-2}}})^6 & r^* \leq -b \end{cases} \tag{1.4.42}$$

then

$$\int_0^{-\infty} f^{\mathrm{III}}_{\gamma,\alpha}(r^*)\,\mathrm{d}r^* = 0$$
$$\int_{-\infty}^0 \int_0^{r^*} f^{\mathrm{III}}_{\gamma,\alpha}(t)\,\mathrm{d}t\,\mathrm{d}r^* = \int_{-\infty}^0 \int_0^{-r^*} (-f^{\mathrm{III}}_{\gamma,\alpha}(t))\,\mathrm{d}t\,\mathrm{d}r^*\,.$$

Remark 1.18 (Existence of constants). This is because the area that is lost under the graph of f^{III} after the replacement to faster decay is less than

$$\int_{-\infty}^{-b_{\gamma,\alpha}} (\frac{b}{r^*})^6\,\mathrm{d}r^* = \frac{1}{5}b_{\gamma,\alpha}$$

which can be accounted for by a shift of $b_{\gamma,\alpha}$ to $b' < \frac{2}{\gamma\alpha}$. The resulting lost of area under the graph of f^{II} is less than

$$\int_{-\infty}^{-b_{\gamma,\alpha}} \big(-\frac{1}{5}\frac{b_{\gamma,\alpha}^6}{r^{*5}}\big)\,\mathrm{d}r^* = \frac{1}{20}b_{\gamma,\alpha}^2$$

which can be compensated by an adjustment of the slope from 1 to $c \leq 1$ on the interval $-b' \leq r^* \leq -\frac{1}{\gamma\alpha}$ (while keeping the area fixed). The gain of area under the graph of f^{II} on the interval $(-b_{\gamma,\alpha}, -\frac{1}{\gamma\alpha})$ alone is enough,

$$\int_{\frac{1}{\gamma\alpha}}^{b_{\gamma,\alpha}} (1-c)(t-\frac{1}{\gamma\alpha})\,\mathrm{d}t = \frac{1}{20}b_{\gamma,\alpha}^2$$

if we choose $c = \frac{3}{8} \geq \frac{1}{4}$. The upper bound on b then follows easily to be $b \leq \frac{4}{\gamma\alpha}$.

The adjusted functions in comparison the the old are also sketched in figures 1.4 and 1.5. Note in particular that for $r^* \leq 0$

$$f^{\mathrm{II}}(r^*) \leq \frac{1}{\gamma\alpha} \tag{1.4.43}$$

$$f^{\mathrm{I}}(r^*) \leq f^{\mathrm{I}}(-r^*) \leq \frac{13}{12}\frac{1}{(\gamma\alpha)^2} \tag{1.4.44}$$

and for $r^* \leq -\frac{1}{\gamma\alpha}$

$$\frac{11}{12}\frac{1}{(\gamma\alpha)^3} \leq |f^0(r^*)| \leq f^0(-r^*) < \frac{3}{2}\frac{1}{(\gamma\alpha)^3}\,. \tag{1.4.45}$$

Remark 1.19. In order to deal with smooth functions one could use (e.g. at the level of second derivatives) a convolution with a Gaussian on the scale given by $\gamma\alpha$ (or finer). I.e. one could define

$$f''_{\gamma,\alpha}(r^*) = \frac{\gamma\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(\gamma\alpha)^2(r^*-t)^2} f^{\mathrm{II}}_{\gamma,\alpha}(t)\,\mathrm{d}t$$

and find $f'''_{\gamma,\alpha} = \frac{\mathrm{d}}{\mathrm{d}r^*}f''_{\gamma,\alpha}$ by differentiation, and ${f'}_{\gamma,\alpha}$ and $f_{\gamma,\alpha}$ by integration with the boundary values ${f'}_{\gamma,\alpha}(-\infty) = 0$, $f_{\gamma,\alpha}(0) = 0$ as above. However, I choose not to do so (as it does not give further insight) and work directly with the step-functions, i.e. define

$$f''_{\gamma,\alpha} = f^{\mathrm{III}}_{\gamma,\alpha}\,.$$

We are now in the position to prove a non-negativity property of the terms occuring in (1.4.21) which we will denote by ${}^0K^{X,1}$,

$$K^{X,1} = \frac{f'}{1-\frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + {}^0K^{X,1}\,. \tag{1.4.46}$$

Proposition 1.20 (Positivity of the current $J^{X_{\gamma,\alpha},1}$)**.** *For* $n \geq 3$,

$$X_{\gamma,\alpha} = f_{\gamma,\alpha}\frac{\partial}{\partial r^*} \qquad (\textit{where we choose } \gamma = 12)\,,$$

and $\phi \in \mathrm{H}^1(S)$ *satisfy*

$$\int_S {}^0K^{X_{\gamma,\alpha},1} \,\mathrm{d}\mu_\gamma \geq 0$$

provided

$$\pi_l \phi = 0 \qquad (0 \leq l < L)$$

for a fixed $L \geq (6\gamma n)^2$.

Proof. By Lemma 1.16

$$\begin{aligned}\int_S {}^0K^{X_{\gamma,\alpha},1} \,\mathrm{d}\mu_\gamma \geq \int_S \Big\{ & L(L+n-2)\frac{f_{\gamma,\alpha}}{r^3}\big(1 - \frac{nm}{r^{n-2}}\big) \\ & - \frac{1}{4}\frac{f'''_{\gamma,\alpha}}{1-\frac{2m}{r^{n-2}}} - \frac{n-1}{2}\frac{f''_{\gamma,\alpha}}{r} - \frac{n-1}{4}\Big[(n-3)+(n-1)\frac{2m}{r^{n-2}}\Big]\frac{f'_{\gamma,\alpha}}{r^2} \\ & - \frac{n-1}{4}\Big[(n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2 - n\frac{2m}{r^{n-2}} - (n-3)\Big]\frac{f_{\gamma,\alpha}}{r^3}\Big\}\phi^2 \,\mathrm{d}\mu_\gamma \,. \end{aligned} \tag{1.4.47}$$

We divide into the five regions

$$-\infty < -\frac{4}{\gamma\alpha} < -\frac{1}{\gamma\alpha} < \frac{1}{\gamma\alpha} < b_{\gamma,\alpha} < \infty \,.$$

Step 1. (near the photon sphere, $|r^*| < \frac{1}{\gamma\alpha}$)

Lemma 1.21. *In the region* $|r^*| < \frac{1}{\gamma\alpha}$ *the corresponding value of* r *lies in the interval*

$$\sqrt[n-2]{\delta n m} < r < \frac{n}{\alpha}$$

where $\delta = \max\{\frac{1}{3}, \frac{4}{3}\frac{2}{n}\}$.

Recalling the graphs of $f_{\gamma,\alpha}$ and its derivatives we then find in the region $|r^*| < \frac{1}{\gamma\alpha}$:

$$\begin{aligned}\int_S {}^0K^{X_{\gamma,\alpha},1} \,\mathrm{d}\mu_\gamma \geq & \\ \geq \int_S \Big\{ & \frac{1}{4} - \frac{1}{2}\frac{\alpha}{\delta^{\frac{1}{n-2}}}\frac{1}{\gamma\alpha} - \frac{1}{4}\frac{\alpha^2}{\delta^{\frac{2}{n-2}}}\frac{1}{n-1}\Big[(n-3)+\frac{1}{\delta}(n-1)\frac{2}{n}\Big]\frac{13}{12}\frac{1}{(\gamma\alpha)^2} \\ & - \frac{1}{4}\frac{\alpha^3}{\delta^{\frac{3}{n-2}}}\frac{2}{\delta n}\frac{3}{2}\frac{1}{(\gamma\alpha)^3}\Big\}\phi^2 \,\mathrm{d}\mu_\gamma \\ \geq & \int_S \Big\{\frac{1}{4} - \frac{1}{2}\frac{3}{\gamma} - \frac{3}{4}\frac{13}{12}\big(\frac{3}{\gamma}\big)^2 - \frac{3}{4}\big(\frac{3}{\gamma}\big)^3\Big\}\phi^2 \,\mathrm{d}\mu_\gamma \\ \geq & \int_S \frac{1}{4}\frac{1}{8}\,\phi^2 \,\mathrm{d}\mu_\gamma \end{aligned}$$

because $\gamma = 12$.

Proof of the Lemma. For the lower bound write

$$\frac{1}{\gamma\alpha} = \int_{r(r^*=-\frac{1}{\gamma\alpha})}^{(nm)^{\frac{1}{n-2}}} \frac{1}{1-\frac{2m}{r^{n-2}}}\,\mathrm{d}r \leq \frac{(nm)^{\frac{1}{n-2}} - r(r^*=-\frac{1}{\gamma\alpha})}{1-\frac{2m}{r(r^*=-\frac{1}{\gamma\alpha})^{n-2}}}\,.$$

If we assume the form $r(r^*=-\frac{1}{\gamma\alpha}) = \sqrt[n-2]{\delta nm}$ for some $\frac{2}{n} < \delta < 1$, then this inequality becomes

$$\frac{1}{\gamma(n-1)} \leq \frac{1-\sqrt[n-2]{\delta}}{1-\frac{2}{n}\frac{1}{\delta}}$$

which is satisfied for example with $\delta = \max\{\frac{1}{3}, \frac{4}{3}\frac{2}{n}\}$ $(\gamma \geq 2)$. The upper bound immediatly follows from

$$\frac{1}{\gamma\alpha} = \int_{(nm)^{\frac{1}{n-2}}}^{r(r^*=\frac{1}{\gamma\alpha})} \frac{1}{1-\frac{2m}{r^{n-2}}}\,\mathrm{d}r \geq r(r^*=\frac{1}{\gamma\alpha}) - \sqrt[n-2]{nm}\,. \qquad \square$$

Step 2. (in the intermediate region, $\frac{1}{\gamma\alpha} \leq r^* \leq \frac{5}{6}\frac{2}{\gamma\alpha}$)

Lemma 1.22. *In the region $\frac{1}{\gamma\alpha} \leq r^* \leq \frac{5}{6}\frac{2}{\gamma\alpha}$ we for the corresponding value of r,*

$$\Big(1+\frac{1}{3\gamma(n-1)}\Big)(nm)^{\frac{1}{n-2}} \leq r \leq \frac{n}{\alpha}\,.$$

Collecting the first and the last term, we find in this region,

$$\begin{aligned}
&\int_S {}^0K^{X_{\gamma,\alpha,1}}\,\mathrm{d}\mu_\gamma \geq \\
&\int_S \Big\{ \frac{\alpha^3}{n^3}\frac{11}{12}\frac{1}{(\gamma\alpha)^3}\Big[\big(1-\frac{nm}{r^{n-2}}\big)L(L+n-2) - \frac{n-1}{4}(n-1)^2\big(\frac{2}{n}\big)^2 + \frac{1}{4}(n-1)(n-3)\Big] \\
&\qquad - \frac{1}{4}\frac{1}{1-\frac{2}{n}} + \frac{\alpha}{3}\frac{1}{3}\frac{1}{\gamma\alpha} - \frac{1}{4}\alpha^2\frac{1}{n-1}\Big[(n-3)+(n-1)\frac{2}{n}\Big]\frac{1}{(\gamma\alpha)^2}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\
&\geq \int_S \Big\{\frac{11}{12}\frac{1}{(n\gamma)^3}\Big[\frac{1}{6\gamma(n-1)}L(L+n-2) - (n-1) + \frac{1}{4}(n-1)(n-3)\Big] \\
&\qquad - \frac{3}{4} - \frac{1}{4}\frac{1}{\gamma^2}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\
&\geq \int_S \Big\{\frac{11}{12}\frac{1}{6}\Big(\frac{(6\gamma n)^2}{\gamma^2 n^2}\Big)^2 - 1\Big\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \phi^2\,\mathrm{d}\mu_\gamma
\end{aligned}$$

because $L \geq (6\gamma n)^2$, where we have used that for $\frac{1}{\gamma\alpha} \leq r^* \leq \frac{5}{6}\frac{2}{\gamma\alpha}$,

$$1-\frac{nm}{r^{n-2}} \geq \frac{1}{6\gamma(n-1)}\,.$$

Proof of the Lemma. For the lower bound note

$$\frac{1}{\gamma\alpha} = \int_{(nm)^{\frac{1}{n-2}}}^{r(r^*=\frac{1}{\gamma\alpha})} \frac{1}{1-\frac{2m}{r^{n-2}}}\,\mathrm{d}r \leq \frac{1}{1-\frac{2}{n}}\big(r - \sqrt[n-2]{nm}\big)$$

to find that

$$r(r^* = \frac{1}{\gamma\alpha}) \geq \Big(n - 1 + \frac{1}{\gamma}\big(1 - \frac{2}{n}\big)\Big)\frac{1}{\alpha}$$
$$\geq \Big(1 + \frac{1}{3\gamma(n-1)}\Big)(nm)^{\frac{1}{n-2}}\,.$$

The upper bound again follows easily from

$$\frac{5}{6}\frac{2}{\gamma\alpha} \geq r(r^* = \frac{5}{6}\frac{2}{\gamma\alpha}) - \sqrt[n-2]{nm}$$

since $\gamma \geq 2$. □

Step 3. (in the asymptotics, $r^* \geq b_{\gamma,\alpha}$)

Given the general fact Prop. B.1 we here only need the weaker statement

Lemma 1.23. *For* $r^* \geq \frac{5}{6}\frac{2}{\gamma\alpha}$,

$$\frac{r}{r^*} \leq 2\gamma n\,.$$

Here

$$\int_S {}^0K^{X_{\gamma,\alpha,1}}\,\mathrm{d}\mu_\gamma \geq$$
$$\int_S \Big\{ \frac{1}{(\gamma\alpha)^3}\Big[\frac{1}{6\gamma(n-1)}L(L+n-2) - \frac{3}{2}(n-1)\Big]\frac{1}{r^3} - \frac{1}{4}\frac{1}{1-\frac{2}{n}}\big(\frac{5}{6}\frac{2}{\gamma\alpha r^*}\big)^6$$
$$- \frac{1}{4}\alpha^2\frac{1}{n-1}\Big[(n-3) + (n-1)\frac{2}{n}\Big]\frac{1}{20}\frac{\big(\frac{5}{6}\frac{2}{\gamma\alpha}\big)^6}{r^{*4}}\Big\}\phi^2\,\mathrm{d}\mu_\gamma$$
$$\geq \int_S \Big[\frac{L^2}{6\gamma^4 n} + \frac{L}{6\gamma^4 n}(n-2) - \frac{3}{2}\frac{1}{\gamma^3}(n-1) - \frac{3}{4}\big(\frac{r}{r^*}\big)^3\big(\frac{5}{6}\frac{2}{\gamma}\big)^6\frac{1}{(\alpha r^*)^3}$$
$$- \frac{1}{4}\frac{1}{20}\big(\frac{r}{r^*}\big)^3\big(\frac{5}{6}\frac{2}{\gamma}\big)^6\frac{1}{\alpha r^*}\Big]\frac{1}{(\alpha r)^3}\phi^2\,\mathrm{d}\mu_\gamma$$
$$\geq \int_S \Big[(6n)^3 - (4n)^3\Big]\frac{1}{(\alpha r)^3}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \big(\frac{n}{\alpha r}\big)^3\phi^2\,\mathrm{d}\mu_\gamma$$
$$\geq \int_S \Bigg(\frac{(nm)^{\frac{1}{n-2}}}{r}\Bigg)^3\phi^2\,\mathrm{d}\mu_\gamma$$

where in the third bound we have again used $L \geq (6\gamma n)^2$ and the Lemma.

Proof of the Lemma. Since

$$r^* \geq r - \sqrt[n-2]{nm}$$

we directly arrive at

$$\frac{r}{r^*} \leq 1 + \frac{(nm)^{\frac{1}{n-2}}}{\frac{5}{6}\frac{2}{\gamma\alpha}} \leq 2\gamma n\,.$$ □

Step 4. (in the intermediate region, $-\frac{4}{\gamma\alpha} \leq r^* \leq -\frac{1}{\gamma\alpha}$)

Recall $\gamma = 12$.

Lemma 1.24. *For $k \leq \gamma$, $k \in \mathbb{N}$,*

$$\Big(1 - \frac{2m}{r^{n-2}}\Big)^{-1}\big|_{r^*=-\frac{k}{\gamma\alpha}} \leq 17$$

and consequently

$$-\Big(1 - \frac{nm}{r^{n-2}}\Big)\big|_{r^*=-\frac{1}{\gamma\alpha}} \geq \frac{1}{20}\frac{1}{2\gamma}\,.$$

In the region $-\frac{4}{\gamma\alpha} \leq r^* \leq -\frac{1}{\gamma\alpha}$ we directly apply the Lemma to see that,

$$\begin{aligned}
&\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \geq \\
&\qquad\int_S\Big\{L(L+n-2)\frac{1}{(nm)^{\frac{3}{n-2}}}\frac{11}{12}\frac{1}{(\gamma\alpha)^3}\frac{1}{20}\frac{1}{2\gamma} - \frac{1}{4}17 - \frac{n-1}{2}\frac{1}{(2m)^{\frac{1}{n-2}}}\frac{1}{\gamma\alpha} \\
&- \frac{n-1}{2}\Big[(n-3)+(n-1)\Big]\frac{1}{(2m)^{\frac{2}{n-2}}}\frac{13}{12}\frac{1}{(\gamma\alpha)^2} - \frac{n-1}{4}\Big[n+(n-3)\Big]\frac{1}{(2m)^{\frac{3}{n-2}}}2\frac{1}{(\gamma\alpha)^3}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\
&\geq \int_S\Big\{\frac{1}{(3\gamma)^4}\frac{1}{(n-1)^3}L(L+n-2) \\
&\qquad - \frac{17}{4} - \frac{3}{2}\frac{1}{2\gamma} - \frac{13}{12}\frac{1}{\gamma^2}\big(\frac{n}{2}\big)^{\frac{2}{n-2}} - \frac{1}{n-1}\frac{1}{\gamma^3}\big(\frac{n}{2}\big)^{\frac{3}{n-2}}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \\
&\qquad \geq \int_S\Big\{2^4 n - \frac{23}{4}\Big\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S \phi^2\,\mathrm{d}\mu_\gamma
\end{aligned}$$

because $L \geq (6\gamma n)^2$.

Proof of the Lemma. According to Prop. B.4 we have for the value of r corresponding to $r^* = -\frac{k}{\gamma\alpha}$ (recall (B.8)):

$$\frac{k}{\gamma\alpha} \geq \frac{(2m)^{\frac{1}{n-2}}}{n-2}\log\frac{q_0\Big(\big(\frac{n}{2}\big)^{\frac{1}{n-2}}\Big)}{q_0\Big(\frac{r}{(2m)^{\frac{1}{n-2}}}\Big)}$$

Thus

$$\begin{aligned}
q_0\Big(\frac{r}{(2m)^{\frac{1}{n-2}}}\Big) &\geq e^{-\frac{k}{\gamma}\frac{n-2}{n-1}(\frac{n}{2})^{\frac{1}{n-2}}}q_0\Big(\big(\frac{n}{2}\big)^{\frac{1}{n-2}}\Big) \\
&\geq \frac{1}{\sqrt{e^3}}\frac{\big(\frac{n}{2}\big)^{\frac{1}{n-2}}-1}{\big(\frac{n}{2}\big)^{\frac{1}{n-2}}+1} \geq \frac{1}{5}\frac{2}{5}\frac{1}{n-2}\log\big(\frac{n}{2}\big)
\end{aligned}$$

Take $r(r^* = -\frac{k}{\gamma\alpha})$ in the form $r = \sqrt[n-2]{2m}(1+\beta)$ then we find for β

$$\beta \geq \big(\frac{2}{5}\big)^2\frac{1}{n-2}\log\big(\frac{n}{2}\big)$$

since $\beta \geq 0$. Therefore

$$\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)\big|_{r^*=-\frac{k}{\gamma\alpha}} \geq 1-\frac{1}{\bigl(1+(\frac{2}{5})^2\frac{1}{n-2}\log\frac{n}{2}\bigr)^{n-2}} \geq 1-e^{-\frac{3}{20}\log\frac{n}{2}} \geq 1-\bigl(\frac{2}{n}\bigr)^{\frac{3}{20}}$$

and

$$\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)^{-1}\big|_{r^*=-\frac{k}{\gamma\alpha}} \leq \frac{\bigl(\frac{n}{2}\bigr)^{\frac{3}{20}}}{\bigl(\frac{n}{2}\bigr)^{\frac{3}{20}}-1} \leq 17\,.$$

Consequently

$$\frac{1}{\gamma\alpha}=\int_{r(r^*=-\frac{1}{\gamma\alpha})}^{(nm)^{\frac{1}{n-2}}}\frac{1}{1-\frac{2m}{r^{n-2}}}\,\mathrm{d}r \leq \frac{1}{1-(\frac{2}{n})^{\frac{3}{20}}}\Bigl(\sqrt[n-2]{nm}-r\Bigr)$$

or

$$r(r^*=-\frac{1}{\gamma\alpha}) \leq \Bigl(1-\frac{1}{17}\frac{1}{\gamma(n-1)}\Bigr)(nm)^{\frac{1}{n-2}}$$

which yields the desired lower bound on

$$-\Bigl(1-\frac{nm}{r^{n-2}}\Bigr)\big|_{r^*=-\frac{1}{\gamma\alpha}} \geq -\Bigl(1-\frac{1}{\bigl(1-\frac{1}{17}\frac{1}{\gamma(n-1)}\bigr)^{n-2}}\Bigr) \geq -1+e^{\frac{1}{20}\frac{n-2}{\gamma(n-1)}} \geq \frac{1}{20}\frac{n-2}{\gamma(n-1)} \geq \frac{1}{20}\frac{1}{2\gamma}\,. \qquad \square$$

Step 5. (near the horizon, $r^* \leq -b$)

Finally we see for $r^* \leq -b$, recalling the adjustment to faster fall-off,

$$\begin{aligned}
&\int_S {}^0K^{X_{\gamma,\alpha},1}\,\mathrm{d}\mu_\gamma \geq \\
&\quad\int_S\Bigl\{L(L+n-2)\frac{1}{(nm)^{\frac{3}{n-2}}}\frac{11}{12}\frac{1}{(\gamma\alpha)^3}\frac{1}{20}\frac{1}{2\gamma}-\frac{1}{4}\bigl(1-\frac{2}{n}\bigr)^5-\frac{n-1}{2}\frac{1}{(2m)^{\frac{1}{n-2}}}\frac{1}{\gamma\alpha}\\
&\qquad-\frac{(n-1)^2}{4}\frac{1}{(2m)^{\frac{1}{n-2}}}\frac{1}{(\gamma\alpha)^2}-\frac{n-1}{4}\Bigl[n+(n-3)\Bigr]\frac{1}{(2m)^{\frac{3}{n-2}}}2\frac{1}{(\gamma\alpha)^3}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma\\
&\geq \int_S\Bigl\{\frac{1}{(3\gamma)^4}\frac{1}{(n-1)^3}L(L+n-2)\\
&\qquad-\frac{1}{4}-\frac{1}{2\gamma}\bigl(\frac{n}{2}\bigr)^{\frac{1}{n-2}}-\frac{1}{(2\gamma)^2}\bigl(\frac{n}{2}\bigr)^{\frac{2}{n-2}}-\frac{4}{n-1}\frac{1}{(2\gamma)^3}\bigl(\frac{n}{2}\bigr)^{\frac{3}{n-2}}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma\\
&\geq \int_S\Bigl\{2^4 n-\frac{5}{4}\Bigr\}\phi^2\,\mathrm{d}\mu_\gamma \geq \int_S\phi^2\,\mathrm{d}\mu_\gamma
\end{aligned}$$

where we have used that here

$$\frac{f'''}{1-\frac{2m}{r^{n-2}}}=\Bigl(1-\frac{2m}{r^{n-2}}\Bigr)^5\Bigl(\frac{b}{(2m)^{\frac{1}{n-2}}}\Bigr)^6 \leq \bigl(1-\frac{2}{n}\bigr)^5 \leq 1\,.$$

□

In fact, we have shown more, because all lower bounds in Step 1-5 are minorized by $\frac{1}{4}\frac{1}{8}\frac{(2m)^{\frac{3}{n-2}}}{r^3}$.

Corollary 1.25. *Let $\phi \in \mathrm{H}^2$ be a solution of the wave equation,*

$$\Box_g \phi = 0\,,$$

satisfying

$$\pi_l \phi = 0 \qquad (0 \leq l < L)$$

on the standard sphere $S = (\mathbb{S}^{n-1}, r^2 \overset{\circ}{\gamma}_{n-1})$ for a fixed $L \geq (6\gamma n)^2$. Then

$$\int_S \Big\{ \frac{1}{4}\frac{1}{8}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2 + \frac{1}{(20\gamma^2)^3}\frac{1}{(n-2)^2(n-1)^6}\Big(1-\frac{2m}{r^{n-2}}\Big)^5\frac{(2m)^{\frac{6}{n-2}}}{r^4}\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big\}\,\mathrm{d}\mu_\gamma \leq \int_S K^{X_{\gamma,\alpha,1}}\,\mathrm{d}\mu_\gamma$$

Proof. It remains to be shown that

$$\frac{1}{20}\frac{1}{(4\cdot 5(n-2))^2}\Big(1-\frac{2m}{r^{n-2}}\Big)^6\frac{b_{\gamma,\alpha}^6}{r^4} \leq f'_{\gamma,\alpha}\,. \qquad (*)$$

First

$$\int_{-\infty}^{r^*}\Big(1-\frac{2m}{r^{n-2}}\Big)^6\mathrm{d}r^* = \int_{(2m)^{\frac{1}{n-2}}}^{r}\Big(1-\frac{2m}{r^{n-2}}\Big)^5\mathrm{d}r$$

because $\mathrm{d}r^*/\,\mathrm{d}r = \big(1-\frac{2m}{r^{n-2}}\big)^{-1}$. Now choose $\sqrt[n-2]{2m} < r_0 < r$ so close to r as to satisfy

$$\frac{r-r_0}{r_0} = \frac{1}{2}\frac{1}{5(n-2)}\Big(1-\frac{2m}{r^{n-2}}\Big)$$

then by the mean value theorem

$$\begin{aligned}\int_{(2m)^{\frac{1}{n-2}}}^{r}\big(1-\frac{2m}{r^{n-2}}\big)^5\mathrm{d}r &\geq \big(1-\frac{2m}{r_0^{n-2}}\big)^5(r-r_0)\\ &\geq \big(1-\frac{2m}{r^{n-2}}\big)^5\Big[1-5(n-2)\frac{1}{1-\frac{2m}{r^{n-2}}}\frac{r-r_0}{r_0}\Big](r-r_0)\\ &\geq \frac{1}{4}\frac{1}{5(n-2)}\big(1-\frac{2m}{r^{n-2}}\big)^6(2m)^{\frac{1}{n-2}}\,.\end{aligned}$$

We conclude for $r^* \leq -b$,

$$\begin{aligned}f'_{\gamma,\alpha}(r^*) &= \int_{-\infty}^{r^*}\int_{-\infty}^{s^*}\big(1-\frac{2m}{r^{n-2}}\big)|_{r^*=s^*}\big(\frac{b}{(2m)^{\frac{1}{n-2}}}\big)^6\mathrm{d}s^*\,\mathrm{d}r^*\\ &\geq \frac{1}{4}\frac{1}{5(n-2)}\int_{-\infty}^{r^*}\big(1-\frac{2m}{r^{n-2}}\big)^6\mathrm{d}r^*\,\frac{b^6}{(2m)^{\frac{5}{n-2}}}\\ &\geq \big(\frac{1}{4}\frac{1}{5(n-2)}\big)^2\big(1-\frac{2m}{r^{n-2}}\big)^6\frac{b^6}{(2m)^{\frac{4}{n-2}}}\\ &\geq \frac{1}{(4\cdot 5(n-2))^2}\big(1-\frac{2m}{r^{n-2}}\big)^6\frac{b_{\gamma,\alpha}^6}{r^4}\,.\end{aligned}$$

Second for $r^* \geq 0$

$$\frac{1}{(4 \cdot 5(n-2))^2} \frac{1}{r^4} = \frac{1}{(4 \cdot 5(n-2))^2} \Big(\frac{r^*}{r}\Big)^4 \frac{1}{r^{*4}} \leq \frac{1}{r^{*4}}$$

Since, thirdly,

$$\frac{b_{\gamma,\alpha}}{r} \leq 1\,,$$

we have established $(*)$ for the regions $r^* \leq -b$, $r^* \geq b_{\gamma,\alpha}$, $-b \leq r^* \leq b_{\gamma,\alpha}$, respectively. ☐

Remark 1.26. This estimate of the zeroth order term ϕ^2 suffices to obtain an estimate for all derivatives using a commutation with the vectorfield T; see Proof of Prop. 1.11 in Section 1.4.4.

1.4.3 Low angular frequencies and commutation

While the current constructed in Section 1.4.2 required a decomposition into spherical harmonics we will now altogether avoid a recourse to the Fourier expansion on the sphere. The key to the positivity property was Poincaré's inequality which states in more generality (see e.g. [31]):

Lemma 1.27 (Poincaré inequality)**.** *Let (S, γ) be a compact Riemannian manifold, and $\phi \in \mathrm{H}^1(S)$ a function on S with mean value*

$$\bar{\phi} = \frac{1}{\int_S \mathrm{d}\mu_\gamma} \int_S \phi \,\mathrm{d}\mu_\gamma\,.$$

Then

$$\int_S (\phi - \bar{\phi})^2 \,\mathrm{d}\mu_\gamma \leq \frac{1}{\lambda_1(S)} \int_S |\nabla\!\!\!/ \phi|^2 \,\mathrm{d}\mu_\gamma$$

where $\lambda_1(S)$ is the first nonzero eigenvalue of the negative Laplacian, $-\Delta\!\!\!/ = -\nabla\!\!\!/^a \nabla\!\!\!/_a$, on S. ($\nabla\!\!\!/$ denotes covariant differentiation on S.)

Now let $(S, \gamma) = (\mathbb{S}^{n-1}, \overset{\circ}{\gamma}_{n-1})$ then we read off from (1.4.28) here

$$\lambda_1(\mathbb{S}^{n-1}) = n - 1\,. \tag{1.4.48}$$

Choose a basis of the Lie algebra of $\mathrm{SO}(n)$,

$$\Omega_i : i = 1, \ldots, \frac{n(n-1)}{2}\,, \tag{1.4.49}$$

and apply Lemma 1.27 to the functions $\Omega_i\phi$ of vanishing mean:

$$\int_{\mathbb{S}^{n-1}} \Omega_i\phi \;\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} = 0\,. \tag{1.4.50}$$

Then we obtain

$$\int_{\mathbb{S}^{n-1}} |\nabla\!\!\!/ \Omega_i\phi|^2 \,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \geq (n-1) \int_{\mathbb{S}^{n-1}} (\Omega_i\phi)^2 \,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \tag{1.4.51}$$

or on $(S,\gamma)=(S_r,\gamma_r)=(\mathbb{S}^{n-1},r^2\overset{\circ}{\gamma}_{n-1})$:

$$\int_{S_r}|\nabla\!\!\!/\,\Omega_i\phi|^2\,\mathrm{d}\mu_{\gamma_r}\geq\frac{n-1}{r^2}\int_{S_r}(\Omega_i\phi)^2\,\mathrm{d}\mu_{\gamma_r}\,. \tag{1.4.52}$$

Also note

$$\sum_{i=1}^{\frac{n(n-1)}{2}}(\Omega_i\phi)^2=r^2\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,. \tag{1.4.53}$$

Second modified current. Recall we are considering vectorfields of the form

$$X=f(r^*)\frac{\partial}{\partial r^*}\,.$$

Define

$$J^{X,2}_\mu=J^{X,1}_\mu+\frac{f'}{f(1-\frac{2m}{r^{n-2}})}\beta\,X_\mu\,\phi^2 \tag{1.4.54}$$

where $\beta=\beta(r^*)$ is a function to be chosen below. Then

$$K^{X,2}=K^{X,1}+\nabla^\mu\Big(\frac{f'}{f(1-\frac{2m}{r^{n-2}})}\beta\,X_\mu\,\phi^2\Big)$$

$$\begin{aligned}
&=\frac{f'}{1-\frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial r^*}+\beta\phi\big)^2+\frac{f}{r}\big(1-\frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\\
&\qquad-\frac{1}{4}\frac{f'''}{1-\frac{2m}{r^{n-2}}}\phi^2+\frac{f''}{1-\frac{2m}{r^{n-2}}}\Big[\beta-\frac{n-1}{2r}\big(1-\frac{2m}{r^{n-2}}\big)\Big]\phi^2\\
&-\frac{f'}{1-\frac{2m}{r^{n-2}}}\Big[\beta^2-\beta'-\frac{n-1}{r}\beta\big(1-\frac{2m}{r^{n-2}}\big)+\frac{n-1}{4r^2}\Big((n-3)+(n-1)\frac{2m}{r^{n-2}}\Big)\big(1-\frac{2m}{r^{n-2}}\big)\Big]\phi^2\\
&\qquad-\frac{n-1}{4}\Big[(n-1)^2\big(\frac{2m}{r^{n-2}}\big)-n\frac{2m}{r^{n-2}}-(n-3)\Big]\frac{f}{r^3}\phi^2
\end{aligned} \tag{1.4.55}$$

Now choose

$$\beta=\frac{n-1}{2r}\big(1-\frac{2m}{r^{n-2}}\big)+\delta \tag{1.4.56}$$

then

$$\beta^2-\beta'-\frac{n-1}{r}\beta\big(1-\frac{2m}{r^{n-2}}\big)+\frac{n-1}{4r^2}\Big((n-3)+(n-1)\frac{2m}{r^{n-2}}\Big)\big(1-\frac{2m}{r^{n-2}}\big)=-\delta'+\delta^2 \tag{1.4.57}$$

and

$$\begin{aligned}
K^{X,2}&=\frac{f'}{1-\frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial r^*}+\beta\phi\big)^2+\frac{f}{r}\big(1-\frac{nm}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\\
&\qquad-\frac{1}{1-\frac{2m}{r^{n-2}}}\Big\{\frac{1}{4}f'''-\delta f''+\big(\delta^2-\delta'\big)f'\Big\}\phi^2\\
&\qquad-\frac{n-1}{4}\Big[(n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2-n\frac{2m}{r^{n-2}}-(n-3)\Big]\frac{f}{r^3}\phi^2
\end{aligned} \tag{1.4.58}$$

Note. Suppose outside a compact interval $[-\alpha,\alpha]\subset\mathbb{R}$ f' is of the form $f'(r^*)=\frac{1}{r^{*2}}$ $(|r^*|>\alpha)$. Then we could choose $\delta=-\frac{1}{r^*}$ $(|r^*|>\alpha)$ so that $\delta f''=\frac{2}{r^{*4}}\geq 0$ and $-\delta'+\delta^2=0$.

Definition of the current $J^{(\alpha)}$. Let $\alpha > 0$ and introduce a shifted coordinate

$$x = r^* - \alpha - \sqrt{\alpha}\,. \tag{1.4.59}$$

The modification we choose is

$$\delta = -\frac{x}{\alpha^2 + x^2} \tag{1.4.60}$$

so that

$$-\delta' + \delta^2 = \frac{\alpha^2}{(\alpha^2 + x^2)^2}\,. \tag{1.4.61}$$

Let

$$f^a = -\frac{C}{\alpha^2 r^{n-1}} \qquad (C > 0) \tag{1.4.62}$$

and

$$(f^b)' = \frac{1}{\alpha^2 + x^2} \qquad (f^b)(r^*) = \int_0^{r^*} \frac{1}{\alpha^2 + x(t^*)^2}\,\mathrm{d}t^*\,. \tag{1.4.63}$$

Note that then

$$(f^a)' + (n-1)\frac{f^a}{r}\bigl(1 - \frac{2m}{r^{n-2}}\bigr) = 0 \tag{1.4.64}$$

and

$$\frac{1}{4}(f^b)''' - \delta(f^b)'' + (\delta^2 - \delta')(f^b)' = -\frac{1}{2}\frac{x^2 - \alpha^2}{(x^2 + \alpha^2)^3}\,. \tag{1.4.65}$$

Our current is built from the multiplier vectorfields

$$X^a = f^a \frac{\partial}{\partial r^*} \qquad X^b = f^b \frac{\partial}{\partial r^*} \tag{1.4.66}$$

by setting

$$J^{(\alpha)}_\mu(\phi) \doteq J^{X^a,0}_\mu(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^{X^b,2}_\mu(\Omega_i\phi) \tag{1.4.67}$$

and will be shown to have the property that its divergence

$$K^{(\alpha)} \doteq \nabla^\mu J^{(\alpha)}_\mu \tag{1.4.68}$$

is nonnegative upon integration over the spheres.

Proposition 1.28 (Positivity of the current $J^{(\alpha)}$)**.** *For $n \geq 3$, and $\phi \in \mathrm{H}^1(S)$*

$$\int_S K^{(\alpha)}\,\mathrm{d}\mu_\gamma \geq 0$$

provided α is chosen sufficently large, and $C(n,m,\alpha)$ set to be $(*)$ below.

Proof. In view of (1.4.64) and (1.4.65)

$$K^{(\alpha)} \geq \frac{(f^a)'}{1 - \frac{2m}{r^{n-2}}}\bigl(\frac{\partial\phi}{\partial r^*}\bigr)^2 + \frac{f^a}{r}\bigl(1 - \frac{nm}{r^{n-2}}\bigr)\bigl|\nabla\!\!\!/\,\phi\bigr|^2_{r^2\mathring{\gamma}_{n-1}}$$

$$+\sum_{i=1}^{\frac{n(n-1)}{2}} \frac{f^b}{r}\Big(1-\frac{nm}{r^{n-2}}\Big)\big|\nabla\Omega_i\phi\big|^2_{r^2\mathring{\gamma}_{n-1}} + \sum_{i=1}^{\frac{n(n-1)}{2}} F\,(\Omega_i\phi)^2$$
$$+\sum_{i=1}^{\frac{n(n-1)}{2}} \frac{n-1}{4r^3}\Big[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\Big(\frac{2m}{r^{n-2}}\Big)^2\Big]\,f^b\,(\Omega_i\phi)^2 \quad (1.4.69)$$

where

$$F \doteq \frac{1}{2}\frac{1}{1-\frac{2m}{r^{n-2}}}\frac{x^2-\alpha^2}{(x^2+\alpha^2)^3}\,. \quad (1.4.70)$$

So by Poincaré's inequality (1.4.52) and (1.4.53)

$$\int_S K^{(\alpha)}\,\mathrm{d}\mu_\gamma \geq \int_S\Big\{\frac{C(n-1)}{\alpha^2 r^n}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2 + \\ +\Big[(n-1)\frac{f^b}{r}\Big(1-\frac{nm}{r^{n-2}}\Big)+F\,r^2+\frac{1}{r}H\Big]\big|\nabla\phi\big|^2_{r^2\mathring{\gamma}_{n-1}}\Big\}\,\mathrm{d}\mu_\gamma \quad (1.4.71)$$

where

$$H \doteq \frac{n-1}{4}\Big[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\Big(\frac{2m}{r^{n-2}}\Big)^2\Big]f^b - \frac{C}{\alpha^2 r^{n-1}}\Big(1-\frac{nm}{r^{n-2}}\Big)\,. \quad (1.4.72)$$

Step 1. $H \geq 0$

It is equivalent to show that

$$\breve{H}(r) \doteq r^{n-1}\,H(r)\,\frac{r^{n-2}}{2m}$$

is nonnegative. We consider $\breve{H}$ to be a function of

$$\rho \doteq \frac{r^{n-2}}{2m}$$

so

$$\breve{H} = \frac{n-1}{4}(2mr)\Big[(n-3)\rho^2+n\rho-(n-1)^2\Big]f^b - \frac{C}{\alpha^2}\Big(\rho-\frac{n}{2}\Big)$$

Note that

$$r = \sqrt[n-2]{nm} \iff \rho = \frac{n}{2} \iff r^* = 0$$

and

$$\breve{H}\Big(\frac{n}{2}\Big) = 0\,.$$

Moreover we choose the constant C such that

$$\frac{\mathrm{d}\breve{H}}{\mathrm{d}\rho}\Big|_{\rho=\frac{n}{2}} = 0\,.$$

$$\frac{\mathrm{d}\breve{H}}{\mathrm{d}\rho} = \frac{n-1}{4}(2mr)\Big[\frac{(n-3)(2n-3)}{n-2}\rho+\frac{n-1}{n-2}n-\frac{(n-1)^2}{n-2}\frac{1}{\rho}\Big]f^b \\ +\frac{n-1}{4(n-2)}\frac{2mr^2}{\rho-1}\Big[(n-3)\rho^2+n\rho-(n-1)^2\Big](f^b)' - \frac{C}{\alpha^2}$$

where we have used

$$\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{r}{(n-2)\rho} \qquad \frac{\mathrm{d}r^*}{\mathrm{d}\rho} = \frac{1}{\rho-1}\frac{r}{n-2}\,.$$

Hence we choose

$$C = \frac{(n-1)^2}{4(n-2)}\frac{(\frac{n}{2})^2-(n-1)}{\frac{n}{2}-1}\,2m\,(nm)^{\frac{2}{n-2}}\,\frac{\alpha^2}{\alpha^2+(\alpha+\sqrt{\alpha})^2}\,. \qquad (*)$$

Note that then also

$$\frac{\mathrm{d}H}{\mathrm{d}r}\Big|_{r=\sqrt[n-2]{nm}} = 0\,.$$

Now returning to the expression for $\breve{H}$ let us denote by $1 \leq \rho_0 \leq \frac{n}{2}$ the value of ρ for which

$$(n-3)\rho_0 + n - (n-1)^2\frac{1}{\rho_0} = 0\,,$$

i.e.

$$\rho_0 = \frac{2(n-1)^2}{n+\sqrt{n^2+4(n-1)^2(n-3)}}\,.$$

We divide into the four regions

$$1 < \rho_0 < \frac{n}{2} < \rho^* < \infty$$

where ρ^* is to be chosen large enough below.

Step 1a. (near the horizon, $1 \leq \rho \leq \rho_0$) Clearly $\breve{H} \geq 0$ termwise, because $f^b \leq 0$.

Step 1b. (near the photon sphere, $\rho_0 \leq \rho \leq \frac{n}{2}$) We show $H = H(r)$ is convex on $r_0 \leq r \leq \sqrt[n-2]{nm}$ where

$$r_0 = \sqrt[n-2]{\frac{4(n-1)^2 m}{n+\sqrt{n^2+4(n-1)^2(n-3)}}}\,.$$

Differentiating twice yields

$$\begin{aligned}\frac{\mathrm{d}^2H}{\mathrm{d}r^2} = \frac{n-1}{4}\frac{1}{\left(1-\frac{2m}{r^{n-2}}\right)^2}(f^b)''\Big[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2\Big] \\ +\frac{n-1}{2}\frac{1}{1-\frac{2m}{r^{n-2}}}(f^b)'(n-2)\Big[2(n-1)^2\frac{2m}{r^{n-2}}-n\Big]\frac{2m}{r^{n-1}} \\ -\frac{n-1}{4}\frac{1}{\left(1-\frac{2m}{r^{n-2}}\right)^2}(f^b)'(n-2)\Big[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2\Big]\frac{2m}{r^{n-1}} \\ +\frac{n-1}{4}(f^b)\Big[(n-2)(n-1)n\frac{2m}{r^n}-2(2n-3)(n-2)(n-1)^2\big(\frac{2m}{r^{n-1}}\big)^2\Big] \\ -\frac{(n-1)nC}{\alpha^{n-1}r^{n+1}}\big(1-\frac{nm}{r^{n-2}}\big)+3\frac{(n-1)(n-2)C}{\alpha^{n-1}r^n}\frac{nm}{r^{n-1}}\end{aligned}$$

Since $(f^b)'' \geq 0$ we further have in this region the bound

$$\frac{\mathrm{d}^2 H}{\mathrm{d}r^2} \geq \frac{n-1}{2}\frac{1}{1-\frac{2m}{r^{n-2}}}\times \\ \times\Big[2(n-1)^2\frac{2m}{r^{n-2}} - n - \frac{1}{2}\frac{1}{1-\frac{2m}{r_0^{n-2}}}\Big((n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\big(\frac{2m}{r^{n-2}}\big)^2\Big)\Big]\times \\ \times\frac{2m}{r^{n-1}}(n-2)(f^b)' \\ +\frac{n-1}{4}\frac{2m}{r^{n-2}}\Big[1-\frac{2(2n-3)(n-1)}{n}\big(\frac{2m}{r^{n-2}}\big)\Big]\frac{(f^b)}{r^2}$$

Since for $n \geq 3$

$$2(n-1)^2\frac{2}{n} - n \\ -\frac{1}{2}\frac{2(n-1)^2}{2(n-1)^2-n-\sqrt{n^2+4(n-1)^2(n-3)}}\Big((n-3)+2-\big(2\frac{n-1}{n}\big)^2\Big) \\ \geq 1$$

and

$$1-\frac{2(2n-3)(n-1)}{n}\frac{2}{n} \leq -1$$

we finally obtain in this region

$$\frac{\mathrm{d}^2 H}{\mathrm{d}r^2} \geq \frac{(n-1)(n-2)}{2r}\frac{1}{\rho-1}(f^b)' > 0\,.$$

Step 1c. (in the intermediate region, $\frac{n}{2} \leq \rho \leq \rho^*$) We show $\breve{H} = \breve{H}(\rho)$ is convex on $\frac{n}{2} \leq \rho \leq \rho^*$ for $r^*(\rho = \rho^*) \leq \alpha$.

$$\frac{\mathrm{d}^2 \breve{H}}{\mathrm{d}\rho^2} = \frac{(n-1)^2}{4(n-2)^2}\frac{2mr}{\rho^2}\Big[(n-3)(2n-3)\rho^2+n\rho+(n-3)(n-1)\Big](f^b) \\ +\frac{(n-1)^2}{4(n-2)^2}\frac{2mr^2}{(\rho-1)^2}\times \\ \times\Big[3(n-3)\rho^2-3(n-5)\rho+(n-1)(n-5)-n\frac{2n-1}{n-1}+3(n-1)\frac{1}{\rho}\Big](f^b)' \\ +\frac{n-1}{4(n-2)^2}\frac{2mr^3}{(\rho-1)^2}\Big[(n-3)\rho^2+n\rho-(n-1)^2\Big](f^b)''$$

Since for $\rho \geq \frac{n}{2}$, and $n \geq 3$,

$$3(n-3)\rho(\rho-1)+6\rho+(n-1)(n-5)-n\frac{2n-1}{n-1}+3(n-1)\frac{1}{\rho} \geq 1$$

and

$$(n-3)\rho^2+n\rho-(n-1)^2 \geq 0$$

we have

$$\frac{\mathrm{d}^2 \breve{H}}{\mathrm{d}\rho^2} \geq \frac{(n-1)^2}{4(n-2)^2}\frac{2mr^2}{(\rho-1)^2}(f^b)' > 0$$

because $(f^b) \geq 0$ for $r^* \geq 0$, and $(f^b)'' \geq 0$ for $x \leq 0$.

Step 1d. (in the asymptotics, $\rho \geq \rho^*$) We show directly $H(r) > 0$ for $r^* \geq R^* \doteq r^*(\rho = \rho^*)$ and ρ^* chosen large enough. Let $r^* \geq R^*$, $R^* \leq \alpha$ then

$$f^b \geq \int_0^{R^*} (f^b)' \, \mathrm{d}r^* = \frac{1}{\alpha} \int_{-\left(1+\frac{1}{\sqrt{\alpha}}\right)}^{\frac{R^*-\alpha-\sqrt{\alpha}}{\alpha}} \frac{1}{1+t^{*2}} \, \mathrm{d}t^* \geq \frac{R^*}{5\alpha^2} \tag{1.4.73}$$

provided $\alpha \geq 1$, and of course

$$f^b \leq \frac{1}{\alpha} \arctan t^* |^0_{-(1+\frac{1}{\sqrt{\alpha}})} \leq \frac{\pi}{2\alpha} \,.$$

Thus

$$\begin{aligned} H = \ & \frac{(n-1)(n-3)}{4} + \Big[\frac{(n-1)n}{4} f^b - \frac{C}{\alpha^2 2m} \frac{1}{r}\Big] \frac{2m}{r^{n-2}} \\ & - \Big[\frac{(n-1)^3}{4} f^b - \frac{Cn}{\alpha^2 4m} \frac{1}{r}\Big] \big(\frac{2m}{r^{n-2}}\big)^2 \\ \geq \ & \frac{1}{\alpha^2} \Big[\frac{(n-1)n}{4} \frac{R^*}{5} - \frac{C}{2m} \frac{1}{r}\Big] \frac{2m}{r^{n-2}} \\ & - \frac{(n-1)^3}{4} \frac{\pi}{2\alpha} \big(\frac{2m}{r^{n-2}}\big)^2 \\ > \ & 0 \end{aligned}$$

for R^* (and consequently α) chosen large enough.

Step 2. (1.4.74)

Since $\big(1 - \frac{nm}{r^{n-2}}\big) f^b \geq 0$ and $F \geq 0$ for $|x| \geq \alpha$ we need to show

$$(n-1)(f^b)\big(1 - \frac{nm}{r^{n-2}}\big) + F\, r^3 \geq 0 \tag{1.4.74}$$

for

$$-\alpha \leq x \leq \alpha \iff \sqrt{\alpha} \leq r^* \leq \sqrt{\alpha} + 2\alpha \,.$$

In this whole region, in view of Prop. B.1,

$$\lim_{\alpha \to \infty} \frac{r^*}{r} = 1$$

$$\lim_{\alpha \to \infty} \big(1 - \frac{2m}{r^{n-2}}\big) = \lim_{\alpha \to \infty} \big(1 - \frac{nm}{r^{n-2}}\big) = 1 \,.$$

$\boxed{n \geq 4:}$ Since

$$f^b(r^*) \geq \int_{\sqrt{\alpha}}^{r^*} \frac{1}{\alpha^2 + x^2} \, \mathrm{d}r^* \geq \frac{x + \alpha}{2\alpha^2} \tag{1.4.75}$$

it suffices to show

$$(n-1) \frac{x+\alpha}{2\alpha^2} + \frac{1}{2} \frac{x^2 - \alpha^2}{(x^2+\alpha^2)^3} r^3 \geq 0 \tag{1.4.76}$$

which is implied by

$$\frac{\alpha - x}{n-1} \frac{(x + \alpha + \sqrt{\alpha})^3}{(x^2 + \alpha^2)^2} \leq 1 \,. \tag{1.4.77}$$

For $-\alpha \leq x \leq 0$

$$(x+\alpha+\sqrt{\alpha})^3 \leq \alpha^3(1+\frac{1}{\sqrt{\alpha}})^3 \leq \frac{4}{3}\alpha^3$$

for α large enough, thus

$$\frac{\alpha - x}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \leq \frac{1}{n-1}\frac{2\alpha}{\alpha^4}\frac{4}{3}\alpha^3 \leq \frac{8}{9}\,. \tag{1.4.78}$$

For $0 \leq x \leq \alpha$ we have to show

$$\frac{\alpha}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \leq 1\,.$$

Since

$$(x+\alpha+\sqrt{\alpha})^3 \leq 2^{\frac{3}{2}}(1+\frac{1}{\sqrt{\alpha}})^3(x^2+\alpha^2)^{\frac{3}{2}}$$

we have for α large enough

$$\frac{\alpha}{n-1}\frac{(x+\alpha+\sqrt{\alpha})^3}{(x^2+\alpha^2)^2} \leq \frac{\alpha}{n-1}\frac{2^{\frac{3}{2}}(1+\frac{1}{\sqrt{\alpha}})^3}{(x^2+\alpha^2)^{\frac{1}{2}}} \leq \frac{2^{\frac{3}{2}}}{3}(1+\frac{1}{\sqrt{\alpha}})^3 < 1\,. \tag{1.4.79}$$

$\boxed{n=3:}$ We see that (1.4.78) and (1.4.79) fail in the case $n=3$, as a consequence of which also (1.4.77) fails to hold. In the case $n=3$, we have to use a better approximation of (1.4.75), see [21] for details. Note also that in view of (1.4.77) the positivity property (1.4.74) is "easily" satisfied for large values of n, which indicates that there may be yet another simplified proof in higher dimensions. □

Given the *strict* inequalities proven in Step 2 of the proof of Prop. 1.28 for α chosen large enough we can keep a fraction of the manifestly nonnegative $|\nabla\!\!\!/\Omega_i\phi|^2$ term in (1.4.69). Furthermore we have obtained control on the $|\nabla\!\!\!/\phi|^2$ term from (1.4.71).

Corollary 1.29. *Let $\phi \in \mathrm{H}^2(S)$ be a solution of the wave equation* (1.1.1)*. Then there exists a constant $C(n,m)$ and a current K such that*

$$\int_S \Big\{ \frac{1}{r^n}(\frac{\partial\phi}{\partial r^*})^2 + \frac{1}{r^{n+1}}(\frac{\partial\phi}{\partial t})^2 + r\big(1-\frac{nm}{r^{n-2}}\big)^2 |\nabla\!\!\!/^2\phi|^2_{r^2\overset{\circ}{\gamma}_{n-1}} + \frac{r^2}{(1-\frac{2m}{r^{n-2}})(1+r^{*2})^2}|\nabla\!\!\!/\phi|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \Big\}\,\mathrm{d}\mu_\gamma \leq C(n,m)\int_S K\,\mathrm{d}\mu_\gamma\,. \tag{1.4.80}$$

Proof. Set $K = K^{(\alpha)} + K^{\mathrm{aux}}$ and choose α large enough.
Here we retrieve the time derivatives with the auxiliary current

$$K^{\mathrm{aux}} = \nabla^\mu J^{\mathrm{aux}}_\mu; \quad J^{\mathrm{aux}} = J^{X^{\mathrm{aux}},0}; \quad X^{\mathrm{aux}} = f^{\mathrm{aux}}\frac{\partial}{\partial r^*}\,,$$

where $f^{\mathrm{aux}} = -\frac{1}{r^n}$ satisfies

$$(f^{\mathrm{aux}})' + (n-1)\frac{f^{\mathrm{aux}}}{r}\big(1-\frac{2m}{r^{n-2}}\big) = \frac{1}{r^{n+1}}\big(1-\frac{2m}{r^{n-2}}\big)\,;$$

for in view of (1.4.9)

$$\frac{1}{r^{n+1}}\big(\frac{\partial\phi}{\partial t}\big)^2 \leq 2\,K^{\mathrm{aux}} + 3\,\frac{1}{r^{n+1}}\big|\nabla\!\!\!/\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,. \qquad \square$$

1.4.4 Boundary terms

In this section we first prove Prop. 1.11 and then a refinement thereof for finite regions, which requires to estimate the boundary terms of the currents introduced in Section 1.4.2 and 1.4.3.

Proof of Prop. 1.11

We can now combine our ealier results Cor. 1.25 and Cor. 1.29 to prove the *integrated local energy decay estimate* (1.4.4); note that there is no restriction on the spherical harmonic number, and that no commutation with angular momentum operators is required.

Proof of Prop. 1.11. Write

$$\phi = \pi_{<L}\phi + \pi_{\geq L}\phi \tag{1.4.81}$$

with

$$\pi_{<L} = \sum_{l=0}^{L-1}\pi_l\phi \qquad \pi_{\geq L} = \sum_{l=L}^{\infty}\pi_l\phi \tag{1.4.82}$$

where $L = (6\gamma n)^2$ is fixed (recall here $\gamma = 12$ from Section 1.4.2).

Step 1. (High spherical harmonics)

By Cor. 1.25

$$\int_{\mathcal{R}(t_0,t_1,u_1^*,v_1^*)} \frac{1}{4}\frac{1}{8}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\pi_{\geq L}\phi\big)^2 \leq \int_{\mathcal{R}(t_0,t_1,u_1^*,v_1^*)} K^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi)\,. \tag{1.4.83}$$

It remains to estimate the boundary terms of the current $J^{X_{\gamma,\alpha},1}$, and to use this estimate to recover all derivatives using a commutation with the Killing vectorfield T.

Step 1a. (Boundary terms) We may assume $|r^*_{0,1}| \geq \frac{4}{\gamma\alpha}$, $r_{0,1}$ entering the definition (1.4.3). Recalling the properties of $f_{\gamma,\alpha}$ away from the photon sphere we find

$$\begin{aligned}
|(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*})| \leq |(J^{X_{\gamma,\alpha}}(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*})| \\
+ \frac{1}{2}\Big|f_{\gamma,\alpha}{}' + (n-1)\frac{f_{\gamma,\alpha}}{r}(1-\frac{2m}{r^{n-2}})\Big|\big(\pi_{\geq L}\phi\big)\big(\frac{\partial \pi_{\geq L}}{\partial v^*}\big) \\
+ \frac{1}{4}\Big|\Big(f_{\gamma,\alpha}{}' + (n-1)\frac{f_{\gamma,\alpha}}{r}(1-\frac{2m}{r^{n-2}})\Big)'\Big|\big(\pi_{\geq L}\phi\big)^2 \\
\leq \frac{n+1}{(\gamma\alpha)^3}\Big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial v^*}\Big) + \frac{1}{(\gamma\alpha)^6}\frac{1}{|r^*|^4}\big(\frac{\partial \pi_{\geq L}\phi}{\partial v^*}\big)^2 \\
+ \frac{1}{(\gamma\alpha)^6}\frac{1}{|r^*|^4}\Big[1+\frac{1}{|r^*|}\Big]\big(\pi_{\geq L}\phi\big)^2 \\
+ \frac{n-1}{(\gamma\alpha)^3}\Big[n + \frac{4}{(\gamma\alpha)^6}\frac{r}{|r^*|^4}\Big]\frac{1}{2r^2}\big(1-\frac{2m}{r^{n-2}}\big)\big(\pi_{\geq L}\phi\big)^2
\end{aligned}$$

and by Lemma 1.16

$$\int_{S_r} \frac{1}{2}\frac{1}{r^2}\big(1-\frac{2m}{r^{n-2}}\big)\Big(\pi_{\geq L}\phi\Big)^2 \leq \frac{1}{(6\gamma n)^4}\int_{S_r}\Big(J^T, \frac{\partial}{\partial v^*}\Big)\,;$$

similarly for

$$\Big|\Big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial u^*}\Big)\Big|.$$

Since also by Lemma B.11 and Lemma 1.16,

$$\begin{aligned}
\int_{\frac{1}{2}(t_0-r_0^*)}^{u_1^*} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\frac{1}{r^{*4}}(\pi_{\geq L}\phi)^2\big|_{v^*=\frac{1}{2}(t_0+r_0^*)} \leq \\
\leq \frac{8}{|r_0^*|^4}\frac{(1+|r_0^*|^2)^2}{|r_0^*|^2}\int_{\frac{1}{2}(t_0-r_0^*)}^{\infty}\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\big(\frac{\partial \pi_{\geq L}\phi}{\partial u^*}\big)^2\,\mathrm{d}u^* \\
+ 2\pi\frac{1+|r_0^*|^2}{|r_0^*|^4}\Big[1+\frac{(nm)^{\frac{2}{n-2}}}{(6\gamma n)^4}\Big]\int_{\frac{1}{2}(t_0-r_0^*)}^{\frac{1}{2}(t_0-r_0^*)+1}\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\Big\{\big|\nabla\!\!\!/\,\pi_{\geq L}\phi\big|^2 + \big(\frac{\partial \pi_{\geq L}\phi}{\partial u^*}\big)^2\Big\}\,\mathrm{d}u^*
\end{aligned}$$

there is a constant $C(n,m)$ (recall $\gamma = 12$, $\alpha = (n-1)/(nm)^{\frac{1}{n-2}}$) such that

$$\begin{aligned}
\int_{\frac{1}{2}(t_0-r_0^*)}^{u_1^*} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\Big|\big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial u^*}\big)\Big| \leq \\
\leq C(n,m)\int_{\frac{1}{2}(t_0-r_0^*)}^{\infty} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\Big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial u^*}\Big)\Big|_{v^*=\frac{1}{2}(t_0+r_0^*)}.
\end{aligned}$$

To establish

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}}\Big|\big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\big)\Big|\, r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \leq \\
\leq C(n,m)\int_{\mathbb{S}^{n-1}}\big(J^T(\pi_{\geq L}\phi), \frac{\partial}{\partial t}\big)\, r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}
\end{aligned}$$

note that

$$\begin{aligned}\Big|\big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi),\frac{\partial}{\partial t}\big)\Big| \leq \Big|\big(J^{X_{\gamma,\alpha}}(\pi_{\geq L}\phi),\frac{\partial}{\partial t}\big)\Big| \\ +\frac{1}{2}\Big|f_{\gamma,\alpha}{}'+(n-1)\frac{f_{\gamma,\alpha}}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big|(\pi_{\geq L}\phi)\big(\frac{\partial\pi_{\geq L}\phi}{\partial t}\big) \leq \\ \leq |f_{\gamma,\alpha}|\,|T(\pi_{\geq L}\phi)(\frac{\partial}{\partial r^*},\frac{\partial}{\partial t})| \\ +\frac{1}{2}\Big[\frac{1}{2}r^2|f_{\gamma,\alpha}{}'|+\frac{3}{2}\frac{n-1}{2}\frac{1}{(\gamma\alpha)^3}\big(1-\frac{2m}{r^{n-2}}\big)^2\Big]\frac{1}{r^2}(\pi_{\geq L}\phi)^2 \\ +\frac{1}{2}\frac{1}{(\gamma\alpha)^2}\Big[1+\frac{3}{2}\frac{n-1}{2}\frac{1}{\gamma\alpha}\Big]\big(\frac{\partial\pi_{\geq L}\phi}{\partial t}\big)^2\end{aligned}$$

and by Lemma 1.16

$$\int_{S_r}\frac{1}{2}\frac{1}{r^2}(\pi_{\geq L}\phi)^2\,\mathrm{d}\mu_\gamma \leq \frac{1}{(6\gamma n)^4}\big(1-\frac{2m}{r_0^{n-2}}\big)^{-1}\int_{S_r}\big(J^T(\pi_{\geq L}\phi),\frac{\partial}{\partial t}\big)\,\mathrm{d}\mu_\gamma$$

which suffices in view of the properties of $f_{\gamma,\alpha}$ in particular that there is a constant $r^2|f_{\gamma,\alpha}{}'|\leq C(n,m)$. For the boundary term

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\Big|\big(J^{X_{\gamma,\alpha},1}(\pi_{\geq L}\phi),\frac{\partial}{\partial v^*}\big)\Big|_{u^*=u_1^*}r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \leq \\ \leq \int_{\mathbb{S}^{n-1}}\Big\{\frac{1}{(\gamma\alpha)^3}\Big[n+1+\frac{1}{2}\frac{1}{(\gamma\alpha)^2}+\frac{n-1}{(6\gamma n)^4}\big(n+\frac{4(nm)^{\frac{1}{n-2}}}{(\gamma\alpha)^2}\big)\Big]\big(J^T(\pi_{\geq L}\phi),\frac{\partial}{\partial v^*}\big) \\ +\frac{1}{(\gamma\alpha)^6}\frac{1}{|r^*|^4}\big(1+\frac{\gamma\alpha}{4}\big)(\pi_{\geq L}\phi)^2\Big\}\,r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\end{aligned}$$

we find (using the boundedness of ϕ on the horizon, see Section 1.5.1) in the limit $u_1^*\to\infty$ a constant $C(n,m)$ such that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\Big|\big(J^{X_{\gamma\alpha},1}(\pi_{\geq L}\phi),\frac{\partial}{\partial v^*}\big)\Big|\,r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}|_{u^*=\infty} \leq \\ \leq C(n,m)\int_{\mathbb{S}^{n-1}}\big(J^T(\pi_{\geq L}\phi),\frac{\partial}{\partial v^*}\big)\,r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}|_{u^*=\infty}\,.\end{aligned}$$

We conclude that there is a constant $C(n,m)$ such that

$$\int_{\mathcal{R}^{\infty}_{\tau_0,r_1}(t_0)}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\pi_{\geq L}\phi\big)^2 \leq C(n,m)\int_{\Sigma_{\tau_0}}\big(J^T(\pi_{\geq L}\phi),n\big)\,; \tag{1.4.84}$$

where $\tau_0=\frac{1}{2}(t_0-r_1^*)$ because

$$\Box_g(\pi_{\geq L}\phi)=0 \qquad K^T(\pi_{\geq L}\phi)=0\,. \tag{1.4.85}$$

Step 1b. (Commutation with T) Since

$$\Box_g\big(T\cdot\pi_{\geq L}\phi\big)=0 \tag{1.4.86}$$

we also have

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial \pi_{\geq L}\phi}{\partial t}\big)^2 \leq C(n,m)\int_{\Sigma_{\tau_0}} \big(J^T(T\cdot\pi_{\geq L}\phi),n\big)\,. \tag{1.4.87}$$

This is enough to control the remaining derivatives, too; for the auxiliary current (B.10) yields

$$K^{\mathrm{aux}} = \phi\,(\partial^\mu h)(\partial_\mu\phi) + h\,\partial^\alpha\phi\,\partial_\alpha\phi \tag{1.4.88}$$

which upon choosing

$$h = \big(1-\frac{2m}{r^{n-2}}\big)\frac{(2m)^{\frac{3}{n-2}}}{r^3} \tag{1.4.89}$$

presents us with

$$K^{\mathrm{aux}} = \phi\frac{\partial h}{\partial r}\frac{\partial\phi}{\partial r^*} - \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\phi}{\partial t}\big)^2 + \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,. \tag{1.4.90}$$

Using Cauchy's inequality for the first term, namely

$$\begin{aligned}\phi\frac{\partial h}{\partial r}\frac{\partial\phi}{\partial r^*} &= (n-2)\phi\frac{2m}{r^{n-1}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\frac{\partial\phi}{\partial r^*} - 3\big(1-\frac{2m}{r^{n-2}}\big)\phi\frac{(2m)^{\frac{3}{n-2}}}{r^4}\frac{\partial\phi}{\partial r^*} \geq \\ &\geq -\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2 - \big(\frac{n-2}{r}\big)^2\big(\frac{2m}{r^{n-2}}\big)^2\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2 - \big(\frac{3}{r}\big)^2\big(1-\frac{2m}{r^{n-2}}\big)^2\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2 \\ &\geq -\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\phi}{\partial r^*}\big)^2 - 2\frac{n^2}{(2m)^{\frac{2}{n-2}}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2\,,\end{aligned} \tag{1.4.91}$$

we can bound

$$K^{\mathrm{aux}} \geq \frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} - \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\phi}{\partial t}\big)^2 - 2\frac{n^2}{(2m)^{\frac{2}{n-2}}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\phi^2\,. \tag{1.4.92}$$

Therefore

$$\begin{aligned}&\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\Big\{\frac{1}{2}\frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\pi_{\geq L}\phi}{\partial r^*}\big)^2 + \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\pi_{\geq L}\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\} \leq \\ &\leq \int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\Big\{K^{\mathrm{aux}}(\pi_{\geq L}\phi) + \frac{(2m)^{\frac{3}{n-2}}}{r^3}\big(\frac{\partial\pi_{\geq L}\phi}{\partial t}\big)^2 + 2\frac{n^2}{(2m)^{\frac{2}{n-2}}}\frac{(2m)^{\frac{3}{n-2}}}{r^3}(\pi_{\geq L}\phi)^2\Big\}\,.\end{aligned} \tag{1.4.93}$$

The boundary terms are controlled using Prop. B.12:

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} K^{\mathrm{aux}}(\pi_{\geq L}\phi) \leq C(n,m)\int_{\Sigma_{\tau_0}}\big(J^T(\pi_{\geq L}\phi),n\big)\,. \tag{1.4.94}$$

Hence

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} \frac{(2m)^{\frac{3}{n-2}}}{r^3}\Big\{\Big(\frac{\partial\pi_{\geq L}\phi}{\partial r^*}\Big)^2+\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\,\pi_{\geq L}\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\}\leq \\ \leq C(n,m)\int_{\Sigma_{\tau_0}}\Big(J^T(\pi_{\geq L}\phi)+J^T(T\cdot\pi_{\geq L}\phi),n\Big). \quad (1.4.95)$$

Step 2. (Low spherical harmonics)

Now recall the $J^{(\alpha)}$ current (1.4.67); we will show in a first step that

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)} K^{(\alpha)}(\phi)\leq C(n,m)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),n\Big). \quad (1.4.96)$$

Then in particular by Cor. 1.29

$$\int_{\mathcal{R}^{\infty}_{r_0,r_1}(t_0)}\Big\{\frac{1}{r^n}\Big(\frac{\partial\pi_{<L}\phi}{\partial r^*}\Big)^2+\frac{1}{r^{n+1}}\Big(\frac{\partial\pi_{<L}\phi}{\partial t}\Big)^2 \\ +\frac{r^2}{(1-\frac{2m}{r^{n-2}})(1+|r^*|^2)^2}\big|\nabla\!\!\!/\,\pi_{<L}\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\}\leq \\ \leq C(n,m)\int_{\Sigma_{\tau_0}}\Big(J^T(\pi_{<L}\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\cdot\pi_{<L}\phi),n\Big). \quad (1.4.97)$$

But in a second step we will show that in fact there exists a constant $C(n)$ such that

$$\int_{\Sigma_{\tau_0}}\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(J^T(\Omega_i\cdot\pi_{<L}\phi),n\Big)\leq C(n)\int_{\Sigma_{\tau_0}}\Big(J^T(\pi_{<L}\phi),n\Big). \quad (1.4.98)$$

Step 2a. (Boundary Terms) The energy identity for $J^{(\alpha)}$ on the domain (1.4.1) implies more explicitly:

$$\int_{\mathcal{R}(t_0,t_1,u_1^*,v_1^*)}K^{(\alpha)}\leq\int_{\frac12(t_0+r_0^*)}^{\frac12(t_1+r_0^*)}\int_{\mathbb{S}^{n-1}}\big|(J^{(\alpha)},\frac{\partial}{\partial v^*})\big|\,r^{n-1}\big|_{u^*=u_1^*}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ +\int_{\frac12(t_1-r_0^*)}^{u_1^*}\int_{\mathbb{S}^{n-1}}\big|(J^{(\alpha)},\frac{\partial}{\partial u^*})\big|\,r^{n-1}\big|_{v^*=\frac12(t_1+r_0^*)}\,\mathrm{d}u^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ +\int_{r_0^*}^{r_1^*}\int_{\mathbb{S}^{n-1}}\big|(J^{(\alpha)},T)\big|\,r^{n-1}\big|_{t=t_1}\,\mathrm{d}r^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ +\int_{\frac12(t_1+r_1^*)}^{v_1^*}\int_{\mathbb{S}^{n-1}}\big|(J^{(\alpha)},\frac{\partial}{\partial v^*})\big|\,r^{n-1}\big|_{u^*=\frac12(t_1-r_1^*)}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ +\int_{\frac12(t_0-r_1^*)}^{\frac12(t_1-r_1^*)}\int_{\mathbb{S}^{n-1}}\big|(J^{(\alpha)},\frac{\partial}{\partial u^*})\big|\,r^{n-1}\big|_{v^*=v_1^*}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}$$

$$+\int_{\frac{1}{2}(t_0-r_0^*)}^{u_1^*}\int_{\mathbb{S}^{n-1}} |(J^{(\alpha)},\frac{\partial}{\partial u^*})|\, r^{n-1}|_{v^*=\frac{1}{2}(t_0+r_0^*)}\,\mathrm{d}u^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}$$
$$+\int_{r_0^*}^{r_1^*}\int_{\mathbb{S}^{n-1}} |(J^{(\alpha)},T)|\, r^{n-1}|_{t=t_0}\,\mathrm{d}r^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}$$
$$+\int_{\frac{1}{2}(t_0+r_1^*)}^{v_1^*}\int_{\mathbb{S}^{n-1}} |(J^{(\alpha)},\frac{\partial}{\partial v^*})|\, r^{n-1}|_{u^*=\frac{1}{2}(t_0-r_1^*)}\,\mathrm{d}v^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}$$

For the boundary integrals on the t-const hypersurfaces, we will use (ii) of the following Lemma.

Lemma 1.30 (Boundary terms of $J^{(\alpha)}$ current on t-const hypersurfaces)**.** *On each* $\bar{\Sigma}_t$

(i) there exists a constant $C(n,m,\alpha)$ *such that*

$$\int_{\mathbb{R}} |(J^{(\alpha)},T)|\, r^{n-1}\,\mathrm{d}r^* \leq C(n,m,\alpha)\int_{\mathbb{R}} (J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi),T)\, r^{n-1}\,\mathrm{d}r^*$$

(ii) for $r\geq r_0$ *a constant* $C(n,m,\alpha,r_0)$ *such that*

$$|(J^{(\alpha)},T)| \leq C(n,m,\alpha,r_0)\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi),T\Big).$$

Proof. Using the definition (1.4.67),

$$(J^{(\alpha)},T) = f^a\big(\frac{\partial\phi}{\partial t}\big)\big(\frac{\partial\phi}{\partial r^*}\big)+\sum_{i=1}^{\frac{n(n-1)}{2}} f^b\big(\frac{\partial\Omega_i\phi}{\partial t}\big)\big(\frac{\partial\Omega_i\phi}{\partial r^*}\big)$$
$$+\frac{1}{4}\sum_{i=1}^{\frac{n(n-1)}{2}}\Big((f^b)'+(n-1)\frac{f^b}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big)2\big(\Omega_i\phi\big)\big(\partial_t\Omega_i\phi\big)$$

because

$$\partial_t\Big((f^b)'+(n-1)\frac{f^b}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big)=0$$

and $g(T,\frac{\partial}{\partial r^*})=0$. By Cauchy's inequality

$$|(J^{(\alpha)},T)| \leq \frac{C}{\alpha^2 r^{n-1}}\Big[\frac{1}{2}\big(\frac{\partial\phi}{\partial t}\big)^2+\frac{1}{2}\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big]$$
$$+\sum_{i=1}^{\frac{n(n-1)}{2}}\frac{\pi}{\alpha}\Big[\frac{1}{2}\big(\frac{\partial\Omega_i\phi}{\partial t}\big)^2+\frac{1}{2}\big(\frac{\partial\Omega_i\phi}{\partial r^*}\big)^2\Big]$$
$$+\frac{1}{4}\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\frac{r}{\alpha^2+x^2}+(n-1)\frac{\pi}{\alpha}\big(1-\frac{2m}{r^{n-2}}\big)\Big)\Big[\frac{1}{r^2}\big(\Omega_i\phi\big)^2+\big(\frac{\partial\Omega_i\phi}{\partial t}\big)^2\Big]$$

which proves (ii) in view of

$$\Big(J^T(\phi),T\Big)=\frac{1}{2}\Big(\frac{\partial\phi}{\partial t}\Big)^2+\frac{1}{2}\Big(\frac{\partial\phi}{\partial r^*}\Big)^2+\frac{1}{2}\Big(1-\frac{2m}{r^{n-2}}\Big)|\nabla\!\!\!/\,\phi|^2\,;$$

here we have also used

$$f^b=\int_0^{r^*}\frac{1}{\alpha^2+(t^*-\alpha-\sqrt{\alpha})^2}\,\mathrm{d}t^*=\frac{1}{\alpha}\arctan x\Big|_{\frac{-\alpha-\sqrt{\alpha}}{\alpha}}^{\frac{r^*-\alpha-\sqrt{\alpha}}{\alpha}}\leq\frac{\pi}{\alpha}\quad(r^*\geq 0)\,.$$

To establish (i) it is enough to infer

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{r}{\alpha^2+x^2}|\nabla\!\!\!/\,\phi|^2\,r^{n-1}\,\mathrm{d}r^*&=\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{-\infty}^{\infty}\frac{r^{n-2}}{\alpha^2+x^2}\big(\Omega_i\phi\big)^2\,\mathrm{d}r^*\\&\leq C\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{-\infty}^{\infty}\Big(\frac{\partial\Omega_i\phi}{\partial r^*}\Big)^2\,r^{n-1}\,\mathrm{d}r^*\\&\leq C\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{-\infty}^{\infty}\Big(J^T(\Omega_i\phi),T\Big)\,r^{n-1}\,\mathrm{d}r^*\,;\end{aligned}$$

this is a standard Hardy inequality, cf. Proof of Prop.10.2 in [20]. □

The following Lemma will be applied to the boundary terms of the $J^{(\alpha)}$-current on the null hypersurfaces in the region $r\leq r_0$.

Lemma 1.31 (Boundary terms of the $J^{(\alpha)}$ current on null hypersurfaces)**.**

(i) On any segment of the outgoing null hypersurface $u^*=u_1^*\geq 0$

$$\Big|\Big(J^{(\alpha)},\frac{\partial}{\partial v^*}\Big)\Big|\leq C(n,m,\alpha)\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),\frac{\partial}{\partial v^*}\Big)\\+\epsilon(u_1^*)\,\Big(J^N(\phi),\frac{\partial}{\partial v^*}\Big)$$

where $C(n,m,\alpha)$ is a constant, and $\epsilon(u_1^*)\to 0$ as $u_1^*\to\infty$.

(ii) Let $v_0^*\geq 1$, and $u_0^*(v^*)$ such that $r(u_0^*(v^*),v^*)=r_0$ (in particular $u_0^*(v_0^*)\geq 1$). Then on the ingoing null hypersurface $v^*=v_0^*$:

$$\int_{u_0^*}^{\infty}\Big|\Big(J^{(\alpha)},\frac{\partial}{\partial u^*}\Big)\Big|\,r^{n-1}\,\mathrm{d}u^*\leq\\\leq C(n,m,\alpha)\int_{u_0^*}^{\infty}\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),\frac{\partial}{\partial u^*}\Big)\,r^{n-1}\,\mathrm{d}u^*$$

Proof. Using the definition (1.4.67) we find

$$
\begin{aligned}
(J^{(\alpha)},\frac{\partial}{\partial u^*}) = f^a\,T(\phi)(\frac{\partial}{\partial r^*},\frac{\partial}{\partial u^*}) + \sum_{i=1}^{\frac{n(n-1)}{2}}\Big\{ f^b\,T(\Omega_i\phi)(\frac{\partial}{\partial r^*},\frac{\partial}{\partial u^*})\\
+\frac{1}{4}\Big((f^b)'+(n-1)\frac{f^b}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big)\,2(\Omega_i\phi)\big(\frac{\partial\Omega_i\phi}{\partial u^*}\big)\\
+\frac{1}{4}\frac{1}{2}\Big((f^b)'+(n-1)\frac{f^b}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big)'\,(\Omega_i\phi)^2-(f^b)'\,\beta\,(\Omega_i\phi)^2\Big\}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
|(J^{(\alpha)},\frac{\partial}{\partial u^*})| \leq \frac{C}{\alpha^2(2m)^{\frac{n-1}{n-2}}}\Big[\frac{1}{2}\big(\frac{\partial\phi}{\partial u^*}\big)^2+\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\phi|^2\Big]\\
+\sum_{i=1}^{\frac{n(n-1)}{2}}\frac{\pi}{\alpha}\Big[\frac{1}{2}\big(\frac{\partial\Omega_i\phi}{\partial u^*}\big)^2+\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\Omega_i\phi|^2\Big]\\
+\sum_{i=1}^{\frac{n(n-1)}{2}}\frac{1}{2}\Big(\frac{r}{\alpha^2+x^2}+(n-1)\frac{\pi}{\alpha}\big(1-\frac{2m}{r^{n-2}}\big)\Big)\frac{1}{2}\big(\frac{\partial\Omega_i\phi}{\partial u^*}\big)^2\\
+\Big(\frac{n-1}{2}\frac{\pi}{\alpha}+\frac{n-1}{4}\frac{r}{\alpha^2+x^2}+\frac{n-1}{4}\frac{\pi}{\alpha}\big(1-\frac{2m}{r^{n-2}}\big)\\
+\frac{(n-1)(n-2)}{4}\frac{\pi}{\alpha}+(n-1)\frac{r}{\alpha^2+x^2}\Big)\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\phi|^2\\
+\Big(\frac{1}{4}\frac{r}{\alpha^2+x^2}+\frac{5}{4}\frac{|x|r^2}{(\alpha^2+x^2)^2}\Big)|\nabla\!\!\!/\phi|^2\,.
\end{aligned}
$$

Similarly for $|(J^{(\alpha)},\frac{\partial}{\partial v^*})|$. Clearly, (ii) now follows from

$$
\begin{aligned}
(J^T,\frac{\partial}{\partial v^*}) = \frac{1}{2}\big(\frac{\partial\phi}{\partial v^*}\big)^2+\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\phi|^2\\
(J^N,\frac{\partial}{\partial v^*}) = \Big[1+\frac{\sigma}{4\kappa}\big(1-\frac{2m}{r^{n-2}}\big)\Big]\,T\big(\frac{2}{1-\frac{2m}{r^{n-2}}}\frac{\partial}{\partial u^*}+\frac{\partial}{\partial t},\frac{\partial}{\partial v^*}\big)\\
\geq 2|\nabla\!\!\!/\phi|^2+\frac{1}{2}\big(\frac{\partial\phi}{\partial v^*}\big)^2\,.
\end{aligned}
$$

In case (i) we only have

$$
\begin{aligned}
(J^T,\frac{\partial}{\partial u^*}) = \frac{1}{2}\big(\frac{\partial\phi}{\partial u^*}\big)^2+\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\phi|^2\\
(J^N,\frac{\partial}{\partial u^*}) \geq \frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\phi|^2\,;
\end{aligned}
$$

but using the Hardy inequality of Lemma B.11:

$$
\begin{aligned}
\int_{u_0^*}^{\infty}\frac{r}{\alpha^2+x^2}|\nabla\!\!\!/\phi|^2\,r^{n-1}\,\mathrm{d}u^* \leq\\
\leq \sum_{i=1}^{\frac{n(n-1)}{2}}\int_{u_0^*}^{\infty}\frac{1+u^{*2}}{\alpha^2+(u^*+\alpha+\sqrt{\alpha}-v^*)^2}r_0^{n-2}\frac{1}{1+u^{*2}}(\Omega_i\phi)^2\,\mathrm{d}u^*
\end{aligned}
$$

$$\leq 8C(n,m,\alpha)\frac{1+u_0^{*2}}{u_0^{*2}}\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{u_0^*}^{\infty}\big(\frac{\partial\Omega_i\phi}{\partial u^*}\big)^2\,r^{n-1}\,\mathrm{d}u^*$$
$$+\,2\pi\,C(n,m,\alpha)\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{u_0^*}^{u_0^*+1}\Big\{(\Omega_i\phi)^2+\big(\frac{\partial\Omega_i\phi}{\partial u^*}\big)^2\Big\}\,\mathrm{d}u^*$$
$$\leq C(n,m,\alpha)\int_{u_0^*}^{\infty}\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),\frac{\partial}{\partial u^*}\Big)\,r^{n-1}\,\mathrm{d}u^*\,.$$

Obviously the same bound holds for

$$\int_{u_0^*}^{\infty}\frac{|x|r^2}{(\alpha^2+x^2)^2}|\nabla\!\!\!/\phi|^2\,r^{n-1}\,\mathrm{d}u^*\,. \qquad \square$$

Step 2b. (Commutation with Ω_i) Since

$$[\Omega_i,\frac{\partial}{\partial t}]=0 \tag{1.4.99}$$

$$\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\frac{\partial}{\partial t}\Omega_i\cdot\pi_{<L}\phi\Big)^2=r^2\big|\nabla\!\!\!/\frac{\partial\pi_{<L}\phi}{\partial t}\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \tag{1.4.100}$$

and since also

$$[\Omega_i,\frac{\partial}{\partial r}]=0 \tag{1.4.101}$$

$$\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\frac{\partial}{\partial r^*}\Omega_i\cdot\pi_{<L}\phi\Big)^2=r^2\Big|\nabla\!\!\!/\frac{\partial\pi_{<L}\phi}{\partial r^*}\Big|^2\,. \tag{1.4.102}$$

Moreover

$$[\pi_l,\frac{\partial}{\partial t}]=0\,, \tag{1.4.103}$$

so that

$$\int_{S_r}\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\frac{\partial}{\partial t}\Omega_i\cdot\pi_{<L}\phi\Big)^2\,\mathrm{d}\mu_\gamma=$$
$$=\int_{S_r}r^2\Big|\nabla\!\!\!/\frac{\partial\pi_{<L}\phi}{\partial t}\Big|^2\,\mathrm{d}\mu_\gamma\leq L(L+n+2)\int_{S_r}\Big(\frac{\partial\pi_{<L}\phi}{\partial t}\Big)^2\,\mathrm{d}\mu_\gamma\,.$$

Since also (cf (1.4.29))

$$\int_{S_r}\big|\nabla\!\!\!/\Omega_i\cdot\pi_{<L}\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,\mathrm{d}\mu_\gamma\leq\frac{L(L+n-2)}{r^2}\int_{S_r}\big(\Omega_i\cdot\pi_{<L}\phi\big)^2\,\mathrm{d}\mu_\gamma$$

we have

$$\int_{S_r}\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\nabla\!\!\!/\Omega_i\cdot\pi_{<L}\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,\mathrm{d}\mu_\gamma\leq L(L+n-2)\int_{S_r}\big|\nabla\!\!\!/\pi_{<L}\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,\mathrm{d}\mu_\gamma\,.$$

Therefore indeed,

$$\int_{S_r}\sum_{i=1}^{\frac{n(n-1)}{2}}\Bigl(J^T(\Omega_i\cdot\pi_{<L}\phi),\frac{\partial}{\partial t}\Bigr)=\frac{1}{2}\sum_{i=1}^{\frac{n(n-1)}{2}}\int_{S_r}\Bigl\{\Bigl(\frac{\partial\Omega_i\cdot\pi_{<L}\phi}{\partial t}\Bigr)^2 + \Bigl(\frac{\partial\Omega_i\cdot\pi_{<L}\phi}{\partial r^*}\Bigr)^2+\bigl(1-\frac{2m}{r^{n-2}}\bigr)\bigl|\nabla\!\!\!/\,\Omega_i\cdot\pi_{<L}\phi\bigr|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Bigr\}\,\mathrm{d}\mu_\gamma\leq$$
$$\leq\frac{1}{2}(6\gamma n)^2\bigl((6\gamma n)^2+n-2\bigr)\int_{S_r}\Bigl(J^T(\pi_{<L}\phi),\frac{\partial}{\partial t}\Bigr)\,\mathrm{d}\mu_\gamma$$

because $L=(6\gamma n)^2$ is fixed; similarly, of course, for $(J^T,\frac{\partial}{\partial u^*})$ and $(J^T,\frac{\partial}{\partial v^*})$.

We conclude the statement of the proposition with the treatment of the two regimes in Step 1 and Step 2 above from

$$\int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)}\Bigl\{\frac{1}{r^n}\bigl(\frac{\partial\phi}{\partial r^*}\bigr)^2+\frac{1}{r^{n+1}}\bigl(\frac{\partial\phi}{\partial t}\bigr)^2+\frac{1}{r^3}\bigl(1-\frac{2m}{r^{n-2}}\bigr)\bigl|\nabla\!\!\!/\,\phi\bigr|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Bigr\}\leq$$
$$\leq 2\int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)}\Bigl\{\frac{1}{r^n}\bigl(\frac{\partial\pi_{<L}\phi}{\partial r^*}\bigr)^2+\frac{1}{r^{n+1}}\bigl(\frac{\partial\pi_{<L}\phi}{\partial t}\bigr)^2+\frac{1}{r^2}\bigl|\nabla\!\!\!/\,\pi_{<L}\phi\bigr|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Bigr\}$$
$$+2\int_{\mathcal{R}^\infty_{r_0,r_1}(t_0)}\Bigl\{\frac{1}{r^3}\bigl(\frac{\partial\pi_{\geq L}\phi}{\partial r^*}\bigr)^2+\frac{1}{r^3}\bigl(\frac{\partial\pi_{\geq L}\phi}{\partial t}\bigr)^2+\frac{1}{r^3}\bigl(1-\frac{2m}{r^{n-2}}\bigr)\bigl|\nabla\!\!\!/\,\pi_{\geq L}\phi\bigr|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Bigr\}\,.$$

□

Refinement for finite regions

Note that in the proof of Prop. 1.11 neither of the currents used for the high or the low spherical harmonic regime requires the use of Hardy inequalities for the boundary integrals in the asymptotic region; indeed in both cases the zeroth order terms ϕ^2 can be estimated by the angular derivatives $|\nabla\!\!\!/\,\phi|^2$, in the case of the current $J^{X_{\gamma,\alpha},1}$ for high angular frequencies by Poincaré's inequality Prop. 1.16, and in the case of the current $J^{(\alpha)}$ for low angular frequencies as a result of the commutation with Ω_i in (1.4.67). Therefore we can in fact state a refinement of Prop. 1.11 for finite regions, i.e. an integrated local energy estimate on bounded domains in terms of the flux through the past boundary of that domain, that will be relevant in Section 1.5.3.

Let

$$^R\mathcal{P}_{\tau_1}^{\tau_2}\doteq\mathcal{R}^\infty_{r_0,R}(2\tau_1+R^*,2\tau_2+R^*)\cap\{r\leq R\}\tag{1.4.104}$$

$$^R\mathcal{D}\!\!\!\!/\,_{\tau_1}^{\tau_2}\doteq\Bigl\{(u^*,v^*):\tau_1\leq u^*\leq\tau_2\,,v^*-u^*\geq R^*\,,v^*\leq\tau_2+R^*\Bigr\}$$
$$=\mathcal{R}^\infty_{r_0,R}(2\tau_1+R^*,2\tau_2+R^*,\tau_2+\frac{1}{2}(R^*-r_0^*),\tau_2+R^*)\setminus{}^R\mathcal{P}_{\tau_1}^{\tau_2}\tag{1.4.105}$$

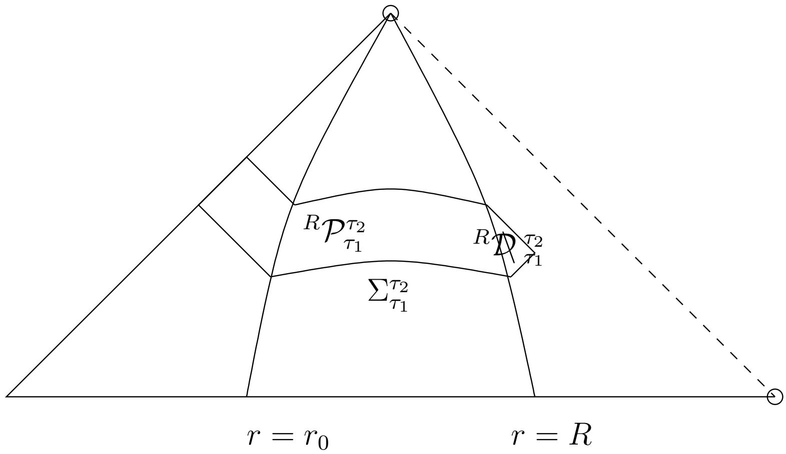

Figure 1.6: The past boundary $\Sigma_{\tau_1}^{\tau_2}$ of ${}^R\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^R\mathcal{D}_{\tau_1}^{\tau_2}$.

and denote by $\Sigma_{\tau_1}^{\tau_2}$ the past boundary of ${}^R\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^R\mathcal{D}_{\tau_1}^{\tau_2}$ (see also figure 1.6):

$$\begin{aligned}\Sigma_{\tau_1}^{\tau_2} \doteq& \partial^- \left({}^R\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^R\mathcal{D}_{\tau_1}^{\tau_2} \right) \\ =& \Big\{ (u^*, v^*) : \ v^* = \tau_1 + \frac{1}{2}(R^* + r_0^*) \,, u^* \geq \tau_1 + \frac{1}{2}(R^* - r_0^*) \Big\} \\ & \cup \Big\{ (u^*, v^*) : \ u^* + v^* = 2\tau_1 + R^* \,, r_0^* \leq v^* - u^* \leq R^* \Big\} \\ & \cup \Big\{ (u^*, v^*) : \ u^* = \tau_1 \,, R^* + \tau_1 \leq v^* \leq R^* + \tau_2 \Big\}\end{aligned} \tag{1.4.106}$$

Proposition 1.32 (Integrated local energy decay on finite regions)**.** *Let ϕ be a solution of the wave equation $\Box_g \phi = 0$, and $R > \sqrt[n-2]{2m}$. Then there exists a constant $C(n,m,R)$, such that*

$$\int_{{}^R\mathcal{P}_{\tau_1}^{\tau_2}} \Big\{ \big(\frac{\partial \phi}{\partial r^*}\big)^2 + \Big(\frac{\partial \phi}{\partial t}\Big)^2 + \big(1 - \frac{2m}{r^{n-2}}\big) \big|\nabla\!\!\!/ \phi\big|^2_{r^2 \overset{\circ}{\gamma}_{n-1}} \Big\} \mathrm{d}\mu_g \\ \leq C(n,m,R) \int_{\Sigma_{\tau_1}^{\tau_2}} \Big(J^T(\phi) + J^T(T \cdot \phi), n \Big) \tag{1.4.107}$$

for any $\tau_2 > \tau_1$.

In view of the remarks above the proof of Prop. 1.32 is of course identical to the proof of Prop. 1.11 given in Section 1.4.4 by replacing the unbounded domain $\mathcal{R}_{r_0,r_1}^{\infty}(2\tau_1 + R^*)$ by the bounded domain ${}^R\mathcal{P}_{\tau_1}^{\tau_2} \cup {}^R\mathcal{D}_{\tau_1}^{\tau_2}$.

However, this estimate does not include the zeroth order term, which we have covered seperately in Prop. 1.14.

Proposition 1.33 (Refinement for zeroth order terms on timelike boundaries)**.** *Let ϕ be solution of the wave equation* (1.1.1)*, and $R > \sqrt[n-2]{8nm}$. Then there is a constant*

$C(n,m,R)$ such that

$$\int_{2\tau'+R^*}^{2\tau+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,\phi^2\rvert_{r=R}\leq$$
$$\leq C(n,m,R)\Big\{\int_{2\tau'+R^*}^{2\tau+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big\{\Big(\frac{\partial\phi}{\partial r^*}\Big)^2+\lvert\nabla\!\!\!/\phi\rvert^2\Big\}\Big\rvert_{r=R}$$
$$+\int_{\Sigma_{\tau'}^{\tau}}\Big(J^T(\phi),n\Big)+\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}r^{n-2}\phi^2\rvert_{(\tau',R^*+\tau)}\Big\} \quad (1.4.108)$$

for all $\tau'<\tau$.

The proof remains the same as for Prop. 1.14 on page 40 with the exception that we consider the energy identity for $J^{X,1}$ on ${}^{R}\mathcal{D}\!\!\!/_{\tau'}^{\tau}$ in place of ${}^{R}\mathcal{D}_{\tau'}^{\tau}$ and use Prop. B.9 instead of Prop. B.5.

1.5 The Decay Argument

We will here prove energy decay for solutions to the wave equation and higher order energy decay of their time derivatives in the interior based on the integrated local energy decay statements of Section 1.4, following the new physical-space approach to decay of [22].

Remark 1.34. Instead one could use the conformal Morawetz vectorfield

$$Z=u^{*2}\frac{\partial}{\partial u^*}+v^{*2}\frac{\partial}{\partial v^*}$$

to prove energy decay of solutions to the wave equation with a rate corresponding to the weights in Z; this method predates the approach followed here, but is included for completeness in Section 1.5.4 (see also [42]). Similary the use of the scaling vectorfield

$$S=v^*\frac{\partial}{\partial v^*}+u^*\frac{\partial}{\partial u^*}$$

should provide an alternative approach to prove higher order energy decay [36]. Here however, we shall avoid the use of multipliers with weights in t.

1.5.1 Uniform Boundedness

A preliminary feature of the solutions to the wave equation (1.1.1) that is necessary to employ the decay mechanism of [22] is the uniform boundedness of their (nondegenerate) energy; this is a consequence of the conservation of the degenerate energy associated to the multiplier T, *and the redshift effect* of Section 1.3, which allows us to control the nondegenerate energy on the horizon.

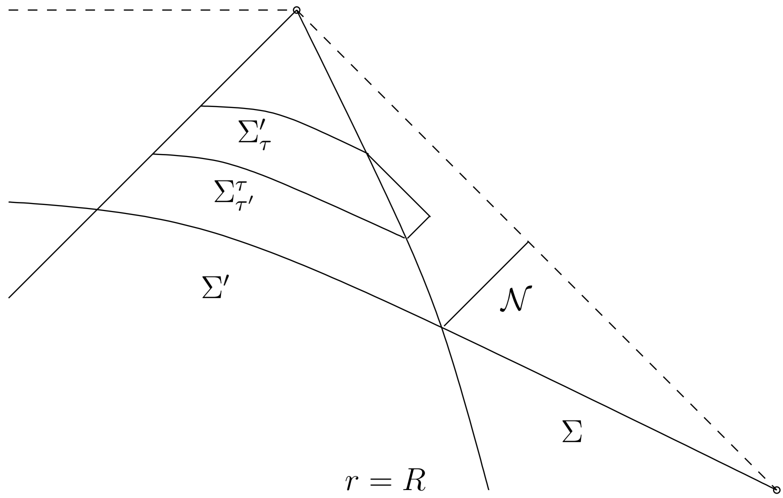

Figure 1.7: The construction of the surfaces Σ'_τ from Σ.

Let Σ be a (spherically symmetric) spacelike hypersurface in $\mathcal{M}$, $\Sigma' = \Sigma \cap \{r \leq R\}$ and $\mathcal{N}$ the outgoing null hypersurface emerging from $\partial\Sigma'$; moreover let

$$\Sigma_\tau = \varphi_\tau\Big((\Sigma' \cup \mathcal{N}) \cap \mathcal{D}\Big)$$

and

$$\Sigma'_\tau = \Sigma_\tau \cap \{r \leq R\}\,, \qquad \Sigma^\tau_{\tau'} = \Sigma_{\tau'} \cap \mathrm{J}^-(\Sigma'_\tau)\,.$$

Proposition 1.35 (Uniform Boundedness)**.** *Let ϕ be a solution of the wave equation* (1.1.1) *with initial data on Σ_0, then there exists a constant $C(\Sigma_0)$ such that*

$$\int_{\Sigma'_\tau} \Big(J^N(\phi), n\Big) \leq C \int_{\Sigma^\tau_0} \Big(J^N(\phi), n\Big) \qquad (\tau > 0)\,. \tag{1.5.1}$$

Proof. One can proceed in analogy to the *local observer's energy estimate* of [17]; indeed, from the energy identity for N on the domain $\mathcal{R}(\tau', \tau) = \cup_{\tau' \leq \overline{\tau} \leq \tau} \Sigma^\tau_{\overline{\tau}}$ it follows

$$\int_{\Sigma'_\tau} (J^N, n) + \int_{\mathcal{R}(\tau', \tau)} K^N \leq \int_{\Sigma^\tau_{\tau'}} (J^N, n) \tag{1.5.2}$$

since $(J^N, n_{\mathcal{H}}) \geq 0$, and $(J^N, n_{\mathcal{N}}) \geq 0$. By Prop. 1.9, namely the redshift effect, K^N is bounded from below by (J^N, n) near the horizon, and from above by (J^T, n) away from the horizon; since also the lapse of the foliation of $\mathcal{R}$ is bounded from above and below we conclude that there are constants $0 < b < B$ only depending on Σ and N such that

$$\int_{\Sigma'_\tau} (J^N, n) + b \int_{\tau'}^{\tau} \mathrm{d}\overline{\tau} \int_{\Sigma^\tau_{\overline{\tau}}} (J^N, n) \leq B \int_{\tau'}^{\tau} \mathrm{d}\overline{\tau} \int_{\Sigma^\tau_{\overline{\tau}}} (J^T, n) + \int_{\Sigma^\tau_{\tau'}} (J^N, n) \leq$$
$$\leq B(\tau - \tau') \int_{\Sigma^\tau_{\tau'}} (J^T, n) + \int_{\Sigma^\tau_{\tau'}} (J^N, n) \tag{1.5.3}$$

where in the last step we have used the energy identity for T on $\mathcal{R}(\tau',\overline{\tau})$ and $K^T = 0$. Thus the desired energy bound follows from the elementary Lemma 1.36. $\square$

Lemma 1.36. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative function, $f \geq 0$, such that for all $t_1 \leq t_2$ and two positive constants $0 < c < C$*

$$f(t_2) + c \int_{t_1}^{t_2} f(t)\,\mathrm{d}t \leq C(t_2 - t_1) + f(t_1)\,,$$

then

$$f(t_2) \leq f(t_1) + \frac{C}{c} \qquad (t_2 \geq t_1)\,.$$

Proof. Define

$$F(t_2,t_1) \doteq \int_{t_1}^{t_2} \bigl(c\,f(t) - B\bigr)\,\mathrm{d}t$$

then

$$f(t_2) + F(t_2,t_1) \leq f(t_1)\,.$$

Consider

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Bigl(F(t_2,t)e^{-c(t_2-t)}\Bigr) = -\Bigl(F_{t_1} + c\,F\Bigr)\,e^{-c(t_2-t)} \geq c\bigl(f(t_2) - \frac{C}{c}\bigr)e^{-b(t_2-t)}$$

because

$$F_{t_1} = -\Bigl(c\,f(t_1) - B\Bigr)\,.$$

Upon integrating on $[t_1,t_2]$

$$f(t_2) - \frac{C}{c} \leq \frac{1}{e^{c(t_2-t_1)} - 1}F(t_2,t_1)$$

we infer with $F(t_2,t_1) \leq f(t_1) - f(t_2)$ that

$$f(t_2) \leq f(t_1)e^{-c(t_2-t_1)} + \frac{C}{c}\Bigl(1 - e^{-c(t_2-t_1)}\Bigr) \leq f(t_1) + \frac{C}{c}\,. \qquad \square$$

1.5.2 Energy decay

In this Section we prove quadratic decay of the nondegenerate energy.

Let

$$\Sigma_{\tau_0} \doteq \partial^- \mathcal{R}^{\infty}_{r_0,R}(t_0) \qquad \tau_0 = \frac{1}{2}(t_0 - R^*) \tag{1.5.4}$$

with $R > \sqrt[n-2]{8nm}$, $t_0 > 0$ and $r_0 \doteq r_0^{(N)}$ according to Prop. 1.7.

Proposition 1.37 (Energy decay)**.** *Let ϕ be a solution of the wave equation* (1.1.1) *with initial data on Σ_{τ_0} satisfying*

$$D \doteq \int_{\tau_0+R^*}^{\infty}\mathrm{d}v \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\mathring{\gamma}_{n-1}} \sum_{k=0}^{1} r^2 \Bigl(\frac{\partial r^{\frac{n-1}{2}}\partial_t^k\phi}{\partial v^*}\Bigr)^2\Big|_{u=\tau_0} + \int_{\Sigma_{\tau_0}} \Bigl(\sum_{k=0}^{2} J^N(T^k\cdot\phi), n\Bigr) < \infty\,, \tag{1.5.5}$$

then there exists a constant $C(n,m,R)$ *such that*

$$\int_{\Sigma_\tau} \Big(J^N(\phi), n \Big) \leq \frac{C\,D}{\tau^2} \qquad (\tau > \tau_0)\,. \tag{1.5.6}$$

The proof is based on a *weighted energy inequality*, derived from the energy identity for the current (1.5.8) on the domain

$$^R\mathcal{D}_{\tau_1}^{\tau_2} = \big\{ (u^*, v^*) : \tau_1 \leq u^* \leq \tau_2, v^* - u^* \geq R^* \big\}\,. \tag{1.5.7}$$

Weighted energy identity. Consider the current

$$\overset{r}{J}_\mu(\phi) = T_{\mu\nu}(\psi) V^\nu \tag{1.5.8}$$

where

$$\psi = r^{\frac{n-1}{2}} \phi \tag{1.5.9}$$

$$V = r^q \frac{\partial}{\partial v^*}\,, \quad q = p + 1 - n\,, \quad p \in \{1,2\}\,. \tag{1.5.10}$$

This may also be viewed as the current to the multiplier vectorfield $r^p \frac{\partial}{\partial v^*}$, modified by the following terms:

$$\begin{aligned}
\overset{r}{J}_\mu(\phi) = T_{\mu\nu}(\phi)\, r^p \big(\frac{\partial}{\partial v^*}\big)^\nu + \big(\frac{n-1}{2}\big)^2 r^{p-2} \big(1 - \frac{2m}{r^{n-2}}\big)(\partial_\mu r)\phi^2 \\
+ \frac{1}{2}\frac{n-1}{2} r^{p-1} (\partial_\mu r) \frac{\partial \phi^2}{\partial v^*} + \frac{1}{2}\frac{n-1}{2} r^{p-1} \big(1 - \frac{2m}{r^{n-2}}\big)\big(\partial_\mu \phi^2\big) \\
- \frac{1}{2}\big(\frac{n-1}{2}\big)^2 r^{p-2} \big(1 - \frac{2m}{r^{n-2}}\big)\big(\frac{\partial}{\partial v^*}\big)_\mu \phi^2 - \frac{1}{2}\frac{n-1}{2}\big(\frac{\partial}{\partial v^*}\big)_\mu r^{p-1} \frac{\partial \phi^2}{\partial r^*}
\end{aligned}$$

If $\Box_g \phi = 0$ then we calculate

$$\begin{aligned}
\Box_g \psi &= -\big(1 - \frac{2m}{r^{n-2}}\big)^{-1} \partial_{u^*}\partial_{v^*}\psi + \frac{n-1}{r}\frac{\partial \psi}{\partial r^*} + \frac{1}{r^2} \overset{\circ}{\Delta}_{n-1} \psi \\
&= \frac{n-1}{2}\big(\frac{n-3}{2} + \frac{n-1}{2}\frac{2m}{r^{n-2}}\big)\frac{1}{r^2}\psi + \frac{n-1}{r}\frac{\partial}{\partial r^*}\psi\,.
\end{aligned} \tag{1.5.11}$$

So the wave equation for ϕ

$$\Box_g \phi = 0$$

is equivalent to the following equation for ψ:

$$\begin{aligned}
&- \partial_{u^*}\partial_{v^*}\psi + \big(1 - \frac{2m}{r^{n-2}}\big)\frac{1}{r^2} \overset{\circ}{\Delta}_{n-1} \psi \\
&\qquad - \frac{n-1}{2}\big(\frac{n-3}{2} + \frac{n-1}{2}\frac{2m}{r^{n-2}}\big)\frac{1}{r^2}\big(1 - \frac{2m}{r^{n-2}}\big)\psi = 0\,.
\end{aligned} \tag{1.5.12}$$

Now,

$$\overset{r}{K}(\phi) = \nabla^\mu \overset{r}{J}_\mu(\phi) = \Box_g(\psi)\, V \cdot \psi + K^V(\psi)\,, \tag{1.5.13}$$

where

$$K^V(\psi) = {}^{(V)}\pi^{\mu\nu}\, T_{\mu\nu}(\psi)\,.$$

Since

$$\begin{aligned}
{}^{(V)}\pi_{u^*u^*} &= 2\,q\,r^{q-1}\big(1-\frac{2m}{r^{n-2}}\big)^2 \\
{}^{(V)}\pi_{v^*v^*} &= 0 \\
{}^{(V)}\pi_{u^*v^*} &= -\big(1-\frac{2m}{r^{n-2}}\big)\,r^{q-1}\Big[q+(n-q-2)\frac{2m}{r^{n-2}}\Big] \\
{}^{(V)}\pi_{aA} &= 0 \\
{}^{(V)}\pi_{AB} &= r^{q-1}\big(1-\frac{2m}{r^{n-2}}\big)\,g_{AB}
\end{aligned} \tag{1.5.14}$$

we find

$$\begin{aligned}
\overset{r}{K}\cdot r^{n-1} = \frac{n-1}{4}\Big(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\Big)\frac{r^p}{r^2}\frac{\partial\psi^2}{\partial v^*}+\frac{p}{2}r^{p-1}\big(\frac{\partial\psi}{\partial v^*}\big)^2 \\
+\frac{1}{2}r^{p-1}\Big[(2-p)+(p-n)\frac{2m}{r^{n-2}}\Big]\big|\nabla\!\!\!/\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,.
\end{aligned} \tag{1.5.15}$$

One may integrate the first term by parts to obtain:

$$\begin{aligned}
\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\;\overset{r}{K}\cdot r^{n-1} = \frac{n-1}{4}\Big(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\Big)\frac{r^p}{r^2}\psi^2\Big|_{u^*+R^*}^{\infty} \\
+\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\Big\{\Big[\frac{n-1}{4}(2-p)\frac{n-3}{2}+\frac{n-1}{2}(n-p)\frac{2m}{r^{n-2}}\Big]\frac{r^p}{r^3}\big(1-\frac{2m}{r^{n-2}}\big)\psi^2 \\
+\frac{p}{2}r^{p-1}\big(\frac{\partial\psi}{\partial v^*}\big)^2+\frac{1}{2}r^{p-1}\Big[(2-p)+(p-n)\frac{2m}{r^{n-2}}\Big]\big|\nabla\!\!\!/\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\}
\end{aligned} \tag{1.5.16}$$

We can now write down the energy identity for the current $\overset{r}{J}$ (see also Appendix B.2):

$$\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}}\overset{r}{K}\;\mathrm{d}\mu_g = \int_{\partial{}^R\mathcal{D}_{\tau_1}^{\tau_2}} {}^*\,\overset{r}{J}$$

Dropping the positive zeroth order terms, we obtain:

$$\begin{aligned}
&\int_{\tau_2+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,\frac{1}{2}r^p\big(\frac{\partial\psi}{\partial v^*}\big)^2\big|_{u^*=\tau_2} \\
&\qquad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times \\
&\qquad\times\Big\{\frac{p}{2}r^{p-1}\big(\frac{\partial\psi}{\partial v^*}\big)^2+\frac{1}{2}r^{p-1}\Big[(2-p)+(p-n)\frac{2m}{r^{n-2}}\Big]\big|\nabla\!\!\!/\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\} \\
&\qquad+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,\frac{1}{2}r^p\big|\nabla\!\!\!/\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\big|_{v^*\to\infty}\leq \\
&\qquad\leq\big(1-\frac{2m}{R^{n-2}}\big)^{-1}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,\frac{1}{2}r^p\big(\frac{\partial\psi}{\partial v^*}\big)^2\big|_{u^*=\tau_1} \\
&\qquad+\int_{2\tau_1+R^*}^{2\tau_2+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big[\frac{1}{4}r^p\big(\frac{\partial\psi}{\partial v^*}\big)^2+\frac{1}{4}r^p\big|\nabla\!\!\!/\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \\
&\qquad\qquad+\frac{n-1}{4}\frac{1}{2}r^p\big(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{R^{n-2}}\big)\frac{1}{r^2}\psi^2\Big]\big|_{r=R}\Big\}
\end{aligned} \tag{1.5.17}$$

Note that the powers of r that appear in the bulk term are 1 less than those that appear in the boundary terms. This allows for a hierarchy of inequalities (1.5.17) for different values of p, the so called *p-hierarchy*.

Proof of Prop. 1.37: In a first step the decay of the solutions at future null infinity will be deduced from the weighted energy inequality, and in a second step the continuation to the event horizon will be inferred from the redshift effect.

Step 1. The p-hierarchy consists of two steps which exploits (1.5.17) first with $p = 2$, then with $p = 1$; but in a zeroth step we need to obtain control on the angular derivatives from (1.5.17) with $p = 1$:

Since

$$1 - (n-1)\frac{2m}{r^{n-2}} > \frac{1}{2} \qquad (r > R)$$

we have from the weighted energy inequality for $p = 1$ on the domain ${}^{r_0'}\mathcal{D}_{\tau'}^{\tau}$ for $\tau > \tau' \geq \tau_0 \doteq \frac{1}{2}(t_0 - R^*)$,

$$\int_{\tau'}^{\tau} \mathrm{d}u^* \int_{u^*+{r_0'}^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \frac{1}{4} \big|\nabla\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \leq$$

$$\leq \big(1 - \frac{2m}{R^{n-2}}\big)^{-1} \int_{\tau'+{r_0'}^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \frac{1}{2} r \big(\frac{\partial\psi}{\partial v^*}\big)^2\Big|_{u^*=\tau'}$$
$$+ C(n,m,R)\big(1 - \frac{2m}{R^{n-2}}\big)^{-1} \int_{\Sigma_{\tau_0}} \Big(J^T(\phi) + J^T(T\cdot\phi), n\Big)\,; \quad (1.5.18)$$

here, we have estimated the boundary integrals as follows:

Choose $r_0' \in (R^*, R^*+1)$ such that

$$\int_{R^*}^{R^*+1} \mathrm{d}r^* \int_{t_0+(r^*-R^*)}^{\infty} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \big(1 - \frac{2m}{r^{n-2}}\big) r^{n-1} \times$$
$$\times \Big\{\frac{1}{r^n}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{1}{r^{n+1}}\big(\frac{\partial\phi}{\partial t}\big)^2 + \frac{1}{r^3}\big(1 - \frac{2m}{r^{n-2}}\big)\big|\nabla\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\}$$
$$= \int_{t_0+({r_0'}^*-R^*)}^{\infty} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \big(1 - \frac{2m}{{r_0'}^{n-2}}\big) {r_0'}^{n-1} \times$$
$$\times \Big\{\frac{1}{{r_0'}^n}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{1}{{r_0'}^{n+1}}\big(\frac{\partial\phi}{\partial t}\big)^2 + \frac{1}{{r_0'}^3}\big(1 - \frac{2m}{{r_0'}^{n-2}}\big)\big|\nabla\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big\}$$

then

$$\int_{2\tau'+{r_0'}^*}^{2\tau+{r_0'}^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big[\frac{1}{4} r^p \big(\frac{\partial\psi}{\partial v^*}\big)^2 + \frac{1}{4} r^p \big|\nabla\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}$$
$$+ \frac{n-1}{4}\frac{1}{2} r^p \Big(\frac{n-3}{2} + \frac{n-1}{2}\frac{2m}{r^{n-2}}\Big)\frac{1}{r^2}\psi^2\Big]\Big|_{r=r_0'} \leq$$

$$\leq \int_{2\tau'+{r'_0}^*}^{2\tau+{r'_0}^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} {r'_0}^{p-2}\Big[\frac{1}{2}\big(\frac{n-1}{2}\big)^2\phi^2 + \frac{1}{2}{r'_0}^2\big(\frac{\partial\phi}{\partial v^*}\big)^2$$
$$+\frac{1}{4}{r'_0}^2\lvert\nabla\!\!\!/\phi\rvert^2_{r^2\overset{\circ}{\gamma}_{n-1}} + \frac{n-1}{4}\frac{1}{2}\Big(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{R^{n-2}}\Big)\phi^2\Big]\Big|_{r=r'_0} r^{n-1} \leq$$
$$\leq C(n,m,R)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big)$$

because by (1.4.24)

$$\int_{{r'_0}^*+2\tau'}^{{r'_0}^*+2\tau} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1}\Big[\frac{1}{4}\big(\frac{\partial\phi}{\partial v^*}\big)^2 + \frac{n-1}{(4r'_0)^2}\big(1-\frac{2m}{r^{n-2}}\big)\phi^2\Big]\Big|_{r=r'_0} \leq$$
$$\leq \int_{{r'_0}^*+2\tau'}^{{r'_0}^*+2\tau} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1}\times$$
$$\times\Big[\frac{1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\lvert\nabla\!\!\!/\phi\rvert^2_{r^2\overset{\circ}{\gamma}_{n-1}} + \frac{n-1}{2}\big(1-\frac{2m}{r^{n-2}}\big)\big(\frac{\partial\phi}{\partial r^*}\big)^2\Big]\Big|_{r=r'_0}$$
$$+ C(n,m,R)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi),n\Big)$$

and by Prop. 1.11 (and the choice of r'_0):

$$\int_{t_0+({r'_0}^*-R^*)}^{\infty} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1}\times$$
$$\times\Big[\frac{1}{{r'_0}^n}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{1}{{r'_0}^3}\big(1-\frac{2m}{{r'_0}^{n-2}}\big)\lvert\nabla\!\!\!/\phi\rvert^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big]\Big|_{r=r'_0} \leq$$
$$\leq C(n,m)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big).$$

Note that for the use of (1.4.24) that with our choice of R

$$(n-3)+n\frac{2m}{r^{n-2}}-(n-1)\big(\frac{2m}{r^{n-2}}\big)^2>0 \qquad (r>R).$$

$\boxed{p=2:}$ For $p=2$ (1.5.17) reads

$$\int_{\tau'}^{\tau}\mathrm{d}u^*\int_{u^*+{r'_0}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big[r\big(\frac{\partial\psi}{\partial v^*}\big)^2-\frac{1}{2}r(n-2)\frac{2m}{r^{n-2}}\lvert\nabla\!\!\!/\psi\rvert^2_{r^2\overset{\circ}{\gamma}_{n-1}}\Big] \leq$$
$$\leq\big(1-\frac{2m}{R^{n-2}}\big)^{-1}\int_{\tau'+{r'_0}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\frac{1}{2}r^2\big(\frac{\partial\psi}{\partial v^*}\big)^2\rvert_{u^*=\tau'}$$
$$+C(n,m,R)\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big).$$

Thus, with the previous estimate (1.5.18),

$$\int_{\tau'}^{\tau}\mathrm{d}u^*\int_{u^*+{r'_0}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r\big(\frac{\partial\psi}{\partial v^*}\big)^2 \leq$$

$$\leq C(n,m,R)\big)^{-1}\Bigg\{\int_{\tau'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \frac{1}{2} r^2 \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau'} + \int_{\Sigma_{\tau_0}} \Big(J^T(\phi) + J^T(T\cdot\phi), n\Big)\Bigg\}. \quad (1.5.19)$$

Let us define

$$\tau_{j+1} = 2\tau_j \quad (j \in \mathbb{N}_0) \qquad \tau_0 = \frac{1}{2}(t_0 - R^*)\,,$$

then there is a sequence $(\tau_j')_{j\in\mathbb{N}_0}$ with $\tau_j' \in (\tau_j, \tau_{j+1})$ $(j \in \mathbb{N}_0)$ such that

$$\int_{\tau_j'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_j'} \leq$$
$$\leq \frac{1}{\tau_j} C(n,m,R)\Bigg[\int_{\tau_j+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^2 \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_j} + \int_{\Sigma_{\tau_0}} \Big(J^T(\phi) + J^T(T\cdot\phi), n\Big)\Bigg]$$

and again by (1.5.17)

$$\int_{\tau_j+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \frac{1}{2} r^2 \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_j} \leq$$
$$\leq C(n,m,R)\Bigg[\int_{\tau_0+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^2 \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \Big(J^T(\phi) + J^T(T\cdot\phi), n\Big)\Bigg].$$

Since $\frac{1}{\tau_j} \leq \frac{1}{\tau_j'}\frac{\tau_{j+1}}{\tau_j} = \frac{2}{\tau_j'}$ we have

$$\int_{\tau_j'+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_j'} \leq$$
$$\leq \frac{C(n,m,R)}{\tau_j'}\Bigg[\int_{\tau_0+r_0'^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^2 \big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \Big(J^T(\phi) + J^T(T\cdot\phi), n\Big)\Bigg]. \quad (1.5.20)$$

$\boxed{p=1:}$ In order to deal with the timelike boundary integrals analogously to the above choose

$$r_j''^* \in (r_0'^*, r_0'^* + 1)$$

such that

$$\int_{2\tau_j'+r_j''^*}^{2\tau_{j+1}'+r_j''^*} \mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1}\times$$
$$\times \left[\frac{1}{r^n}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{1}{r^{n+1}}\big(\frac{\partial\phi}{\partial t}\big)^2 + \frac{1}{r^3}\big(1 - \frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\right]|_{r=r_j''} \leq$$

$$\leq C(n,m)\int_{\Sigma_{\tau'_j}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big).$$

Then, proceeding as before,

$$\int_{2\tau'_j+r''^*_j}^{2\tau'_{j+1}+r''^*_j}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times$$
$$\times\left[\frac{1}{4}r\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\frac{1}{4}r\big|\nabla\!\!\!\!/\,\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}+\frac{n-1}{4}\frac{1}{2}r\Big(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\Big)\frac{1}{r^2}\psi^2\right]\Big|_{r=r''_j}\leq$$
$$\leq C(n,m,R)\int_{\Sigma_{\tau'_j}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big).\quad(1.5.21)$$

Now apply (1.5.17) to the region ${}^{r''_j}\mathcal{D}_{\tau'_j}^{\tau'_{j+1}}$ to obtain:

$$\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}u^*\int_{u^*+r''^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times$$
$$\times\left[\frac{1}{2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\frac{1}{4}\big|\nabla\!\!\!\!/\,\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\right]\leq$$
$$\leq\Big(1-\frac{2m}{R^{n-2}}\Big)^{-1}\int_{\tau'_j+r''^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\frac{1}{2}r\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau'_j}$$
$$+C(n,m,R)\int_{\Sigma_{\tau'_j}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big)$$

By virtue of the result (1.5.20) from the case $p=2$, this yields

$$\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}u^*\int_{u^*+r''^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\left[\frac{1}{2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\frac{1}{4}\big|\nabla\!\!\!\!/\,\psi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\right]\leq$$
$$\leq\frac{C(n,m,R)}{\tau'_j}\left[\int_{\tau_0+r'^*_0}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^2\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau_0}+\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big)\right]$$
$$+C(n,m,R)\int_{\Sigma_{\tau'_j}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big).\quad(1.5.22)$$

Step 2. Our aim is to prove decay for the *non-degenerate* energy. Let us first find an estimate for

$$\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma_\tau}\Big(J^N(\phi),n\Big)=$$
$$=\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma_\tau\cap\{r\leq r''_j\}}\Big(J^N(\phi),n\Big)+\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma_\tau\cap\{r\geq r''_j\}}\Big(J^T(\phi),n\Big).$$

The estimate of the first term is exactly the content of Cor. 1.13, and for the second term

$$\int_{\tau'_j}^{\tau'_{j+1}}\mathrm{d}\tau\int_{\Sigma'_\tau\cap\{r\geq r''_j\}}\Big(J^T(\phi),n\Big)=$$

$$= \int_{\tau_j'}^{\tau_{j+1}'} \mathrm{d}u^* \int_{u^*+r_j''^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \, r^{n-1} \Big[\frac{1}{2} \big(\frac{\partial\phi}{\partial v^*}\big)^2 + \frac{1}{2}\big(1 - \frac{2m}{r^{n-2}}\big) |\nabla\phi|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \Big]$$

we can use (1.5.22) once we have turned it into an estimate for the derivatives of ϕ. Note that

$$\int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \, \big(\frac{\partial\psi}{\partial v^*}\big)^2 =$$

$$= \int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \Big[\frac{n-1}{2} \big(1 - \frac{2m}{r^{n-2}}\big) r^{\frac{n-3}{2}} \frac{\partial}{\partial v^*} \Big(r^{\frac{n-1}{2}} \phi^2 \Big) + r^{n-1} \big(\frac{\partial\phi}{\partial v^*}\big)^2 \Big]$$

$$= -\frac{n-1}{2}\big(1 - \frac{2m}{r^{n-2}}\big) \, r^{n-2} \, \phi^2|_{v^*=u^*+r_0'^*}$$

$$+ \int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \Big\{ -\frac{n-1}{2}\big(1 - \frac{2m}{r^{n-2}}\big) \Big[(n-2)\frac{2m}{r^n} + \frac{n-3}{2}\big(1 - \frac{2m}{r^{n-2}}\big)\frac{1}{r^2} \Big] \phi^2 + \big(\frac{\partial\phi}{\partial v^*}\big)^2 \Big\} \, r^{n-1}$$

and by Lemma B.6

$$\int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \frac{1}{r^2} \phi^2 \, r^{n-1} \leq$$

$$\leq C(n,m) \int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \, \big(\frac{\partial\psi}{\partial v^*}\big)^2 + C(n,m) \, r^{n-1} \phi^2|_{(u^*, v^*=u^*+r_0'^*)} \, .$$

Thus

$$\int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \Big(\frac{\partial\phi}{\partial v^*}\Big)^2 r^{n-1} \leq C(n,m,R) \Big[\phi^2|_{(u^*, u^*+r_0'^*)} + \int_{u^*+r_0'^*}^{\infty} \mathrm{d}v^* \big(\frac{\partial\psi}{\partial v^*}\big)^2 \Big] \, ,$$

and finally in view of (1.5.21)

$$\int_{\tau_j'}^{\tau_{j+1}'} \mathrm{d}u^* \int_{u^*+r_j''^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \big(\frac{\partial\phi}{\partial v^*}\big)^2 r^{n-1} \leq$$

$$\leq C(n,m,R) \int_{\tau_j'}^{\tau_{j+1}'} \mathrm{d}u^* \int_{u^*+r_j''^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \big(\frac{\partial\psi}{\partial v^*}\big)^2$$

$$+ C(n,m,R) \int_{\Sigma_{\tau_j'}} \Big(J^T(\phi) + J^T(T\cdot\phi), n \Big) \, . \quad (1.5.23)$$

Therefore, putting the estimates for the two terms back together,

$$\int_{\tau_j'}^{\tau_{j+1}'} \mathrm{d}\tau \int_{\Sigma_\tau} \Big(J^N(\phi), n \Big) \leq$$

$$\leq C(n,m,R) \int_{\tau_j'}^{\tau_{j+1}'} \mathrm{d}u^* \int_{u^*+r_j''^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{ \big(\frac{\partial\psi}{\partial v^*}\big)^2 + |\nabla\psi|^2_{r^2\overset{\circ}{\gamma}_{n-1}} \Big\}$$

$$+ C(n,m) \int_{\Sigma_{\tau_j'}} \Big(J^N(\phi) + J^T(T\cdot\phi), n \Big) \leq$$

$$\leq \frac{C(n,m,R)}{\tau_j'}\Bigg[\int_{\tau_0+r_0'^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^2\big(\frac{\partial\psi}{\partial v^*}\big)^2|_{u^*=\tau_0}+\int_{\Sigma_{\tau_0}}\Big(J^T(\phi)+J^T(T\cdot\phi),n\Big)\Bigg] \\ +C(n,m,R)\int_{\Sigma_{\tau_j'}}\Big(J^N(\phi)+J^T(T\cdot\phi),n\Big) \quad (1.5.24)$$

where we have now used (1.5.22). The same inequality holds for τ_{j+2}' in place of τ_{j+1}', by adding the inequalities corresponding to the intervals $[\tau_j',\tau_{j+1}']$ and $[\tau_{j+1}',\tau_{j+2}']$ and using Prop. 1.35 for the last term. So there is a sequence

$$(\tau_j'')_{j\in\mathbb{N}} \qquad \tau_j''\in\big(\tau_j',\tau_{j+2}'\big)$$

such that

$$\int_{\tau_j'}^{\tau_{j+2}'}\mathrm{d}\tau\int_{\Sigma_\tau}\Big(J^N(\phi),n\Big)\geq\tau_{j+1}\int_{\Sigma_{\tau_j''}}\Big(J^N(\phi),n\Big)$$

and since $\frac{1}{\tau_{j+1}}\leq\frac{1}{\tau_j''}\frac{\tau_{j+3}}{\tau_{j+1}}=\frac{4}{\tau_j''}$ we have

$$\int_{\Sigma_{\tau_j''}}\Big(J^N(\phi),n\Big)\leq\frac{4}{\tau_j''}\int_{\tau_j}^{\tau_{j+2}'}\mathrm{d}\tau\int_{\Sigma_\tau'}\Big(J^N(\phi),n\Big). \quad (1.5.25)$$

Now for any given $\tau>\tau_0$ we may choose

$$j^*=\max\{j\in\mathbb{N}:\tau_j''\leq\tau\}$$

so that by (1.5.1)

$$\int_{\Sigma_\tau}\Big(J^N(\phi),n\Big)\leq C\int_{\Sigma_{\tau_{j^*}''}}\Big(J^N(\phi),n\Big)$$

with $\frac{\tau}{\tau_{j^*}''}\leq\frac{\tau_{j^*+1}''}{\tau_{j^*}''}\leq 2^4$. In particular we may estimate the last integral in (1.5.24)

$$\int_{\Sigma_{\tau_j'}}\Big(J^N(\phi)+J^T(T\cdot\phi),n\Big)\leq\frac{C}{\tau'_j}\int_{\tau_{j-1}'}^{\tau_{j+1}'}\mathrm{d}\tau\int_{\Sigma_\tau}\Big(J^N(\phi)+J^N(T\cdot\phi),n\Big)$$

to see that in fact we have

$$\int_{\tau_j'}^{\tau_{j+2}'}\mathrm{d}\tau\int_{\Sigma_\tau}\Big(J^N(\phi),n\Big)\leq \\ \leq\frac{C(n,m,R)}{\tau_j'}\Bigg[\int_{\tau_0+r_0'^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big\{r^2\big(\frac{\partial r^{\frac{n-1}{2}}\phi}{\partial v^*}\big)^2+r^2\big(\frac{\partial r^{\frac{n-1}{2}}\frac{\partial\phi}{\partial t}}{\partial v^*}\big)\Big\} \\ +\int_{\Sigma_{\tau_0}}\Big(J^N(\phi)+J^N(T\cdot\phi)+J^T(T^2\phi),n\Big)\Bigg]. \quad (1.5.26)$$

Again with the sequence $(\tau_j'')_{j\in\mathbb{N}}$

$$\int_{\Sigma_{\tau_j''}}\Big(J^N(\phi),n\Big)\leq\frac{1}{\tau_{j+1}}\int_{\tau_j'}^{\tau_{j+2}'}\mathrm{d}\tau\int_{\Sigma_\tau}\Big(J^N(\phi),n\Big)$$

and since $\frac{1}{\tau_{j+1}}\frac{1}{\tau'_j} \leq \frac{2^5}{\tau''^2_j}$ we obtain by virtue of Prop. 1.35 our final result:

$$\int_{\Sigma_\tau}\Big(J^N(\phi),n\Big) \leq$$
$$\leq \frac{C(n,m,R)}{\tau^2}\Bigg[\int_{\tau_0+R^*}^\infty \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{ r^2\big(\frac{\partial r^{\frac{n-1}{2}}\phi}{\partial v^*}\big)^2 + r^2\big(\frac{\partial r^{\frac{n-1}{2}}\frac{\partial\phi}{\partial t}}{\partial v^*}\big)^2\Big\}$$
$$+\int_{\Sigma_{\tau_0}}\Big(J^N(\phi)+J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\Big)\Bigg] \quad (1.5.27)$$

□

1.5.3 Improved interior decay of the first order energy

In this Section we prove an energy estimate for the first order energy which improves the decay rate as compared to Prop. 1.37 in a bounded radial region.

Remark 1.38. The argument largely depends on the asymptotic properties of the space-time, and is similar and slightly easier in Minkowski space, see Appendix A.

Proposition 1.39 (Improved interior first order energy decay)**.** *Let* $0 < \delta < \frac{1}{2}$, $R > \sqrt[n-2]{\frac{8nm}{\delta}}$*, and let* ϕ *be a solution of the wave equation* (1.1.1) *with initial data on* Σ_{τ_1} ($\tau_1 > 0$) *satisfying*

$$D \doteq \int_{\tau_1+R^*}^\infty \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Big\{\sum_{k=0}^{1} r^{4-\delta}\Big(\frac{\partial(T^k\cdot\chi)}{\partial v^*}\Big)^2$$
$$+\sum_{k=0}^{4} r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2 + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{5} J^N(T^k\cdot\phi)+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi),n\Big) < \infty\,. \quad (1.5.28)$$

Then there exists a constant $C(n,m,\delta,R)$ *such that*

$$\int_{\Sigma'_\tau}\Big(J^N(T\cdot\phi),n\Big) \leq \frac{C\,D}{\tau^{4-2\delta}} \qquad (\tau > \tau_1) \quad (1.5.29)$$

where $\Sigma'_\tau = \Sigma_\tau \cap \{r \leq R\}$.

In addition to the weighted energy identity arising from the multiplier $r^p\frac{\partial}{\partial v^*}$ that was used to prove Prop. 1.37 we will here also use a commutation with $\frac{\partial}{\partial v^*}$ to obtain the energy decay for $\frac{\partial\phi}{\partial t}$ of Prop. 1.39.

Weighted energy and commutation. Consider the current

$$\overset{v}{J}_\mu(\phi) \doteq T_{\mu\nu}(\chi)V^\nu\,, \tag{1.5.30}$$

where now

$$\begin{gathered}\chi \doteq \partial_{v^*}\psi = \frac{\partial(r^{\frac{n-1}{2}}\phi)}{\partial v^*} \qquad V = r^q\frac{\partial}{\partial v^*} \\ q = p-(n-1) \qquad 2<p<4 \qquad \delta = 4-p\,.\end{gathered} \tag{1.5.31}$$

Notation. To make the dependence on p explicit, we denote by

$$\overset{v}{K}_p(\phi) \doteq \nabla^\mu \overset{v}{J}_\mu(\phi)\,. \tag{1.5.32}$$

The error terms for $\overset{v}{K}$ arise from the fact that χ is not a solution of (1.1.1); here, similarly to (1.5.11), we find:

$$\begin{aligned}\Box_g\chi = &-\frac{n-1}{2r^3}\left\{(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\Big(\frac{2m}{r^{n-2}}\Big)^2\right\}\psi \\ &+\frac{n-1}{4r^2}\Big[(n-3)+(n-1)\frac{2m}{r^{n-2}}\Big]\chi+\frac{1}{r}\Big[2-n\frac{2m}{r^{n-2}}\Big]\triangle\!\!\!\!/\,\psi+\frac{n-1}{r}\frac{\partial\chi}{\partial r^*}\end{aligned} \tag{1.5.33}$$

Hence

$$\begin{aligned}\overset{v}{K}_p(\phi) =&\Box(\chi)\,V\cdot\chi+K^V(\chi)\\ =&\frac{1}{2}p\,r^{q-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+\frac{1}{2}\Big[(2-p)-(n-p)\frac{2m}{r^{n-2}}\Big]\,r^{q-1}\big|\nabla\!\!\!\!/\,\chi\big|^2 &(1.5.34)\\ &-\frac{n-1}{2}r^{q-3}\left[(n-3)+n\frac{2m}{r^{n-2}}-(n-1)^2\Big(\frac{2m}{r^{n-2}}\Big)^2\right]\psi\frac{\partial^2\psi}{\partial v^{*2}} &(1.5.35)\\ &+\frac{n-1}{8}r^{q-2}\Big[(n-3)+(n-1)\frac{2m}{r^{n-2}}\Big]\frac{\partial\chi^2}{\partial v^*} &(1.5.36)\\ &+r^{q-1}\Big[2-n\frac{2m}{r^{n-2}}\Big](\triangle\!\!\!\!/\,\psi)\Big(\frac{\partial\chi}{\partial v^*}\Big) &(1.5.37)\end{aligned}$$

which is not positive definite. However, we have

$$\begin{aligned}\frac{1}{4}p\,r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 \leq \overset{v}{K}_p(\phi)\cdot r^{n-1}&+\frac{1}{2}\Big[(p-2)+(n-p)\frac{2m}{r^{n-2}}\Big]r^{p-1}\big|\nabla\!\!\!\!/\,\chi\big|^2\\ &+\frac{(n-1)^2(n-2)^2}{2}r^{(p-2)-1}\frac{1}{r^2}\psi^2+\frac{4}{p}r^{(p-2)-1}r^2\Big(\triangle\!\!\!\!/\,\psi\Big)^2\\ &-\frac{n-1}{8}r^{p-2}\Big[(n-3)+(n-1)\frac{2m}{r^{n-2}}\Big]\frac{\partial\chi^2}{\partial v^*}\,,\end{aligned} \tag{1.5.38}$$

where we have used that

$$n-2>n-3+n\frac{2m}{r^{n-2}}-(n-1)^2\Big(\frac{2m}{r^{n-2}}\Big)^2\geq n-3 \tag{1.5.39}$$

(is decreasing) on $r > \sqrt[n-2]{4nm}$. The key insight here is that we are able to control all other terms on the right hand side of (1.5.38) by the current $\overset{r}{J}$ of Section 1.5.2 with $p-2$ in the role of p, i.e.

$$\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 \leq$$
$$\leq C(n,m,\delta,p,R) \int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} \Big\{ \overset{v}{K}_p(\phi) + \overset{r}{K}_{p-2}(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} \overset{r}{K}_{p-2}(\Omega_i\phi)\Big\}$$
$$+C(n,m,\delta,p,R)\int_{2\tau_1+R^*}^{2\tau_2+R^*}\mathrm{d}t \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{\psi^2 + \Big(\frac{\partial\psi}{\partial v^*}\Big)^2 + \sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\Omega_i\psi\Big)^2 + \sum_{i=1}^{\frac{n(n-1)}{2}}\big|\nabla\!\!\!/\,\Omega_i\psi\big|^2\Big\}\Big|_{r=R}\,. \tag{1.5.40}$$

Indeed, the first term $|\nabla\!\!\!/\,\partial_{v^*}\psi|^2$ can be integrated by parts twice (such that we can absorb the resulting $\partial_{v^*}\chi$ term in the left hand side):

$$\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{p-1}\big|\nabla\!\!\!/\,\chi\big|^2 =$$
$$= -\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{p-1} \triangle\!\!\!/\,\partial_{v^*}\psi\cdot\partial_{v^*}\psi$$
$$= -\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{p-1}\, \triangle\!\!\!/\,\psi\, \frac{\partial\psi}{\partial v^*}\Big|_{u^*+R^*}^{\infty}$$
$$+ \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{(p-1)r^{p-2}\big(1-\frac{2m}{r^{n-2}}\big)(\triangle\!\!\!/\,\psi)(\frac{\partial\psi}{\partial v^*})$$
$$+ r^{p-1}(\triangle\!\!\!/\,\psi)\frac{\partial\chi}{\partial v^*} + r^{p-1}\frac{2}{r}\big(1-\frac{2m}{r^{n-2}}\big)(\triangle\!\!\!/\,\psi)\frac{\partial\psi}{\partial v^*}\Big\} \leq$$
$$\leq \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{p-1}\,(\triangle\!\!\!/\,\psi)\,(\frac{\partial\psi}{\partial v^*})\Big|_{v^*=u^*+R^*}$$
$$+ \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{\Big(p-1+\frac{n-2}{p}+2\Big) r^{(p-2)-1}\Big(\triangle\!\!\!/\,\psi\Big)^2 r^2$$
$$+ (p-1+2) r^{(p-2)-1}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 + \frac{1}{2}\frac{p}{4}\frac{2}{n-2} r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2\Big\} \tag{1.5.41}$$

The second term in (1.5.38) is controlled by the Hardy inequality

$$\frac{1}{2}\int_{u^*+R^*}^{\infty} \mathrm{d}v^*\, r^{(p-2)-1}\frac{1}{r^2}\psi^2 \leq \frac{1}{4-p}\frac{1}{R^{4-p}}\frac{1}{1-\frac{2m}{R^{n-2}}}\psi^2|_{(u^*,u^*+R^*)}$$
$$+ \frac{2}{(4-p)^2(1-\frac{2m}{R^{n-2}})^2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\, r^{(p-2)-1}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2, \tag{1.5.42}$$

and the third term simply by the following commutation with Ω_i:

Lemma 1.40. *For any function $\phi \in \mathrm{H}^2(S_r)$ we have $\triangle\!\!\!/_{\,r^2\overset{\circ}{\gamma}_{n-1}}\phi \in \mathrm{L}^2(S_r)$, and there exists a constant $C > 0$ such that it holds*

$$\int_{\mathbb{S}^{n-1}} \Big(\triangle\!\!\!/\,\psi\Big)^2 r^2\, \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \leq C \int_{\mathbb{S}^{n-1}} \Big\{\sum_{i=1}^{\frac{n(n-1)}{2}}\Big|\nabla\!\!\!/(\Omega_i\psi)\Big|^2 + \Big|\nabla\!\!\!/\,\psi\Big|^2\Big\}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,. \tag{1.5.43}$$

The last term in (1.5.38) we can rearrange as follows:

$$
\begin{aligned}
-\frac{n-1}{8}r^{p-2}\Big[n-3+(n-1)\frac{2m}{r^{n-2}}\Big]\frac{\partial\chi^2}{\partial v^*} = \\
= -\frac{\partial}{\partial v^*}\Big\{\frac{n-1}{8}r^{p-2}\Big[(n-3)+(n-1)\frac{2m}{r^{n-2}}\Big]\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big\} \\
+\frac{n-1}{8}r^{(p-2)-1}\Big[(p-2)(n-3)+(n-1)\Big((p-2)+(n-2)\Big)\frac{2m}{r^{n-2}}\Big](1-\frac{2m}{r^{n-2}})\Big(\frac{\partial\psi}{\partial v^*}\Big)^2
\end{aligned}
\tag{1.5.44}
$$

Therefore (see also Appendix B.2)

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\times\frac{1}{8}p\,r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 \leq \\
\leq \frac{1}{2}\frac{1}{1-\frac{2m}{R^{n-2}}}\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}}\overset{v}{K}_p(\phi)\,\mathrm{d}\mu_g \\
+C(n,p,\delta,R)\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\Big\{r^{p-2}\Big(\mathring{\triangle}\psi\Big)^2r^2+r^{p-2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\psi^2\Big\}\big|_{v^*=u^*+R^*} \\
+C(p,n,\delta,R)\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\times \\
\times\Big\{r^{(p-2)-1}\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\mathring{\nabla}\Omega_i\psi\big|^2+r^{(p-2)-1}\big|\mathring{\nabla}\psi\big|^2+r^{(p-2)-1}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big\} \\
-\frac{n-1}{8}\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\times r^{p-2}\Big[n+3+(n-1)\frac{2m}{r^{n-2}}\Big]\frac{\partial\chi^2}{\partial v^*}.
\end{aligned}
\tag{1.5.45}
$$

Now, recall (1.5.15), and note that

$$
\delta-\big(n-(2-\delta)\big)\frac{2m}{r^{n-2}}>\frac{\delta}{2}\qquad(r>\sqrt[n-2]{\frac{4nm}{\delta}}),
\tag{1.5.46}
$$

to see that

$$
\begin{aligned}
r^{(p-2)-1}\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\mathring{\nabla}r^{\frac{n-1}{2}}\Omega_i\phi\big|^2 \leq \frac{4}{\delta}\sum_{i=1}^{\frac{n(n-1)}{2}}\overset{r}{K}_{p-2}(\Omega_i\phi)r^{n-1} \\
-\frac{4}{\delta}\sum_{i=1}^{\frac{n(n-1)}{2}}\frac{n-1}{4}\Big[\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}}\Big]r^{(p-2)-2}\frac{\partial\Big(r^{\frac{n-1}{2}}\Omega_i\phi\Big)^2}{\partial v^*}.
\end{aligned}
\tag{1.5.47}
$$

So

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\times p\,r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 \leq \\
\leq C(n,m,\delta,p,R)\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}}\Big\{\overset{v}{K}_p(\phi)+\overset{r}{K}_{p-2}(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}\overset{r}{K}_{p-2}(\Omega_i\phi)\Big\} \\
-C(n,m,\delta,p)\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\mathring{\gamma}_{n-1}}\times\frac{n-1}{8}\Big[n-3+(n-1)\frac{2m}{r^{n-2}}\Big]\times \\
\times\Big\{\frac{r^{p-2}}{r^2}\frac{\partial\psi^2}{\partial v^*}+\frac{r^{p-2}}{r^2}\sum_{i=1}^{\frac{n(n-1)}{2}}\frac{\partial\big(\Omega_i\psi\big)^2}{\partial v^*}+\frac{r^p}{r^2}\frac{\partial\chi^2}{\partial v^*}\Big\}
\end{aligned}
$$

$$+ C(n,m,\delta,p,R)\int_{2\tau_1+R^*}^{2\tau_2+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big\{\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\nabla\!\!\!/\,\Omega_i\phi\big|^2+\big|\nabla\!\!\!/\,\phi\big|^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\psi^2\Big\}\Big|_{r=R} \tag{1.5.48}$$

which upon integration by parts yields (1.5.40); note that the $\partial_{v^*}\psi^2$ and $\partial_{v^*}(\Omega_i\psi)^2$ terms generate boundary terms at infinity and zeroth order bulk terms *with the right sign* by (1.5.16) while the $\partial_{v^*}\chi^2$ is reduced to a $(\partial_{v^*}\psi)^2$ term by (1.5.44).

By virtue of Stokes' theorem (B.5) and in view of (B.6) we conclude

$$\begin{aligned}\int_{\tau_2+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^p\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+r^{p-2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^{p-2}\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_2}\\
+\int_{\tau_1}^{\tau_2}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times r^{p-1}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2\leq\\
\leq C(n,m,\delta,p,R)\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\\
\times\Big\{r^p\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+r^{p-2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^{p-2}\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}\\
+C(n,m,\delta,p,R)\int_{2\tau_1+R^*}^{2\tau_2+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2\\
+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big]+\big|\nabla\!\!\!/\,\chi\big|^2+\big|\nabla\!\!\!/\,\psi\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\nabla\!\!\!/\,\Omega_i\psi\big|^2\Big\}\Big|_{r=R}.\end{aligned} \tag{1.5.49}$$

Proof of Prop. 1.39. We shall use this *weighted energy inequality* for χ to proceed in a hierarchy of four steps.

$\boxed{p=4-\delta:}$ Let $\tau_1>0$, and $\tau_{j+1}=2\tau_j\quad(j\in\mathbb{N})$. In a first step we use (1.5.49) with $p=4-\delta$ and (1.5.17) with $p=2$ as an estimate for the spacetime integral of $\partial_{v^*}\chi$, $\partial_{v^*}\psi$, and $\partial_{v^*}(\Omega_j\psi)$ on ${}^R\mathcal{D}_{\tau_j}^{\tau_{j+1}}$, and in a second step as an estimate for the corresponding integral on the future boundary of ${}^R\mathcal{D}_{\tau_1}^{\tau_j}$:

$$\begin{aligned}\int_{\tau_j}^{\tau_{j+1}}\mathrm{d}u^*\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{3-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+r\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\leq\\
\leq C(n,m,\delta,R)\int_{\tau_j+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_j}\\
+C(n,m,\delta,R)\int_{2\tau_j+R^*}^{2\tau_{j+1}+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2\\
+\big|\nabla\!\!\!/\,\psi\big|^2+\big|\nabla\!\!\!/\,\frac{\partial\psi}{\partial v^*}\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/\,\Omega_i\psi\big|^2\Big]\Big\}\Big|_{r=R}\leq\end{aligned}$$

$$\leq C(n,m,\delta,R)\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+C(n,m,\delta,R)\int_{2\tau_1+R^*}^{2\tau_{j+1}+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2$$
$$+\big|\nabla\psi\big|^2+\big|\nabla\frac{\partial\psi}{\partial v^*}\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2+\big|\nabla\Omega_i\psi\big|^2\Big]\Big\}\Big|_{r=R}\quad(1.5.50)$$

Thus by the mean value theorem of integration we obtain a sequence $\tau_j'\in(\tau_j,\tau_{j+1})\quad(j\in\mathbb{N})$ such that the corresponding integral from the left hand side on $u^*=\tau_j'$ is bounded by $\tau_j^{-1}\times$ the right hand side of (1.5.50).

$\boxed{p=3-\delta:}$ Next we shall use (1.5.49) with $p=3-\delta$ on ${}^{R_j}\mathcal{D}_{\tau'_{2j-1}}^{\tau'_{2j+1}}$, (with $R_j^*\in(R^*,R^*+1)\quad(j\in\mathbb{N})$ chosen appropriately below). However, the quantity we are actually interested in is not $\partial_{v^*}\chi$, but rather

$$\Big(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\Big)^2=\Big(\frac{\partial T\cdot r^{\frac{n-1}{2}}\phi}{\partial v^*}\Big)^2=\Big(\frac{1}{2}\frac{\partial^2\psi}{\partial v^{*2}}+\frac{1}{2}\frac{\partial^2\psi}{\partial u^*\partial v^*}\Big)^2=$$
$$\overset{(1.5.12)}{=}\Big(\frac{1}{2}\frac{\partial^2\psi}{\partial v^{*2}}+\frac{1}{2}(1-\frac{2m}{r^{n-2}})\not\Delta\psi-\frac{1}{2}\frac{n-1}{2}(\frac{n-3}{2}+\frac{n-1}{2}\frac{2m}{r^{n-2}})\frac{1}{r^2}(1-\frac{2m}{r^{n-2}})\psi\Big)^2$$
$$\leq\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2+\Big(\not\Delta\psi\Big)^2+\frac{n-1}{4}2(n-2)\frac{1}{r^4}\psi^2.\quad(1.5.51)$$

Using the simple Hardy inequality

$$\frac{1}{2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\,r^{2-\delta}\frac{1}{r^4}\psi^2\leq$$
$$\leq\frac{1}{1-\frac{2m}{R^{n-2}}}\frac{1}{r^{1+\delta}}\psi^2(u^*,u^*+R^*)+\frac{2}{\big(1-\frac{2m}{R^{n-2}}\big)^2}\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\frac{1}{r^\delta}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\quad(1.5.52)$$

and again the commutation introduced in Lemma 1.40 we obtain

$$\int_{\tau'_{2j-1}}^{\tau'_{2j+1}}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times r^{2-\delta}\Big(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\Big)^2\leq$$
$$\leq\int_{\tau'_{2j-1}}^{\tau'_{2j+1}}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{2-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$
$$+\frac{C}{r^\delta}\sum_{i=1}^{\frac{n(n-1)}{2}}\big|\nabla\Omega_i\psi\big|^2+\frac{C}{r^\delta}\big|\nabla\psi\big|^2+\frac{(n-1)(n-2)}{2}\frac{2}{\big(1-\frac{2m}{R^{n-2}}\big)^2}r^{-\delta}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2\Big\}$$
$$+\frac{1}{1-\frac{2m}{R^{n-2}}}\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\frac{1}{r^{1+\delta}}\psi^2\Big\}\Big|_{r=R_j}\leq$$
$$\leq C(n,m,\delta)\int_{\tau'_{2j-1}}^{\tau'_{2j+1}}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{2-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$
$$+\overset{r}{K}_{1-\delta}(\phi)r^{n-1}+\sum_{i=1}^{\frac{n(n-1)}{2}}\overset{r}{K}_{1-\delta}(\Omega_i\phi)r^{n-1}\Big\}$$

$$+ C(n,m,\delta)\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\psi^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\Omega_i\psi\Big)^2\Big\}\Big|_{r=R_j} \quad (1.5.53)$$

where in the last step we have again used (1.5.16). Furthermore, by now applying (1.5.49) with $p=3-\delta$,

$$\int_{\tau'_{2j-1}}^{\tau'_{2j+1}}\mathrm{d}u^*\int_{u^*+R_j{}^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times r^{2-\delta}\Big(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\Big)^2\leq$$
$$\leq C(n,m,\delta,R)\int_{\tau'_{2j-1}+R_j^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{3-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$
$$+r\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\sum_{i=1}^{\frac{n(n-1)}{2}}r\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau'_{2j-1}}$$
$$+C(n,m,\delta,R)\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2$$
$$+\big|\nabla\!\!\!/\psi\big|^2+\big|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/\Omega_i\psi\big|^2\Big]\Big\}\Big|_{r=R_j}, \quad (1.5.54)$$

we obtain a sequence $\tau''_j\in(\tau'_{2j-1},\tau'_{2j+1})\quad(j\in\mathbb{N})$ such that in view of the previous step:

$$\int_{\tau''_j+R_j^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{2-\delta}\Big(\frac{\partial(r^{\frac{n-1}{2}}T\cdot\phi)}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau''_j}\leq$$
$$\leq\frac{C(n,m,\delta,R)}{\tau_{2j}\,\tau_{2j-1}}\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$
$$+r^2\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+r^2\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+\frac{C(n,m,\delta,R)}{\tau_{2j}\,\tau_{2j-1}}\int_{2\tau_1+R^*}^{2\tau_{2j+1}+R^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2$$
$$+\big|\nabla\!\!\!/\psi\big|^2+\big|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/\Omega_i\psi\big|^2\Big\}\Big|_{r=R}$$
$$+C(n,m,\delta,R)\int_{2\tau'_{2j-1}+R_j^*}^{2\tau'_{2j+1}+R_j^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2$$
$$+\big|\nabla\!\!\!/\psi\big|^2+\big|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/\Omega_i\psi\big|^2\Big\}\Big|_{r=R_j} \quad (1.5.55)$$

Now, by writing out the derivatives of $\psi=r^{\frac{n-1}{2}}\phi$, and using (1.5.12), we calculate that

$$\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big\{\psi^2+\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+\Big(\frac{\partial^2\psi}{\partial v^{*2}}\Big)^2$$
$$+\big|\nabla\!\!\!/\psi\big|^2+\big|\nabla\!\!\!/\frac{\partial\psi}{\partial v^*}\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\Big(\Omega_i\psi\Big)^2+\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/\Omega_i\psi\big|^2\Big]\Big\}\Big|_{r=R}\leq$$

$$\leq C(R)\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\Big\{\phi^2+\Big(\frac{\partial\phi}{\partial v^*}\Big)^2+\Big(\frac{\partial\phi}{\partial u^*}\Big)^2 + \Big(\frac{\partial T\cdot\phi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/\phi\big|^2+\sum_{i=1}^{\frac{n(n-1)}{2}}\Big[\big|\nabla\!\!\!/\Omega_i\phi\big|^2+\Big(\frac{\partial\Omega_i\phi}{\partial v^*}\Big)^2\Big]\Big\}\Big|_{r=R}; \quad (1.5.56)$$

by applying Prop. 1.11 first to the domain ${}^{r_1}\mathcal{D}_{\tau_1}^{\tau_{2j+1}}\subset\mathcal{R}^\infty_{r_0,r_1}(2\tau_1+r_1^*)$ where $r_1>\sqrt[n-2]{\frac{4nm}{\delta}}$ to fix the radius R and then to the domain ${}^{r(r^*=R^*+1)}\mathcal{D}_{\tau'_{2j-1}}^{\tau'_{2j+1}}\setminus{}^{R}\mathcal{D}_{\tau'_{2j-1}}^{\tau'_{2j+1}}\subset\mathcal{R}^\infty_{r_0,R}(2\tau'_{2j-1}+R^*)$ to fix the radii R_j ($j\in\mathbb{N}$) by using the mean value theorem for the integration in r^* this yields (see also Appendix B.2)

$$\int_{\tau''_j+R^*_j}^\infty \mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{2-\delta}\Big(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau''_j}\leq$$
$$\leq\frac{C(n,m,\delta,R)}{(\tau''_j)^2}\Big\{\int_{\tau_1+R^*}^\infty\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 + r^2\Big(\frac{\partial\psi}{\partial v^*}\Big)^2+r^2\sum_{i=1}^{\frac{n(n-1)}{2}}\Big(\frac{\partial\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\Big(J^T(\phi)+J^T(T\cdot\phi)+J^T(T^2\cdot\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}\big[J^T(\Omega_i\phi)+J^T(T\cdot\Omega_i\phi)\big],n\Big)\Big\}$$
$$+C(n,m,\delta,R)\int_{\Sigma_{\tau_{2j-1}}}\Big(J^T(\phi)+J^T(T\cdot\phi)+J^T(T^2\cdot\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}}\big[J^T(\Omega_i\phi)+J^T(T\cdot\Omega_i\phi)\big],n\Big). \quad (1.5.57)$$

Therefore, by Prop. 1.37:

$$\int_{\tau''_j+R^*_j}^\infty \mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{2-\delta}\Big(\frac{\partial r^{\frac{n-1}{2}}T\cdot\phi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau''_j}\leq$$
$$\leq\frac{C(n,m,\delta,R)}{(\tau''_j)^2}\Big\{\int_{\tau_1+R^*}^\infty\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2 + \sum_{k=0}^{3}r^2\Big(\frac{\partial T^k\cdot\psi}{\partial v^*}\Big)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\} \quad (1.5.58)$$

Remark 1.41. This statement should be compared to the assumptions of Prop. 1.37 (1.5.5), from which all that one can deduce with (1.5.17) is

$$\int_{\tau+R^*}^\infty\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^2\Big(\frac{\partial(r^{\frac{n-1}{2}}T\cdot\phi)}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau}<\infty \qquad (\tau>\tau_0). \quad (1.5.59)$$

We shall now proceed along the lines of the proof of Prop. 1.37 in Section 1.5.2, just that we have (1.5.58) as a starting point for the solution $T \cdot \phi$ of (1.1.1), (and (1.5.6)); however, as opposed to Prop. 1.37 the hierarchy does not descend from $p = 2$ but $p < 2$, which introduces a degeneracy in the last step, and requires the refinement of Prop. 1.11 to Prop. 1.32, and Prop. 1.14 to Prop. 1.33, see Section 1.4.4.

Lemma 1.42 (Pointwise decay under special assumptions)**.** *Let ϕ be a solution of the wave equation* (1.1.1)*, with initial data on Σ_{τ_1} ($\tau_1 > 0$) satisfying*

$$D \doteq \int_{\tau_1+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Big\{ r^{4-\delta}\Big(\frac{\partial \chi}{\partial v^*}\Big)^2 + \sum_{k=0}^{3} r^2 \Big(\frac{\partial T^k \psi}{\partial v^*}\Big)^2 + \sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2 \Big(\frac{\partial T^k \Omega_i \psi}{\partial v^*}\Big)^2 \Big\}\Big|_{u^*=\tau_1} + \int_{\Sigma_{\tau_1}} \Big(\sum_{k=0}^{4} J^N(T^k\cdot\phi) + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi), n\Big) < \infty$$

for some $\delta > 0$ and

$$\int_{\tau'+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times r^{2-\delta}\Big(\frac{\partial T\cdot\psi}{\partial v^*}\Big)^2\Big|_{u^*=\tau'} \leq \frac{C(n,m,\delta,R)\,D}{\tau'^2} \tag{$*$}$$

for some $\tau' > \tau_1$. Then there is a constant $C(n, m, \delta, R)$ such that

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1-\frac{\delta}{2}} \big(T\cdot\phi\big)^2\big|_{(u^*=\tau', v^*=R^*+\tau)} \leq \frac{C\,D}{\tau'^2}$$

for all $\tau > \tau'$.

Remark 1.43. Note the gain in powers of r in comparison to the boundary term arising in Prop. 1.33.

Proof. First, integrating from infinity,

$$(T\cdot\phi)(\tau', R^*+\tau') = -\int_{\tau'+R^*}^{\infty} \frac{\partial (T\cdot\phi)}{\partial v^*}\,\mathrm{d}v^*$$

and then by Cauchy's inequality,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} (T\cdot\phi)^2(\tau', R+\tau') &\leq \int_{R^*+\tau'}^{\infty} \frac{1}{r^{n-1}}\,\mathrm{d}v^* \times \int_{R+\tau'}^{\infty}\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big(\frac{\partial(T\cdot\phi)}{\partial v^*}\Big)^2 r^{n-1}\,\mathrm{d}v^* \\ &\leq \frac{1}{2}\Big(1 - \frac{2m}{r^{n-2}|_{(u^*=\tau', v^*=R^*+\tau')}}\Big)^{-1} \frac{1}{n-2}\frac{1}{r^{n-2}} \times \\ &\qquad \times C(m,n) \int_{R^*+\tau'}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1}\Big(\frac{\partial(T\cdot\phi)}{\partial v^*}\Big)^2 .\end{aligned}$$

Therefore, by Prop. 1.37,

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \bigl(r^{n-2}(T\cdot\phi)^2\bigr)(\tau',R^*+\tau') \leq \frac{C(n,m)}{1-\frac{2m}{R^{n-2}}}\int_{\Sigma_{\tau'}}\Bigl(J^T(T\cdot\phi),n\Bigr) \leq \frac{C(n,m,R)}{{\tau'}^2}\,D\,. \quad (**)$$

Now

$$\begin{aligned}
r^{n-1}\int_{\mathbb{S}^{n-1}} &\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} (T\cdot\phi)^2\,(\tau',R^*+\tau) = \\
&= \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \bigl(r^{n-1}(T\cdot\phi)^2\bigr)(\tau',R^*+\tau') + \int_{R^*+\tau'}^{R^*+\tau}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} 2T\cdot\psi\frac{\partial T\cdot\psi}{\partial v^*} \leq \\
&\leq R^{n-1}\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}(T\cdot\phi)^2(\tau',R^*+\tau') \\
&\quad + 2r^{\frac{\delta}{2}}\big|_{\substack{(u^*=\tau',\\ v^*=R^*+\tau)}}\sqrt{\int_{R^*+\tau'}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\frac{1}{r^2}(T\cdot\phi)^2r^{n-1}}\times \\
&\quad \times\sqrt{\int_{R^*+\tau'}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{2-\delta}\Bigl(\frac{\partial T\cdot\psi}{\partial v^*}\Bigr)^2}\,,
\end{aligned}$$

which proves the pointwise estimate of the Lemma in view of the Hardy inequality of Lemma B.6, Prop. 1.37, the assumption $(*)$ and $(**)$. $\square$

$\boxed{p=2-\delta:}$ By the weighted energy inequality with $p=2-\delta$ and $r^{\frac{n-1}{2}}T\cdot\phi$ in the role of ψ, see (1.5.16) in particular,

$$\begin{aligned}
\int_{\tau''_{2j-1}}^{\tau''_{2j+1}}&\mathrm{d}u^*\int_{u^*+R'^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times r^{1-\delta}\Bigl(\frac{\partial T\cdot\psi}{\partial v^*}\Bigr)^2 \leq \\
&\leq C(n,m)\int_{\tau''_{2j-1}}^{\tau''_{2j+1}}\mathrm{d}u^*\int_{u^*+R'^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\overset{r}{K}_{2-\delta}\,(T\cdot\phi) \\
&\quad + C(n,m)\int_{2\tau''_{2j-1}+R'^*_j}^{2\tau''_{2j+1}+R'^*_j}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Bigl\{\Bigl(T\cdot\psi\Bigr)^2\Bigr\}\Big|_{r=R'_j} \leq \\
&\leq C(n,m,R)\int_{\tau''_{2j-1}+R'^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Bigl\{r^{2-\delta}\Bigl(\frac{\partial T\cdot\psi}{\partial v^*}\Bigr)^2\Bigr\}\Big|_{u^*=\tau''_{2j-1}} \\
&\quad + C(n,m)\int_{2\tau''_{2j-1}+R'^*_j}^{2\tau''_{2j+1}+R'^*_j}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Bigl\{\Bigl(T\cdot\psi\Bigr)^2+\Bigl(\frac{\partial T\cdot\psi}{\partial v^*}\Bigr)^2+\bigl|\nabla\!\!\!/ T\cdot\psi\bigr|^2\Bigr\}\Big|_{r=R'_j}
\end{aligned} \tag{1.5.60}$$

where we choose $R'^*_j\in(R^*+1,R^*+2)$ such that Prop. 1.11 applied to the domain ${}^{r(r^*=R^*+2)}\mathcal{D}_{\tau''_{2j-1}}^{\tau''_{2j+1}}\setminus{}^{r(r^*=R^*+1)}\mathcal{D}_{\tau''_{2j-1}}^{\tau''_{2j+1}}$ yields an estimate for the integral on the timelike boundary above in terms of the first and second order energies on $\Sigma_{\tau''_{2j-1}}$ which in turn decays by Prop. 1.37. Therefore there exists a sequence $\tau'''_j\in(\tau''_{2j-1},\tau''_{2j+1})$ $(j\in\mathbb{N})$ such that

$$\int_{\tau'''_j+R'^*_j}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Bigl\{r^{1-\delta}\Bigl(\frac{\partial T\cdot\psi}{\partial v^*}\Bigr)^2\Bigr\}\Big|_{u^*=\tau'''_j}\leq$$

$$\leq \frac{C(n,m,\delta,R)}{(\tau_j''')^3}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$

$$+\sum_{k=0}^{3}r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$

$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\}. \quad (1.5.61)$$

$\boxed{p=1-\delta:}$ Since, by integrating by parts,

$$\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\frac{1}{r^\delta}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2=$$

$$=\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\frac{1}{r^\delta}\Big\{\frac{n-1}{2r}r^{\frac{n-1}{2}}\big(1-\frac{2m}{r^{n-2}}\big)\frac{\partial\big(r^{\frac{n-1}{2}}\phi^2\big)}{\partial v^*}+r^{n-1}\Big(\frac{\partial\phi}{\partial v^*}\Big)^2\Big\}$$

$$=\frac{1}{r^\delta}\frac{n-1}{2r}\big(1-\frac{2m}{r^{n-2}}\big)\psi^2|_{u^*+R^*}^{\infty}+\int_{u^*+R^*}^{\infty}\mathrm{d}v^*\Big\{\frac{\delta}{r^{1+\delta}}\frac{n-1}{2r}\big(1-\frac{2m}{r^{n-2}}\big)^2\psi^2$$

$$+\frac{1}{r^\delta}\frac{n-1}{2r^2}\big(1-\frac{2m}{r^{n-2}}\big)\psi^2\Big[(n-2)+\big(1-\frac{2m}{r^{n-2}}\big)\frac{n-3}{2}\Big]+\frac{1}{r^\delta}\Big(\frac{\partial\phi}{\partial v^*}\Big)^2r^{n-1}\Big\} \quad (1.5.62)$$

we have by (1.5.15) that also (with $R_j''^*\in(R^*+2,R^*+3)$),

$$\int_{\tau_{2j-1}'''}^{\tau_{2j+1}'''}\mathrm{d}u^*\int_{u^*+R_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\frac{1}{r^\delta}\Big\{\Big(\frac{\partial T\cdot\phi}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/ T\cdot\phi\big|^2\Big\}r^{n-1}\leq$$

$$\leq C(n,m)\Big\{\int_{\tau_{2j-1}'''}^{\tau_{2j+1}'''}\mathrm{d}u^*\int_{u^*+R_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\overset{r}{K}_{1-\delta}(T\cdot\phi)\cdot r^{n-1}$$

$$+\int_{2\tau_{2j-1}'''+R_j''^*}^{2\tau_{2j+1}'''+R_j''^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\Big(T\cdot\psi\Big)^2\Big\}\Big|_{r=R_j''}\Big\}. \quad (1.5.63)$$

By virtue of Stokes theorem (B.5), (B.6) and our previous result (1.5.61) we obtain

$$\int_{\tau_{2j-1}'''}^{\tau_{2j+1}'''}\mathrm{d}u^*\int_{u^*+R_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\frac{1}{r^\delta}\Big(J^T(T\cdot\phi),\frac{\partial}{\partial v^*}\Big)r^{n-1}\leq$$

$$\leq C(n,m)\Big\{\int_{\tau_{2j-1}'''+R_j''^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{1-\delta}\Big(\frac{\partial(T\cdot\psi)}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_{2j-1}'''}$$

$$+\int_{2\tau_{2j-1}'''+R_j''^*}^{2\tau_{2j+1}'''+R_j''^*}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\Big(\frac{\partial(T\cdot\psi)}{\partial v^*}\Big)^2+\big|\nabla\!\!\!/ T\cdot\psi\big|^2+\Big(T\cdot\psi\Big)^2\Big\}\Big|_{r=R_j''}\Big\}$$

$$\leq \frac{C(n,m,\delta,R)}{(\tau_j''')^3}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$

$$+\sum_{k=0}^{3}r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$

$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\}$$

$$+ C(n,m,R)\Big\{\int_{\Sigma^{\tau'''_{2j+1}}_{\tau''_{2(2j-1)-1}}}\Big(J^T(T\cdot\phi)+J^T(T^2\cdot\phi),n\Big)$$
$$+\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-2}\Big(T\cdot\phi\Big)^2\big|_{(u^*=\tau''_{2(2j-1)-1},\ v^*=R''^*_j+\tau'''_{2j+1})}\Big\}\,, \quad (1.5.64)$$

where in the last inequality we have used Prop. 1.33, and then chosen R''_j ($j\in\mathbb{N}$) suitably by Prop. 1.32; furthermore the inequality still holds if we add the integral of the nondegenerate energy on ${}^{R''_j}\mathcal{P}^{\tau'''_{2j+1}}_{\tau'''_{2j-1}}$ on the left hand side and replace J^T by J^N in the first term of the integral on $\Sigma^{\tau'''_{2j+1}}_{\tau''_{2(2j-1)-1}}$ on the right hand side. The last two terms on the right hand side of (1.5.64) in fact decay with almost the same rate as the first; for first note here that we could have used Prop. 1.14 and Cor. 1.13 instead, and then employ Prop. 1.37 to obtain in any case that

$$\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}\tau\int_{\Sigma_\tau}\frac{1}{r^\delta}\Big(J^N(T\cdot\phi),n\Big)\leq$$
$$\leq\frac{C(n,m,\delta,R)}{(\tau'''_{2j-1})^2}\Big\{\int_{\tau_1+R^*}^\infty\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$
$$+\sum_{k=0}^{3}r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\}\,. \quad (1.5.65)$$

It then follows that there exists a sequence $\tau''''_j\in(\tau'''_{2j-1},\tau'''_{2j+1})$ such that

$$\int_{\Sigma^{\tau'''_{2(j+2)+1}}_{\tau''''_j}}\Big(J^N(T\cdot\phi),n\Big)\leq r^\delta\big|_{(u^*=\tau''''_j,\ v^*=R''^*_{j+2}+\tau'''_{2(j+2)+1})}\int_{\Sigma_{\tau''''_j}}\frac{1}{r^\delta}\Big(J^N(T\cdot\phi),n\Big)\leq$$
$$\leq\frac{C(n,m,\delta,R)}{(\tau''''_j)^{3-\delta}}\Big\{\int_{\tau_1+R^*}^\infty\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$
$$+\sum_{k=0}^{3}r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\} \quad (1.5.66)$$

because $\tau''''_j(\tau'''_{2(j+2)+1}-\tau''''_j)^{-1}\leq 1$. And second the assumptions of Lemma 1.42 are satisfied in view of (1.5.58) on $u^*=\tau''_j$ ($j\in\mathbb{N}$) which yields

$$\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-2}\Big(T\cdot\phi\Big)^2\big|_{(u^*=\tau''_{2(2j-1)-1},v^*=R''^*_j+\tau'''_{2j+1})}\leq$$

$$\leq \frac{C(n,m,\delta,R)}{(\tau'''_{2j-1})^{3-\frac{\delta}{2}}}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2$$

$$+\sum_{k=0}^{3}r^2\Big(\frac{\partial T^k\cdot\psi}{\partial v^*}\Big)^2+\sum_{k=0}^{2}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$

$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{4}J^N(T^k\cdot\phi)+\sum_{k=0}^{3}J^N(T^k\Omega_i\phi),n\Big)\Big\} \quad (1.5.67)$$

because also $\tau'''_{2j-1}(\tau'''_{2j+1}-\tau''_{2(2j-1)-1})^{-1}\leq C$. We shall now return to (1.5.64) (and its extension that includes the nondegenerate energy on ${}^{R''_j}\mathcal{P}^{\tau'''_{2j+1}}_{\tau'''_{2j-1}}$) to find that, after inserting (1.5.66) and using Prop. 1.35,

$$\int_{\Sigma^{\tau'''_{2j+1}}_{\tau''_{2(2j-1)-1}}}\Big(J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\Big)\leq C\int_{\Sigma^{\tau'''_{2j+1}}_{\tau'''_{(2j-1)-1}}}\Big(J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\Big)\leq$$

$$\leq C\int_{\Sigma^{\tau'''_{2j+1}}_{\tau''''_{j-2}}}\Big(J^N(T\cdot\phi)+J^T(T^2\cdot\phi),n\Big)\leq$$

$$\leq\frac{C(n,m,\delta,R)}{(\tau''''_{j-2})^{3-\delta}}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v^*}\Big)^2+r^{4-\delta}\Big(\frac{\partial(T\cdot\chi)}{\partial v^*}\Big)^2$$

$$+\sum_{k=0}^{4}r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$

$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{5}J^N(T^k\cdot\phi)+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\} \quad (1.5.68)$$

and using (1.5.67), that there exists (another) sequence $\tau''''_j\in(\tau'''_{2j-1},\tau'''_{2j+1})$ ($j\in\mathbb{N}$) such that

$$\int_{\Sigma_{\tau''''_j}}\frac{1}{r^\delta}\Big(J^N(T\cdot\phi),n\Big)\leq$$

$$\leq\frac{C(n,m,\delta,R)}{(\tau''''_j)^{4-\delta}}\Big\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\times\Big\{\sum_{k=0}^{1}r^{4-\delta}\Big(\frac{\partial(T^k\cdot\chi)}{\partial v^*}\Big)^2$$

$$+\sum_{k=0}^{4}r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2+\sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}}r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_1}$$

$$+\int_{\Sigma_{\tau_1}}\Big(\sum_{k=0}^{5}J^N(T^k\cdot\phi)+\sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(T^k\Omega_i\phi),n\Big)\Big\}. \quad (1.5.69)$$

So for any $\tau>\tau_1$ we can choose $j\in\mathbb{N}$ such that $\tau\in(\tau'''_{2j-1},\tau'''_{2j+1})$ to obtain finally by Prop. 1.35 that

$$\int_{\Sigma_\tau\cap\{r\leq R\}}\Big(J^N(T\cdot\phi),n\Big)\leq$$

$$\leq \int_{\Sigma_{\tau''''_{j-1}}^{\tau'''_{2j+1}}} \Bigl(J^N(T\cdot\phi),n\Bigr) \leq r^\delta\big|_{(u^*=\tau''''_{j-1},\, v^*=R^*+\tau'''_{2j+1})} \int_{\Sigma_{\tau''''_{j-1}}} \frac{1}{r^\delta}\Bigl(J^N(T\cdot\phi),n\Bigr)$$

$$\leq \frac{C(n,m,\delta,R)}{\tau^{4-2\delta}}\Biggl\{\int_{\tau_1+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Biggl\{\sum_{k=0}^{1} r^{4-\delta}\Bigl(\frac{\partial(T^k\cdot\chi)}{\partial v^*}\Bigr)^2$$

$$+\sum_{k=0}^{4} r^2\Bigl(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Bigr)^2 + \sum_{k=0}^{3}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Bigl(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Bigr)^2\Biggr\}\Big|_{u^*=\tau_1}$$

$$+\int_{\Sigma_{\tau_1}}\Bigl(\sum_{k=0}^{5} J^N(T^k\cdot\phi) + \sum_{k=0}^{4}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi),n\Bigr)\Biggr\}. \quad (1.5.70)$$

□

Remark 1.44. Note that for the removal of the restriction to dyadic sequences in the last step of the proof, (1.5.69) - (1.5.70), we could have equally obtained a decay estimate for the energy flux through $\Sigma_\tau \cap \{r^* \leq R^* + \tau^k\}$ (with $k \in \mathbb{N}$) by replacing $\Sigma_{\tau''''_{j-1}}^{\tau'''_{2j+1}}$ by $\Sigma_{\tau''''_{j-1}}^{\tau''''_{j-1}+\tau^k}$ in the first estimate in (1.5.70); if $\delta > 0$ for a chosen $k \in \mathbb{N}$ is restricted to $\delta < (1+k)^{-1}$ we then still obtain a decay rate of $\tau^{4-(1+k)\delta}$ for the energy flux through $\Sigma_\tau \cap \{r^* \leq R^* + \tau^k\}$.

1.5.4 Digression: Conformal energy decay

In this section we present an alternative proof of energy decay using the conformal Morawetz vectorfield. This is the conventional way to prove energy decay — it was first applied in the context of the Schwarzschild spacetime in [20] —, but is now superseded by the physical space approach [22] applied in Section 1.5.2. In particular it has the disadvantage of introducing weights in t into the argument, and makes the dependence on the initial data less transparent.

Let $\tilde{\Sigma}$ be a spacelike hypersurface in $J^+(u^*=1, v^*=1)$ (see figure 1.8). And

$$\tilde{\Sigma}_\tau \doteq \varphi_\tau(\tilde{\Sigma} \cap \mathscr{D}).$$

Proposition 1.45 (Conformal energy decay)**.** *Let ϕ be a solution of the wave equation* (1.1.1) *with initial data on the spacelike hypersurface $t=0$ such that*

$$D^{(Z)} \doteq \int_{\overline{\Sigma}_0}\Bigl(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i\phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i\Omega_j\phi) + \sum_{i,j,k=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\Omega_j\Omega_k\phi),n\Bigr)$$
$$+\int_{\overline{\Sigma}_0}\Bigl(J^{Z,1}(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i\phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i\Omega_j\phi),n\Bigr) < \infty\,,$$

and let $\tilde{\Sigma}$ be chosen as above. Then there exists a constant $C(n,m)$ such that

$$\int_{\tilde{\Sigma}_\tau}\Bigl(J^N(\phi),n\Bigr) \leq \frac{C\,D}{\tau^2} \qquad (\tau > 0)\,.$$

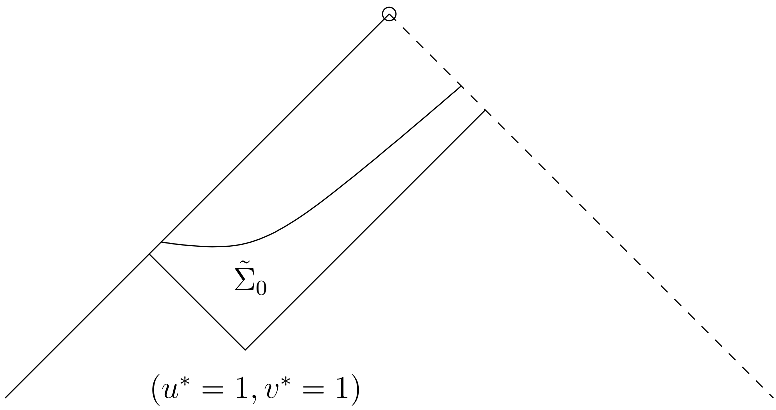

Figure 1.8: A spacelike hypersurface $\tilde{\Sigma}_0$.

Morawetz vectorfield Z. In Minkowski space

$$Z = u^{*2}\frac{\partial}{\partial u^*} + v^{*2}\frac{\partial}{\partial v^*} \tag{1.5.71}$$

is a conformal Killing vectorfield which has the weights that allow us to prove energy decay. Here the trace-free part of the deformation tensor

$$^{(Z)}\hat{\pi}_{\mu\nu} = {}^{(Z)}\pi_{\mu\nu} - \frac{1}{n+1}g_{\mu\nu}\,\mathrm{tr}\,^{(Z)}\pi$$

does not vanish, but we may still write:

$$\nabla^\mu J^Z_\mu = {}^{(Z)}\hat{\pi}^{\mu\nu}T_{\mu\nu} + \frac{1}{n+1}\ \mathrm{tr}\,^{(Z)}\pi\ \mathrm{tr}\,T\,. \tag{1.5.72}$$

Calculation of $^{(Z)}\pi$. First,

$$\begin{aligned}
&{}^{(Z)}\pi_{u^*u^*} = 0 \qquad {}^{(Z)}\pi_{v^*v^*} = 0\\
&{}^{(Z)}\pi_{u^*v^*} = -2t\left(1 - \frac{2m}{r^{n-2}}\right) - (n-2)\,t\,r^*\,\frac{2m}{r^{n-1}}\left(1-\frac{2m}{r^{n-2}}\right)\\
&{}^{(Z)}\pi_{AB} = \frac{tr^*}{r}\left(1-\frac{2m}{r^{n-2}}\right)r^2(\mathring{\gamma}_{n-1})_{AB}\,.
\end{aligned}$$

Second, with

$$\mathrm{tr}\,^{(Z)}\pi = 2t + (n-2)tr^*\frac{2m}{r^{n-1}} + (n-1)\frac{t\,r^*}{r}\left(1-\frac{2m}{r^{n-2}}\right)$$

and denoting by

$$^{(Z)}\tilde{\pi} = 2t\Big(\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)-1\Big) - (n-2)tr^*\frac{2m}{r^{n-1}} \tag{1.5.73}$$

we find

$$\begin{aligned}
{}^{(Z)}\hat{\pi}_{ab} &= -\frac{1}{2}\frac{n-1}{n+1}{}^{(Z)}\tilde{\pi}\,g_{ab} \quad (a,b = u^*, v^*)\\
{}^{(Z)}\hat{\pi}_{AB} &= \frac{1}{n+1}{}^{(Z)}\tilde{\pi}\,g_{AB} \quad (A,B = 1,\dots,n-1)\,.
\end{aligned}$$

In the above

$$\operatorname{tr} T = -\frac{n-1}{4}\Box(\phi^2)$$

so we are lead to consider

$$\nabla^\mu\Big(J^Z_\mu + \frac{1}{4}\frac{n-1}{n+1}\operatorname{tr}{}^{(Z)}\pi\,\partial_\mu\big(\phi^2\big) - \frac{1}{4}\frac{n-1}{n+1}\partial_\mu\big(\operatorname{tr}{}^{(Z)}\pi\big)\,\phi^2\Big) = \\ = {}^{(Z)}\hat{\pi}^{\mu\nu}T_{\mu\nu} - \frac{1}{4}\frac{n-1}{n+1}\Box\big(\operatorname{tr}{}^{(Z)}\pi\big)\,\phi^2\,.$$

Now

$${}^{(Z)}\hat{\pi}^{\mu\nu}T_{\mu\nu} = \frac{1}{2}\frac{n-1}{n+1}\frac{{}^{(Z)}\tilde{\pi}}{1-\frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial u^*}\big)\big(\frac{\partial\phi}{\partial v^*}\big) + \frac{1}{n+1}{}^{(Z)}\tilde{\pi}\,\big|\nabla\!\!\!/\phi\big|^2_{r^2\mathring{\gamma}_{n-1}}$$

and thus we see

$${}^{(Z)}\hat{\pi}^{\mu\nu}T_{\mu\nu} + \frac{1}{2}\frac{n-1}{n+1}{}^{(Z)}\tilde{\pi}\,\partial^\alpha\phi\,\partial_\alpha\phi = \frac{1}{2}{}^{(Z)}\tilde{\pi}\,\big|\nabla\!\!\!/\phi\big|^2_{r^2\mathring{\gamma}_{n-1}}\,.$$

So if we define

$$J^{Z,1}_\mu \doteq J^Z_\mu + \frac{1}{4}\frac{n-1}{n+1}\Big(\operatorname{tr}{}^{(Z)}\pi + {}^{(Z)}\tilde{\pi}\Big)\partial_\mu\big(\phi^2\big) - \frac{1}{4}\frac{n-1}{n+1}\partial_\mu\Big(\operatorname{tr}{}^{(Z)}\pi + {}^{(Z)}\tilde{\pi}\Big)\phi^2$$

then

$$\nabla^\mu J^{Z,1}_\mu = \frac{1}{2}{}^{(Z)}\tilde{\pi}\,\big|\nabla\!\!\!/\phi\big|^2_{r^2\mathring{\gamma}_{n-1}} - \frac{1}{4}\frac{n-1}{n+1}\Box\Big(\operatorname{tr}{}^{(Z)}\pi + {}^{(Z)}\tilde{\pi}\Big)\,\phi^2\,.$$

Note that finally

$$\operatorname{tr}{}^{(Z)}\pi + {}^{(Z)}\tilde{\pi} = (n+1)\frac{tr^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)$$

and

$$\Box\Big(\frac{tr^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big) = \\ = \Big[-\frac{1}{1-\frac{2m}{r^{n-2}}}\big(\frac{\partial}{\partial t}-\frac{\partial}{\partial r^*}\big)\big(\frac{\partial}{\partial t}+\frac{\partial}{\partial r^*}\big) - \frac{n-1}{2r}\big(-2\frac{\partial}{\partial r^*}\big)\Big]\Big(\frac{tr^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big) \\ = -(n-3)\frac{t}{r^2}\big(1-\frac{2m}{r^{n-2}}\big)\Big(\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)-1\Big) - (n-2)\frac{t}{r}\frac{2m}{r^{n-1}}\Big(3\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)-2\Big) \\ + (n-2)^2\big(\frac{2m}{r^{n-1}}\big)^2\frac{tr^*}{r}\,.$$

We conclude with

$$J^{Z,1}_\mu = J^Z_\mu + \frac{n-1}{4}\frac{t\,r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\partial_\mu\big(\phi^2\big) - \frac{n-1}{4}\partial_\mu\Big(\frac{t\,r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\Big)\phi^2 \tag{1.5.74}$$

we have

$$K^{Z,1} = \frac{1}{2}\Big[2t\Big(\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)-1\Big) - (n-2)t\,r^*\frac{2m}{r^{n-1}}\Big]\big|\nabla\!\!\!/\phi\big|^2_{r^2\mathring{\gamma}_{n-1}} \\ + \Big\{\frac{(n-3)(n-1)}{4}\frac{t}{r^2}\big(1-\frac{2m}{r^{n-2}}\big)\Big(\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)-1\Big) \\ + \frac{(n-2)(n-1)}{4}\frac{t}{r}\frac{2m}{r^{n-1}}\Big(3\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)-2\Big) \\ - \frac{(n-1)(n-2)^2}{4}\big(\frac{2m}{r^{n-1}}\big)^2\frac{t\,r^*}{r}\Big\}\phi^2\,. \tag{1.5.75}$$

Remark 1.46 (Minkowski space). If we set $m = 0$ and $r^* = r$ then $K^{Z,1} = 0$.

While $K^{Z,1}$ does not vanish identically the following Lemma shows that at least near the horizon and at infinity it has a sign.

Lemma 1.47. *There are constant values of the radius*

$$\sqrt[n-2]{2m} < r_0(n,m) < R(n,m) < \infty$$

such that

$$K^{Z,1} \geq 0 \quad (r \leq r_0) \tag{1.5.76}$$

$$\text{and} \quad K^{Z,1} \geq \frac{n-1}{8}\frac{2mt}{r^n}\phi^2 \quad (r \geq R)\,. \tag{1.5.77}$$

Proof. A suitable rearrangement of the terms in (1.5.75) is in powers of $\frac{2m}{r^{n-2}}$:

$$\begin{aligned} K^{Z,1} &= -t\Big[1 + \frac{r^*}{2r}\big(n\frac{2m}{r^{n-2}} - 2\big)\Big]|\nabla\phi|^2 \\ &- \frac{n-1}{4}\frac{t}{r^2}\Big[(n-1)\frac{2m}{r^{n-2}} + (n-3) + \frac{r^*}{r}\Big((n^2+1)\big(\frac{2m}{r^{n-2}}\big)^2 - n\frac{2m}{r^{n-2}} - (n-3)\Big)\Big]\phi^2 \end{aligned} \tag{1.5.78}$$

(1.5.76) is clear by inspection as $r^* \to -\infty$ ($r \to \sqrt[n-2]{2m}$). For (1.5.77) note that the coefficient to ϕ^2 tends to

$$-\frac{n-1}{4}\frac{t}{r^2}\Big[-\frac{2m}{r^{n-2}} + (n^2+1)\big(\frac{2m}{r^{n-2}}\big)^2\Big] \searrow 0 \quad (r \to \infty)\,.$$

It remains to be shown the nonnegativity of the coefficient to $|\nabla\phi|^2$ in the asymptotic region $r \to \infty$. Here we insert

$$r^* = r - (nm)^{\frac{1}{n-2}} + (2m)^{\frac{1}{n-2}} \int_{(\frac{n}{2})^{\frac{1}{n-2}}}^{\frac{r}{\sqrt[n-2]{2m}}} \frac{1}{x^{n-2}-1}\,\mathrm{d}x$$

to obtain the limit

$$-t\Big[1 + \frac{r^*}{2r}\Big(n\frac{2m}{r^{n-2}} - 2\Big)\Big] \overset{(r\to\infty)}{\longrightarrow} \lim_{y\to\infty}\frac{t}{y}\int_{(\frac{n}{2})^{\frac{1}{n-2}}}^{y}\frac{1}{x^{n-2}-1}\,\mathrm{d}x$$

which approaches zero from above. $\square$

The next proposition shows that the current $J^{Z,1}$ gives rise to nonnegative boundary terms on t-const surfaces with the desired weights.

Proposition 1.48. *For* $n \geq 4$,

$$\begin{aligned} \int_{\mathbb{R}} (J^{Z,1}, T)\, r^{n-1}\,\mathrm{d}r^* \geq \int_{\mathbb{R}} \frac{1}{4}\Big\{\frac{1}{5}\Big(u^{*2}\big(\frac{\partial\phi}{\partial u^*}\big)^2 + v^{*2}\big(\frac{\partial\phi}{\partial v^*}\big)^2\Big) \\ + \big(t^2 + r^{*2}\big)\big(1 - \frac{2m}{r^{n-2}}\big)|\nabla\phi|^2_{r^2\mathring{\gamma}_{n-1}} + \frac{n-1}{2}\frac{3n-10}{7}\big(1 - \frac{2m}{r^{n-2}}\big)\frac{r^{*2}+t^2}{r^2}\phi^2\Big\}\, r^{n-1}\,\mathrm{d}r^*\,. \end{aligned}$$

Proof. We compute

$$(J^Z,T)=\frac{1}{2}u^{*2}\Big(\frac{\partial\phi}{\partial u^*}\Big)^2+\frac{1}{2}v^{*2}\Big(\frac{\partial\phi}{\partial v^*}\Big)^2+\frac{1}{2}\big(u^{*2}+v^{*2}\big)\big(1-\frac{2m}{r^{n-2}}\big)\big|\nabla\!\!\!/\phi\big|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\,.$$

Consider the vectorfields

$$\begin{aligned}S&=v^*\frac{\partial}{\partial v^*}+u^*\frac{\partial}{\partial u^*}=t\frac{\partial}{\partial t}+r^*\frac{\partial}{\partial r^*}\\ \underline{S}&=v^*\frac{\partial}{\partial v^*}-u^*\frac{\partial}{\partial u^*}=t\frac{\partial}{\partial r^*}+r^*\frac{\partial}{\partial t}\,.\end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{R}}(J^{Z,1},T)\,r^{n-1}\,\mathrm{d}r^*=\int_{\mathbb{R}}\Big\{&(J^Z,T)+\frac{n-1}{4}\frac{tr^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)2\phi\frac{\partial\phi}{\partial t}\\ &-\frac{n-1}{4}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\phi^2\Big\}\,r^{n-1}\,\mathrm{d}r^*\\ =\int_{\mathbb{R}}\frac{1}{4}\Big\{&(S\cdot\phi)^2+(\underline{S}\cdot\phi)^2+(t^2+r^{*2})\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\phi|^2\\ &+\frac{n-1}{2}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\,2\phi(S\cdot\phi)\\ &-\frac{n-1}{2}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\,2\phi\,r^*\frac{\partial\phi}{\partial r^*}\\ &+\frac{n-1}{2}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\,2\phi\,\frac{t}{r^*}\big(\underline{S}\cdot\phi\big)\\ &-\frac{n-1}{2}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)2\phi\,\frac{t^2}{r^*}\frac{\partial\phi}{\partial r^*}\\ &-(n-1)\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)\,\phi^2\Big\}\,r^{n-1}\,\mathrm{d}r^*\,.\end{aligned}$$

We can integrate by parts

$$\begin{aligned}-\int_{\mathbb{R}}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)2\phi\,r^*\frac{\partial\phi}{\partial r^*}\,r^{n-1}\,\mathrm{d}r^*&=\int_{\mathbb{R}}\frac{\partial}{\partial r^*}\Big(r^{*2}r^{n-2}\big(1-\frac{2m}{r^{n-2}}\big)\Big)\phi^2\,\mathrm{d}r^*\\ &=\int_{\mathbb{R}}\big(1-\frac{2m}{r^{n-2}}\big)\Big\{2\frac{r^*}{r}+(n-2)\big(\frac{r^*}{r}\big)^2\Big\}r^{n-1}\phi^2\,\mathrm{d}r^*\end{aligned}$$

and

$$\begin{aligned}-\int_{\mathbb{R}}\frac{r^*}{r}\big(1-\frac{2m}{r^{n-2}}\big)2\phi\,\frac{t^2}{r^*}\frac{\partial\phi}{\partial r^*}\,r^{n-1}\,\mathrm{d}r^*&=\int_{\mathbb{R}}\frac{\partial}{\partial r^*}\Big(r^{n-2}t^2\big(1-\frac{2m}{r^{n-2}}\big)\Big)\phi^2\,\mathrm{d}r^*\\ &=\int_{\mathbb{R}}\big(1-\frac{2m}{r^{n-2}}\big)(n-2)\frac{t^2}{r^2}\phi^2\,r^{n-1}\,\mathrm{d}r^*\end{aligned}$$

to obtain

$$\int_{\mathbb{R}}(J^{Z,1},T)\,r^{n-1}\,\mathrm{d}r^*=$$

$$= \int_{\mathbb{R}} \frac{1}{4} \Big\{ \frac{2m}{r^{n-2}} \Big[(S\cdot\phi)^2 + (\underline{S}\cdot\phi)^2 \Big] + \big(t^2 + r^{*2}\big)\big(1 - \frac{2m}{r^{n-2}}\big) |\nabla\phi|^2$$
$$+ \big(1 - \frac{2m}{r^{n-2}}\big) \Big[(S\cdot\phi)^2 + (n-1)(S\cdot\phi)\frac{r^*}{r}\phi + \frac{n-1}{2}(n-2)\frac{r^{*2}}{r^2}\phi^2$$
$$+ (\underline{S}\cdot\phi)^2 + (n-1)(\underline{S}\cdot\phi)\frac{t}{r}\phi + \frac{n-1}{2}(n-2)\frac{t^2}{r^2}\phi^2 \Big] \Big\} r^{n-1}\,\mathrm{d}r^* \,.$$

By Cauchy's inequality

$$(S\cdot\phi)^2 + (n-1)(S\cdot\phi)\frac{r^*}{r}\phi + \frac{n-1}{2}(n-2)\frac{r^{*2}}{r^2}\phi^2 \geq$$
$$\geq (1-2\epsilon)(S\cdot\phi)^2 + \frac{n-1}{2}\Big((n-2) - \frac{1}{2\epsilon}\frac{n-1}{2}\Big)\frac{r^{*2}}{r^2}\phi^2 \,; \quad (1.5.79)$$

here we choose $\epsilon = \frac{7}{16}$. So

$$\int_{\mathbb{R}} (J^{Z,1}, T)\, r^{n-1}\,\mathrm{d}r^* \geq$$
$$\geq \int_{\mathbb{R}} \frac{1}{4} \Big\{ \frac{2m}{r^{n-2}} \Big((S\cdot\phi)^2 + (\underline{S}\cdot\phi)^2 \Big) + \big(t^2 + r^{*2}\big)\big(1 - \frac{2m}{r^{n-2}}\big)|\nabla\phi|^2$$
$$+ \big(1 - \frac{2m}{r^{n-2}}\big)\frac{1}{8}\Big[(S\cdot\phi)^2 + (\underline{S}\cdot\phi)^2 + \frac{n-1}{2}\frac{8}{7}(3n-10)\frac{r^{*2}+t^2}{r^2}\phi^2 \Big] \Big\} r^{n-1}\,\mathrm{d}r^*$$

and the stated inequality follows. □

Remark 1.49 (Case $n=3$ in Prop 1.48). If we set $\epsilon = \frac{1}{2}$ in (1.5.79) we merely obtain in the $n=3$-dimensional case

$$\int_{\mathbb{R}} (J^{Z,1}, T)\, r^2\,\mathrm{d}r^* \geq$$
$$\geq \int_{\mathbb{R}} \frac{1}{4} \Big\{ 2\frac{2m}{r} \Big(u^{*2}\big(\frac{\partial\phi}{\partial u^*}\big)^2 + v^{*2}\big(\frac{\partial\phi}{\partial v^*}\big)^2 \Big) + \big(t^2 + r^{*2}\big)\big(1 - \frac{2m}{r^{n-2}}\big)|\nabla\phi|^2 \Big\} r^2\,\mathrm{d}r^* \,.$$

The degeneracy at infinity can however be removed by splitting the $t\frac{\partial\phi}{\partial t}$-term differently before the integration by parts. (See [20] Prop 10.10.)

The last fact related to the Morawetz vectorfield is that the bulk term in a t-const slab can be controlled with the $J^{(\alpha)}$ current (see Cor. 1.29) in a finite annular region.

Proposition 1.50. *For $n \geq 3$, there are constant radial values*

$$\sqrt[n-2]{2m} < r_0(n,m) < r_1(n,m) < \infty$$

and a constant $C(n,m) > 0$ such that

$$-\int_{\mathbb{R}} K^{Z,1}\, r^{n-1}\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^* \leq$$
$$\leq C(n,m)\, t \int_{r_0^*}^{r_1^*} \Big\{ \frac{1}{r^n}\big(\frac{\partial\phi}{\partial r^*}\big)^2 + \frac{r^2}{(1-\frac{2m}{r^{n-2}})(1+r^{*2})^2}|\nabla\phi|^2 \Big\} r^{n-1}\big(1 - \frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^* \,.$$

Lemma 1.51. *Let $\phi \in \mathrm{C}^1(\mathbb{R})$ with $\frac{1}{\sqrt{x}}\phi(x), \frac{\mathrm{d}\phi}{\mathrm{d}x}(x) \in \mathrm{L}^2_{loc}(\mathbb{R})$, and let $a, b \in \mathbb{R}$ with $a < b$. Then for any $\varepsilon > 0$, there exist $c(\varepsilon, b, a) > b, C(\varepsilon, b, a) > 0$ such that*

$$\int_a^b \phi^2 \,\mathrm{d}x \leq \varepsilon \int_b^c \frac{1}{(y-a)+(b-a)}\phi^2(y)\,\mathrm{d}y + C\int_a^c \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 \mathrm{d}x\,.$$

Proof of the Lemma. Take $x \in [1,2], y \in [2,\infty)$ then

$$\phi^2(x) - \phi^2(y) \leq 2\int_x^y |\phi\frac{\mathrm{d}\phi}{\mathrm{d}z}|\,\mathrm{d}z \leq 2\Big(\int_x^y \frac{1}{z}\phi^2(z)\,\mathrm{d}z\Big)^{\frac{1}{2}}\Big(\int_x^y z\big(\frac{\mathrm{d}\phi}{\mathrm{d}z}\big)^2\,\mathrm{d}z\Big)^{\frac{1}{2}}.$$

Now apply Cauchy's inequality and integrate over $x \in [1,2]$:

$$\int_1^2 \phi^2(x)\,\mathrm{d}x \leq 2\epsilon\int_1^y \frac{1}{z}\phi^2(z)\,\mathrm{d}z + \frac{1}{2\epsilon}\int_1^y z\big(\frac{\mathrm{d}\phi}{\mathrm{d}z}\big)^2\,\mathrm{d}z + \phi^2(y)$$

Next divide by y, choose $d < (\log 2)^{-1}$ and integrate over $y \in [2, e^{\frac{1}{d}}]$:

$$\int_2^{e^{\frac{1}{d}}} \frac{1}{y}\,\mathrm{d}y \int_1^2 \phi^2(x)\,\mathrm{d}x \leq$$
$$\leq \epsilon\int_1^{e^{\frac{1}{d}}} \frac{1}{x}\phi^2(x)\,\mathrm{d}x + \frac{1}{4\epsilon}e^{\frac{2}{d}}\int_1^{e^{\frac{1}{d}}} x\big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x + \int_2^{e^{\frac{1}{d}}} \frac{1}{y}\phi^2(y)\,\mathrm{d}y$$

where we have replaced $\epsilon e^{\frac{1}{d}}$ by ϵ. Writing

$$\int_1^{e^{\frac{1}{d}}} \frac{1}{x}\phi^2(x)\,\mathrm{d}x \leq \int_1^2 \phi^2(x)\,\mathrm{d}x + \int_2^{e^{\frac{1}{d}}} \frac{1}{x}\phi^2(x)\,\mathrm{d}x$$

and absorbing the first term in the left hand side we obtain with a choice of

$$d \leq \frac{1}{4\log 2 + 2} \qquad \epsilon \leq \frac{1}{2}$$

$$\int_1^2 \phi^2(x)\,\mathrm{d}x \leq 2d\int_2^{e^{\frac{1}{d}}} \frac{1}{y}\phi^2(y)\,\mathrm{d}y + \frac{de^{\frac{3}{d}}}{3}\int_1^{e^{\frac{1}{d}}} \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x\,.$$

Hence

$$\int_a^b \phi^2(x)\,\mathrm{d}x = \int_1^2 \phi^2(t(b-a)+2a-b)(b-a)\,\mathrm{d}t \leq$$
$$\leq 2d(b-a)\int_b^{e^{\frac{1}{d}}(b-a)+2a-b} \frac{1}{(y-a)+(b-a)}\phi^2(y)\,\mathrm{d}y$$
$$+ \frac{de^{\frac{3}{d}}}{3}(b-a)^2\int_a^{e^{\frac{1}{d}}(b-a)+2a-b} \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x\,.$$

So given $0 < \varepsilon < 2(b-a)(4\log 2 + 2)$ we choose $d = \varepsilon\,(2(b-a))^{-1}$ and then have in fact

$$c = (b-a)e^{\frac{2(b-a)}{\varepsilon}} + 2a - b \qquad C = \frac{\varepsilon(b-a)}{6}e^{\frac{6(b-a)}{\varepsilon}}\,. \qquad \square$$

Proof. In view of Lemma 1.47 and (1.5.78) we have

$$\int_{\mathbb{R}} K^{Z,1}\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^* \geq$$
$$\geq -B(n,m)\, t \int_{r_0^*}^{R^*} \Big\{|\nabla\!\!\!/\phi|^2+\phi^2\Big\}\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^*$$
$$+\frac{n-1}{8}2mt\int_{R^*}^{\infty}\frac{1}{r^n}\phi^2\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^*$$

for some constant $B(n,m)$. Using the Lemma we infer that

$$\int_{r_0^*}^{R^*}\phi^2\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^* \leq$$
$$\leq R^{n-1}\big(1-\frac{2m}{R^{n-2}}\big)\,\varepsilon\int_{R^*}^{r_1^*}\frac{r}{(r^*-r_0^*)+(R^*-r_0^*)}\frac{1}{r}\phi^2\,\mathrm{d}r^* + R^{n-1}C\int_{r_0^*}^{r_1^*}\big(\frac{\partial\phi}{\partial r^*}\big)^2\,\mathrm{d}r^*$$

where $r_1 = r_1(\varepsilon, r_0, R) > R(n,m)$ and $C(\varepsilon, r_0, R) > 0$. In fact choose

$$\varepsilon = \frac{2m}{8}\frac{1}{B(n,m)R^{n-1}}$$

then since $\sup_{r\geq R}\frac{r}{r^*-r_0^*+R^*-r_0^*}\leq 1$ (if not choose r_0 smaller and R bigger) we can compensate with the last term in the region $r \geq R$:

$$\int_{\mathbb{R}} K^{Z,1}\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^* \geq$$
$$\geq -B(n,m)\frac{(1+R^{*2})^2}{r_0^2}\, t\int_{r_0^*}^{R^*}\frac{r^2}{(1+r^{*2})^2}|\nabla\!\!\!/\phi|^2_{r^2\overset{\circ}{\gamma}_{n-1}}\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^*$$
$$+\frac{n-2}{8}\int_{R^*}^{r_1^*}\frac{2mt}{r^n}\phi^2\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^*$$
$$-R^{n-1}r_1\, B(n,m)C\left(1-\frac{2m}{r_0^{n-2}}\right)^{-1} t\int_{r_0^*}^{r_1^*}\frac{1}{r^n}\big(\frac{\partial\phi}{\partial r^*}\big)^2\, r^{n-1}\big(1-\frac{2m}{r^{n-2}}\big)\,\mathrm{d}r^*$$

where

$$r_1^* = (R^*-r_0^*)(n,m)e^{\frac{2m}{8}B(n,m)R(n,m)^{n-1}2(R^*-r_0^*)(n,m)} + 2r_0^* - R^*$$

and

$$C = \frac{1}{6}\frac{2m}{8}\frac{(R^*-r_0^*)(n,m)}{B(n,m)R(n,m)^{n-1}}e^{\frac{8}{2m}B(n,m)R(n,m)^{n-1}6(R^*-r_0^*)(n,m)}\,. \qquad \square$$

Proof of Prop. 1.45.

Step 1. $\frac{1}{t}$-decay on t-const hypersurfaces. We begin our argument using a t-const foliation (see figure 1.9)

$$\overline{\Sigma}_t = \Big\{(u,v) : u\leq 0, v\geq 0, \frac{2}{n-2}(2m)^{\frac{1}{n-2}}\operatorname{artanh}\big(\frac{u+v}{v-u}\big) = t\Big\}$$

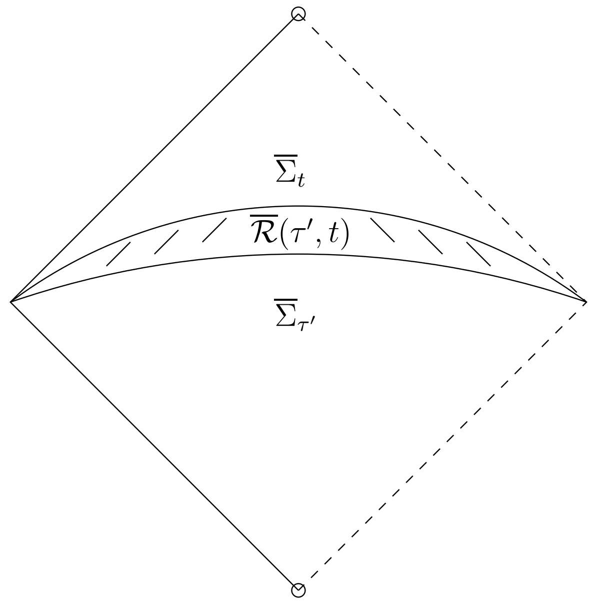

Figure 1.9: The t-const foliation.

and denote the corresponding spacetime slab by

$$\overline{\mathcal{R}}(t', t) = \bigcup_{t' \leq \bar{t} \leq t} \Sigma_{\bar{t}} .$$

Recall the current (1.4.67). The energy identity for $J^{(\alpha)}$ reads

$$\int_{\overline{\Sigma}_t} (J^{(\alpha)}, n) + \int_{\overline{\mathcal{R}}(t',t)} K^{(\alpha)} = \int_{\overline{\Sigma}_{t'}} (J^{(\alpha)}, n) \tag{1.5.80}$$

where

$$n = \frac{1}{\left(1 - \frac{2m}{r^{n-2}}\right)^{\frac{1}{2}}} \frac{\partial}{\partial t} .$$

Note in general that

$$\int_{\overline{\Sigma}_t} (J, n) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} (J, T)\, r^{n-1}\, \mathrm{d}r^*\, \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}$$

because $\mathrm{d}\mu_{g|_{\overline{\Sigma}_t}} = \left(1 - \frac{2m}{r^{n-2}}\right)^{\frac{1}{2}} r^{n-1}\, \mathrm{d}r^* \wedge \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}$.

We conclude from (1.5.80) in conjunction with Lemma 1.30 — where we have already addressed the boundary terms of the $J^{(\alpha)}$-current on t-const hypersurfaces — that

$$\int_{\overline{\mathcal{R}}(t',t)} K^{(\alpha)} \leq C(n, m, \alpha) \int_{\overline{\Sigma}_{t'}} \Big(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i \phi), n \Big) \qquad (\tau' > t) . \tag{1.5.81}$$

Choose $\sqrt[n-2]{2m} < \tilde{r}_1 < \tilde{r}_2 < \infty$ such that

$$\frac{1}{2}\big(t_0 - \tilde{r}_2^*\big) \geq 1 \qquad \frac{1}{2}\big(t_0 + \tilde{r}_1^*\big) \geq 1 \tag{1.5.82}$$

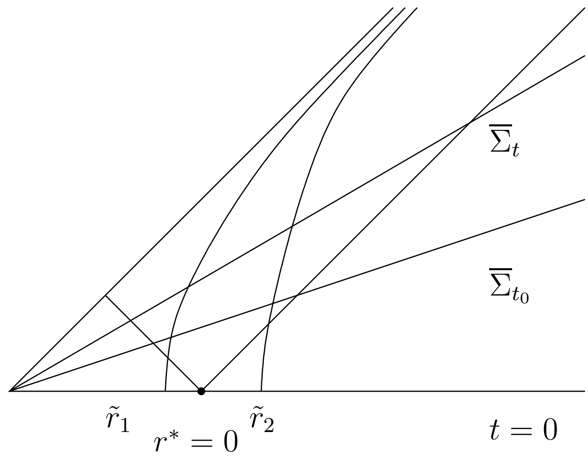

Figure 1.10: Construction of the triple $(t_0, \tilde{r}_1, \tilde{r}_2)$.

for some fixed $t_0 > 0$ (see figure 1.10). Then for $t \geq t_0$

$$\int_{\tilde{r}_1^*}^{\tilde{r}_2^*} \big(\frac{\partial\phi}{\partial u^*}\big)^2 \,\mathrm{d}r^* \leq \big(\frac{2}{t-\tilde{r}_2^*}\big)^2 \int_{\tilde{r}_1^*}^{\tilde{r}_2^*} u^{*2} \big(\frac{\partial\phi}{\partial u^*}\big)^2 \,\mathrm{d}r^*$$

and

$$\int_{\tilde{r}_1^*}^{\tilde{r}_2^*} \big(\frac{\partial\phi}{\partial v^*}\big)^2 \leq \big(\frac{2}{t+\tilde{r}_1^*}\big)^2 \int_{\tilde{r}_1^*}^{\tilde{r}_2^*} v^{*2} \big(\frac{\partial\phi}{\partial v^*}\big)^2 \,\mathrm{d}r^* \,.$$

Therefore by Prop. 1.48

$$\begin{aligned}\int_{\tilde{r}_1^*}^{\tilde{r}_2^*} (J^T, T)\, r^{n-1}\,\mathrm{d}r^* \leq \Big(\frac{20}{(t-\tilde{r}_2^*)^2} + \frac{20}{(t+\tilde{r}_1^*)^2} + \frac{2}{t^2}\Big) \times \\ \times \int_{\tilde{r}_1^*}^{\tilde{r}_2^*} \frac{1}{4}\Big\{\frac{1}{5}\Big(u^{*2}\big(\frac{\partial\phi}{\partial u^*}\big)^2 + v^{*2}\big(\frac{\partial\phi}{\partial v^*}\big)^2\Big) + \big(t^2 + r^{*2}\big)\big(1-\frac{2m}{r^{n-2}}\big)|\nabla\!\!\!/\,\phi|^2\Big\} r^{n-1}\,\mathrm{d}r^* \\ \leq \frac{C(t_0)}{t^2}\int_{\mathbb{R}} (J^{Z,1}, T)\, r^{n-1}\,\mathrm{d}r^* \quad (t > t_0)\,.\end{aligned}$$

Note that the larger $t_0 > 0$, the larger one may choose the interval $[\tilde{r}_1^*, \tilde{r}_2^*] \subset \mathbb{R}$. On the other hand Prop. 1.50 yields when combined with Cor. 1.29

$$\begin{aligned}-\int_{\overline{\mathcal{R}}(t_0,t_1)} K^{Z,1} = -\int_{t_0}^{t_1}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} K^{Z,1}\big(1-\frac{2m}{r^{n-2}}\big)\, r^{n-1}\,\mathrm{d}t\,\mathrm{d}r^*\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \\ \leq C(n,m)\int_{\overline{\mathcal{R}}(t_0,t_1)\cap\{r_0\leq r\leq r_1\}} t\,K^{(\alpha)}\,. \quad (1.5.83)\end{aligned}$$

Therefore by the energy identity for $J^{Z,1}$:

$$\begin{aligned}\int_{\overline{\Sigma}_t\cap\{\tilde{r}_1\leq r\leq\tilde{r}_2\}} (J^T, n) \leq \frac{C(t_0)}{t^2}\int_{\overline{\Sigma}_t} (J^{Z,1}, n) \\ \leq \frac{C(t_0)}{t^2}\Big[\int_{\overline{\Sigma}_{t_0}} (J^{Z,1}, n) + C(n,m)\, t\int_{\overline{\mathcal{R}}(t_0,t)} K^{(\alpha)}\Big]\end{aligned}$$

$$\leq \frac{C(n,m,\alpha,t_0)}{t}\Big[\int_{\overline{\Sigma}_{t_0}}(J^{Z,1},n)+\int_{\overline{\Sigma}_{t_0}}\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),n\Big)\Big]$$
$$(t>t_0) \quad (1.5.84)$$

Step 2. $\frac{1}{\tau}$-decay on the hypersurfaces $\tilde{\Sigma}_\tau\cap\{r\leq\overline{r}_2(\tau)\}$. The spacelike hypersurfaces relative to which we expect the local observer's energy to decay terminate at future null infinity (and on the horizon to the future of the bifurcation sphere). Let $\tilde{\Sigma}_0$ be a spacelike hypersurface in $J^+(u^*=1,v^*=1)$ and first choose $\sqrt[n-2]{2m}<\tilde{r}_1\leq r_0(n,m)$ and then t_0 and $\tilde{r}_2$ such that $(t_0,\tilde{r}_1^*)\in\tilde{\Sigma}_0$, $(t_0,\tilde{r}_2^*)\in\overline{\Sigma}_0$. Clearly, $(t_0,\tilde{r}_1,\tilde{r}_2)$ satisfy (1.5.82) by construction. Note that by definition of

$$\tilde{\Sigma}_\tau=\varphi_\tau(\tilde{\Sigma}_0)\,,$$

$\tilde{\Sigma}_\tau$ and $\Sigma_{t_0+\tau}$ will intersect at $r=\tilde{r}_1$ and $r=\tilde{r}_2$ for all $\tau\geq 0$. Also denote

$$\tilde{\mathcal{R}}(\tau',\tau)=\bigcup_{\tau'\leq\overline{\tau}\leq\tau}\tilde{\Sigma}_{\overline{\tau}}\,.$$

By conservation of J^T flux,

$$\int_{\tilde{\Sigma}_\tau\cap\{\tilde{r}_1\leq r\leq\tilde{r}_2\}}(J^T,n)=\int_{\overline{\Sigma}_{t_0+\tau}\cap\{\tilde{r}_1\leq r\leq\tilde{r}_2\}}(J^T,n)\,. \tag{1.5.85}$$

We may assume that $K^N\geq b\,(J^N,n)$ on $\tilde{\Sigma}_\tau\cap\{r\leq\tilde{r}_1\}$ for all $\tau\geq 0$, as well as $\tilde{r}_2\geq r_1(n,m)$; for otherwise choose $\tilde{r}_1$ closer to the horizon, see also Prop. 1.7. Then

$$\begin{aligned}\int_{\tilde{\mathcal{R}}(\tau',\tau)\cap\{r\leq\tilde{r}_1\}}(J^N,n)&\leq\frac{1}{b}\int_{\tilde{\mathcal{R}}(\tau',\tau)\cap\{r\leq\tilde{r}_1\}}K^N\\&\leq\frac{1}{b}\int_{\tilde{\Sigma}_{\tau'}\cap\{r\leq R_{\tau',\tau}\}}(J^N,n)+\frac{1}{b}\int_{\overline{\mathcal{R}}(0,t_0+\tau)\cap\{r\geq\tilde{r}_1\}}|K^N|\end{aligned} \tag{1.5.86}$$

where $(t_0+\tau,R^*_{\tau',\tau})\in\tilde{\Sigma}_{\tau'}$ (see figure 1.11). We can use the $J^{(\alpha)}$ current to control the spacetime integral:

$$\begin{aligned}\int_{\overline{\mathcal{R}}(0,t_0+\tau)\cap\{r\geq\tilde{r}_1\}}|K^N|&\leq C(n,m,\sigma,\tilde{r}_1)\int_{\overline{\mathcal{R}}(0,t_0+\tau)}\Big\{K^{(\alpha)}+K^{\mathrm{aux}}\Big\}\\&\leq C(n,m,\alpha,\sigma,\tilde{r}_1)\int_{\overline{\Sigma}_0}\Big(J^T(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),n\Big)\end{aligned}$$

By virtue of the uniform boundedness result of Section 1.5.1 applied to the first term in (1.5.86) (recall also $N=T$ for $r\geq r_1^N$)

$$\int_{\tilde{\mathcal{R}}(\tau',\tau)\cap\{r\leq\tilde{r}_1\}}(J^N,n)\leq C(n,m,\alpha,b,\sigma,\tilde{r}_1)\int_{\overline{\Sigma}_0}\Big(J^N(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),n\Big) \tag{1.5.87}$$

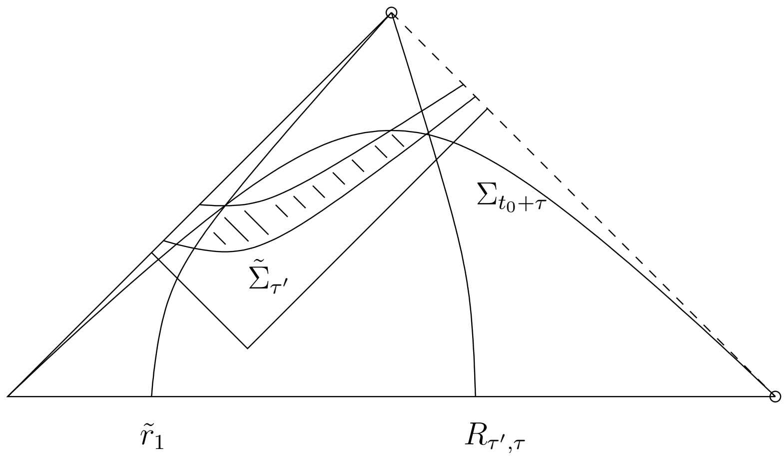

Figure 1.11: Illustration of (1.5.86).

Lemma 1.52. *On* $\overline{\Sigma}_t \cap \{\tilde{r}_1 \leq r \leq \tilde{r}_2\}$,

$$\int_S |K^N| \,\mathrm{d}\mu_\gamma \leq C(n, m, \sigma, \tilde{r}_1, r_1^N) \int_S \Big\{ K^{(\alpha)} + K^{aux} \Big\} \,\mathrm{d}\mu_\gamma \,.$$

Proof. The statement is immediate from Cor. 1.29. In fact for $r_0^N \leq r \leq r_1^N$

$$|K^N| = |{}^{(N)}\pi^{\mu\nu} T_{\mu\nu}| \leq |{}^{(N)}\pi^{\mu\nu}||\partial_\mu\phi||\partial_\nu\phi| + \frac{1}{2}|\operatorname{tr}{}^{(N)}\pi||\partial^\alpha\phi||\partial_\alpha\phi| \leq$$
$$\leq B\left(J^T, n\right), \quad (1.5.88)$$

whereas $K^N = K^T = 0$ for $r \geq r_1^N$. For $r \leq r_0^N$ one may use

$$N = \Big[1 + \frac{\sigma}{4\kappa}\big(1 - \frac{2m}{r^{n-2}}\big)\Big]\Big(\frac{2}{1 - \frac{2m}{r^{n-2}}}\frac{\partial}{\partial u^*} + \frac{\partial}{\partial t}\Big)$$

to calculate

$$\begin{aligned}K^N = (n-2)\frac{\sigma}{4\kappa}\frac{2m}{r^{n-1}}\big(\frac{\partial\phi}{\partial t}\big)\big(\frac{\partial\phi}{\partial r^*}\big) + (n-2)\frac{2m}{r^{n-1}}\big(1 - \frac{2m}{r^{n-2}}\big)^{-1}\big(\frac{\partial\phi}{\partial u^*}\big)^2 \\ + (n-2)\frac{\sigma}{4\kappa}\frac{2m}{r^{n-1}}|\nabla\!\!\!/\phi|^2 + \frac{n-3}{r}\Big[1 + \frac{\sigma}{4\kappa}\big(1 - \frac{2m}{r^{n-2}}\big)\Big]|\nabla\!\!\!/\phi|^2 \\ - \frac{n-1}{r}\Big[1 + \frac{\sigma}{4\kappa}\big(1 - \frac{2m}{r^{n-2}}\big)\Big]\frac{1}{1 - \frac{2m}{r^{n-2}}}\big(\frac{\partial\phi}{\partial u^*}\big)\big(\frac{\partial\phi}{\partial v^*}\big)\,.\end{aligned}$$

to find the bound

$$\begin{aligned}|K^N| \leq \frac{n-2}{2}\frac{\sigma}{4\kappa}(2m)^{-\frac{1}{n-2}}\Big[\big(\frac{\partial\phi}{\partial t}\big)^2 + \big(\frac{\partial\phi}{\partial r^*}\big)^2\Big] \\ + 2(n-2)(2m)^{-\frac{1}{n-2}}\big(1 - \frac{2m}{\tilde{r}_1^{n-2}}\big)^{-1}\Big[\big(\frac{\partial\phi}{\partial t}\big)^2 + \big(\frac{\partial\phi}{\partial r^*}\big)^2\Big] \\ + (n-2)\frac{\sigma}{4\kappa}(2m)^{-\frac{1}{n-2}}|\nabla\!\!\!/\phi|^2 + \frac{n-3}{(2m)^{\frac{1}{n-2}}}\Big[1 + \frac{\sigma}{4\kappa}\Big]|\nabla\!\!\!/\phi|^2 \\ + \frac{n-1}{(2m)^{\frac{1}{n-2}}}\Big[1 + \frac{\sigma}{4\kappa}\Big]\big(1 - \frac{2m}{\tilde{r}_1^{n-2}}\big)^{-1}\Big[\big(\frac{\partial\phi}{\partial t}\big)^2 + \big(\frac{\partial\phi}{\partial r^*}\big)^2\Big]\end{aligned}$$

because $r \geq \tilde{r}_1$. $\square$

The spacetime integral estimate (1.5.87) near the horizon is the starting point of the following pigeonhole argument.

Let $\tau_1 > 0$, $\tau_{n+1} = 2\,\tau_n$ ($n \in \mathbb{N}$) then there is a sequence $(\tau_n')_{n\in\mathbb{N}},\ \tau_n' \in (\tau_n, \tau_{n+1})$ such that

$$\int_{\tau_n}^{\tau_{n+1}} \mathrm{d}\overline{\tau} \int_{\tilde{\Sigma}_{\overline{\tau}}\cap\{r\leq\tilde{r}_1\}} (J^N, n) = \tau_n \int_{\tilde{\Sigma}_{\tau_n'}\cap\{r\leq\tilde{r}_1\}} (J^N, n)$$

and we have for all $n \in \mathbb{N}$

$$\int_{\tau_n}^{\tau_{n+1}} \mathrm{d}\overline{\tau} \int_{\tilde{\Sigma}_{\overline{\tau}}\cap\{r\leq\tilde{r}_1\}} (J^N, n) \leq C(n, m, \alpha, b, \sigma, \tilde{r}_1, \tilde{\Sigma}_0) \int_{\overline{\Sigma}_0} \Big(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), n \Big) .$$

Therefore

$$\int_{\tilde{\Sigma}_{\tau_n'}\cap\{r\leq\tilde{r}_1\}} (J^N, n) \leq$$
$$\leq \frac{C(n, m, \alpha, b, \sigma, \tilde{r}_1, \tilde{\Sigma}_0)}{\tau_n'} \int_{\overline{\Sigma}_0} \Big(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), n \Big) \quad (n \geq 2) . \tag{1.5.89}$$

Recall (1.5.85) which implies

$$\int_{\tilde{\Sigma}_{\tau}\cap\{\tilde{r}_1\leq r\leq\tilde{r}_2\}} (J^N, n) \leq$$
$$\leq \frac{C(n, m, \alpha, t_0, \tilde{r}_1)}{(t_0 + \tau)} \Big\{ \int_{\overline{\Sigma}_{t_0}} \big(J^{Z,1}(\phi), n \big) + \int_{\overline{\Sigma}_{t_0}} \Big(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), n \Big) \Big\} \tag{1.5.90}$$

Next define for $\overline{\tau} \geq 0$, $\overline{r}_2(\overline{\tau})$ by $\big(t_0 + \tau + \overline{\tau}, \overline{r}_2^*(\overline{\tau})\big) \in \tilde{\Sigma}_\tau$ (see figure 1.12) then

$$\int_{\tilde{\Sigma}_{\tau}\cap\{\tilde{r}_2\leq r\leq\overline{r}_2(\overline{\tau})\}} (J^T, n) \leq \int_{\overline{\Sigma}_{t_0+\tau+\overline{\tau}}\cap J^+(\tilde{\Sigma}_\tau)} (J^T, n) \leq$$
$$\leq \frac{C(t_0)}{(t_0 + \tau)^2} \int_{\overline{\Sigma}_{t_0+\tau+\overline{\tau}}} (J^{Z,1}(\phi), n) \tag{1.5.91}$$

because

$$\min_{\overline{\Sigma}_{t_0+\tau+\overline{\tau}}\cap J^+(\tilde{\Sigma}_\tau)} u^{*2} \geq \min_{\tilde{\Sigma}_\tau} u^{*2} \qquad \min_{\overline{\Sigma}_{t_0+\tau+\overline{\tau}}\cap J^+(\tilde{\Sigma}_\tau)} v^{*2} \geq \min_{\tilde{\Sigma}_\tau} v^{*2} \qquad .$$

However again

$$\int_{\overline{\Sigma}_{t_0+\tau+\overline{\tau}}} (J^{Z,1}, n) \leq$$
$$\leq C(n, m, \alpha, t_0)\,(t_0 + \tau + \overline{\tau}) \Big[\int_{\overline{\Sigma}_{t_0}} (J^{Z,1}, n) + \int_{\overline{\Sigma}_{t_0}} \Big(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), n \Big) \Big]. \tag{1.5.92}$$

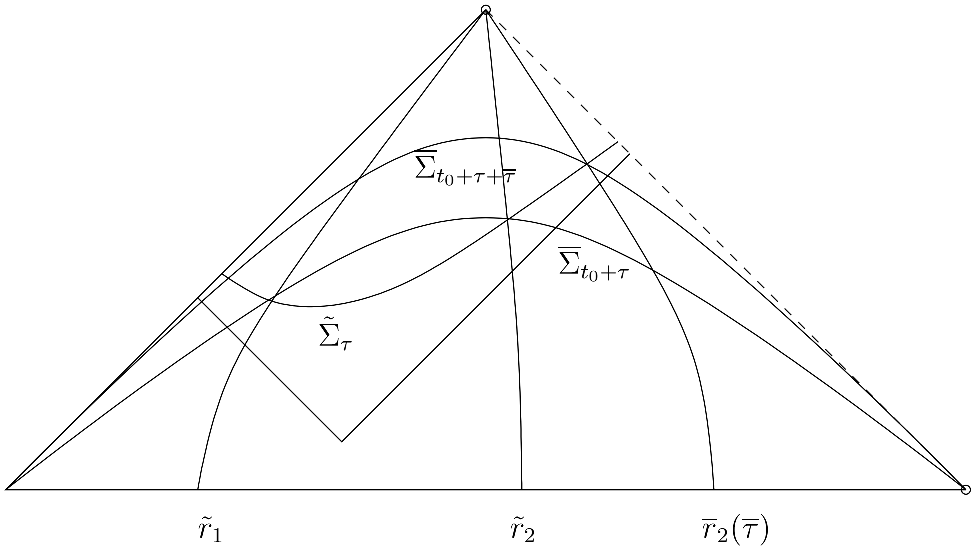

Figure 1.12: Extension to the region $\tilde{r}_2 \leq r \leq \overline{r}_2(\overline{\tau})$.

Now choose $\overline{\tau} = c\,(t_0+\tau)$ then with (1.5.90)

$$\int_{\tilde{\Sigma}_\tau \cap \{\tilde{r}_1 \leq r \leq \overline{r}_2(\tau)\}} (J^N, n) \leq$$
$$\leq \frac{C(n,m,\alpha,t_0,\tilde{r}_1)}{(t_0+\tau)} \int_{\overline{\Sigma}_{t_0}} \Big(J^{Z,1}(\phi) + J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i \phi), n \Big)\,; \quad (1.5.93)$$

note we can arrange for $\overline{r}_2^*(\tau) \geq \tilde{r}_2^* + \tau$. For any $\tau \geq \tau_3$ we may choose $\tau' = \max\big\{\tau_n' : \tau_n' \leq \tau\big\}$ so by the local observer's energy estimate

$$\int_{\tilde{\Sigma}_\tau \cap \{r \leq \tilde{r}_1\}} (J^N, n) \leq C \int_{\tilde{\Sigma}_{\tau'} \cap \{r \leq \overline{r}_2(\tau')\}} (J^N, n)$$

and using (1.5.89) and (1.5.93) we finally obtain

$$\int_{\tilde{\Sigma}_\tau \cap \{r \leq \overline{r}_2(\tau)\}} (J^N, n) \leq \frac{C(n,m,\alpha,b,\sigma,t_0,\tilde{r}_1,\tilde{\Sigma}_0)}{\tau} \times$$
$$\times \left\{ \int_{\overline{\Sigma}_{t_0}} \big(J^{Z,1}(\phi), n \big) + \int_{\overline{\Sigma}_0} \Big(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i \phi), n \Big) \right\} \qquad (\tau \geq 4\tau_1)\,. \quad (1.5.94)$$

Step 3. $\frac{1}{\tau^2}$-decay on the hypersurfaces $\tilde{\Sigma}_\tau$. The aim is here to improve (1.5.94) to the extend that $\frac{1}{\tau}$ will be replaced by $\frac{1}{\tau^2}$ and the restriction on $r \leq \overline{r}_2(\tau)$ will be removed. Recall the regions (1.4.1), and consider in particular

$$\mathcal{R}(t_0,t_1,u_1^*) \doteq \mathcal{R}_{r_0,r_1}(t_0,t_1,u_1^*,\frac{1}{2}(t_1+r_1^*))$$
$$\mathcal{R}^\infty(t_0,t_1) \doteq \bigcup_{u_1^* \geq \frac{1}{2}(t_1 - r_0^*)} \mathcal{R}(t_0,t_1,u_1^*)\,.$$

As we have already seen in Step 2a of the proof of Prop. 1.11 in Section 1.4.4 we obtain

$$\int_{\mathcal{R}^\infty(t_0,t_1)} K^{(\alpha)} \leq C(n,m,\alpha) \int_{\partial^- \mathcal{R}^\infty(t_0,t_1)} \Big(J^T(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\phi), n \Big) \tag{1.5.95}$$

where $\partial^-\mathcal{R}^\infty(t_0,t_1)$ is the past boundary of $\mathcal{R}^\infty(t_0,t_1)$.

Now cover the region $\{r_0(n,m) \leq r \leq r_1(n,m)\}$ dyadically with $\mathcal{R}^\infty(t_n,t_{n+1})$ $(n \in \mathbb{N})$ where $t_1 > 0$, $t_{n+1} = 2\,t_n$ $(n \in \mathbb{N})$ and also denote

$$\tilde{\Sigma}_n \doteq \partial^- \mathcal{R}^\infty(t_n,t_{n+1})\,.$$

Then for $N \in \mathbb{N}$ large enough $\tilde{\Sigma}_n \subset J^+(u^* = 1, v^* = 1)$ $(n \geq N+1)$ thus by the previous estimate (1.5.95) and the earlier $\frac{1}{\tau}$ decay result (1.5.94) (recall $\tilde{r}_1 \leq r_0(n,m)$, $\tilde{r}_2 \geq r_1(n,m)$, $\overline{r}_2^*(\tau) \geq \tilde{r}_2^* + \tau$) and choose $\tau_1 < \frac{t_N}{4}$:

$$\int_{\{t_n \leq t \leq t_{n+1}\}\cap\{r_0 \leq r \leq r_1\}} K^{(\alpha)} \leq \frac{C D_2^{(Z)}}{(t_n - t_N)} \qquad (n \geq N+1) \tag{1.5.96}$$

where $C = C(n,m,\alpha,b,\sigma,t_0,\tilde{\Sigma}_0)$ and

$$D_2^{(Z)} = \int_{\overline{\Sigma}_{t_0}} \Big(J^{Z,1}(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i\phi), n \Big) + \int_{\overline{\Sigma}_0} \Big(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i\phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\Omega_j\phi), n \Big)\,. \tag{1.5.97}$$

By summing over $n \in \mathbb{N}$,

$$\int_{\{t_0 \leq t \leq t_1'\}\cap\{r_0\leq r\leq r_1\}} t\,K^{(\alpha)} \leq \leq t_N \int_{\{t_0 \leq t \leq t_{N+1}\}} K^{(\alpha)} + \sum_{n=N+1}^{[t_1']} 2 \int_{\{t_n \leq t \leq t_{n+1}\}} t_n\,K^{(\alpha)} \leq t_N\,C D_2^{(Z)} + 2\,C D_2^{(Z)} [t_1']$$

where $[t_1'] \in \mathbb{N}$ is the smallest number such that $t_{[t_1']} \geq t_1'$; observe that since

$$t_{[t_1']} = 2^{[t_1']} t_1$$
$$[t_1'] = \frac{\log t_{[t_1']} - \log t_1}{2} \leq 1 + \frac{\log t_1'}{\log 2} \qquad \text{if } t_1 > 1\,.$$

Hence

$$\int_{\{t_0 \leq t \leq t_1\}\cap\{r_0\leq r\leq r_1\}} t\,K^{(\alpha)} \leq (1 + \log t_1)\,C D_2^{(Z)}\,, \tag{1.5.98}$$

with $C, D_2^{(Z)}$ as above, and we immediatly improve (1.5.83) to

$$-\int_{\mathcal{R}(t_0,t_1)} K^{Z,1} \leq (1 + \log t_1)\,C D_2^{(Z)}\,, \tag{1.5.99}$$

with the consequence that also (1.5.84) is replaced by

$$\int_{\overline{\Sigma}_t\cap\{\tilde r_1\le r\le\tilde r_2\}}(J^T,n)\le\frac{1+\log t}{t^2}\,CD_2^{(Z)} \tag{1.5.100}$$

with $C, D_2^{(Z)}$ given as above by (1.5.97). With the same argument as before we extend this to

$$\int_{\tilde\Sigma_\tau\cap\{\tilde r_1\le r\le\overline{r}_2(\tau)\}}(J^N,n)\le\frac{1+\log(t_0+\tau)}{(t_0+\tau)^2}\,CD_2^{(Z)}\,. \tag{1.5.101}$$

Next we refine (1.5.87). By Cor. 1.29, (1.5.95) and the $\frac{1}{\tau}$-decay result (1.5.94)

$$\begin{aligned}
&\int_{\tilde{\mathcal{R}}(\tau',\tau)\cap\{r\le\tilde r_1\}}(J^N,n)\le\\
&\qquad\le\frac{1}{b}\int_{\substack{\tilde{\mathcal{R}}(\tau',\tau)\cap\{r\le\tilde r_1\}\\ \cap\{v^*\le\frac12(t_0+\tau'+\tilde r_1^*)\}}}K^N+\int_{\substack{\{r\le\tilde r_1\}\cap\\ \{\frac12(t_0+\tau'+\tilde r_1^*)\le v^*\le\frac12(t_0+\tau+\tilde r_1^*)\}}}(J^N,n)\\
&\qquad\le\frac{1}{b}\int_{\tilde\Sigma_{\tau'}\cap\{r\le\tilde r_1\}}(J^N,n)+C(n,m)\int_{\mathcal{R}^\infty(t_0+\tau',t_0+\tau)}\Big\{K^{(\alpha)}+K^{\mathrm{aux}}\Big\}\\
&\qquad\le C(n,m,\alpha,b)\int_{\tilde\Sigma_{\tau'}\cap\{r\le\overline{r}_2(\tau')\}}\Big(J^N(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^T(\Omega_i\phi),n\Big)\\
&\qquad\le\frac{C(n,m,\alpha,b,\sigma,t_0,\tilde\Sigma_0)}{\tau'}\Big\{\int_{\overline{\Sigma}_{t_0}}\Big(J^{Z,1}(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^{Z,1}(\Omega_i\phi),n\Big)\\
&\qquad\qquad+\int_{\overline{\Sigma}_0}\Big(J^N(\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(\Omega_i\phi)+\sum_{i=1}^{\frac{n(n-1)}{2}}J^N(\Omega_i\Omega_j\phi),n\Big)\Big\}
\end{aligned} \tag{1.5.102}$$

or with $C, D_2^{(Z)}$ as above

$$\int_{\tilde R(\tau',\tau)\cap\{r\le\tilde r_1\}}(J^N,n)\le\frac{CD_2^{(Z)}}{\tau'}\,. \tag{1.5.103}$$

Proceeding as above let $\tau_1>0$, $\tau_{n+1}=2\,\tau_n$ ($n\in\mathbb{N}$), then there is a sequence $(\tau_n')_{n\in\mathbb{N}}$, $\tau_n'\in(\tau_n,\tau_{n+1})$ such that

$$\begin{aligned}
\int_{\tilde\Sigma_{\tau_n'}\cap\{r\le\tilde r_1\}}(J^N,n)&=\frac{1}{\tau_n}\int_{\tau_n}^{\tau_{n+1}}\mathrm{d}\overline{\tau}\int_{\tilde\Sigma_{\overline{\tau}}\cap\{r\le\tilde r_1\}}(J^N,n)\\
&\le\frac{1}{\tau_n^2}CD_2^{(Z)}\le\frac{16}{{\tau_n'}^2}CD_2^{(Z)}
\end{aligned} \tag{1.5.104}$$

and for any $\tau\ge\tau_3$ we may choose $\tau'=\max\{\tau_n':\tau_n'\le\tau\}$ so that

$$\int_{\tilde\Sigma_\tau\cap\{r\le\tilde r_1\}}(J^N,n)\le C\int_{\tilde\Sigma_{\tau'}\cap\{r\le\overline{r}_2(\tau')\}}(J^N,n)\,.$$

Thus

$$\int_{\tilde\Sigma_\tau\cap\{r\le\overline{r}_2(\tau)\}}(J^N,n)\le\frac{1+\log(t_0+\tau)}{(t_0+\tau)^2}C\,D_2^{(Z)}\,. \tag{1.5.105}$$

One can repeat this step to remove the $1+\log(t_0+\tau)$-term and the restriction $r \leq \overline{r}_2(\tau)$. Namely, now using (1.5.105) yields in place of (1.5.96)

$$\int_{\substack{\{t_n \leq t \leq t_{n+1}\} \\ \cap\{r_0 \leq r \leq r_1\}}} K^{(\alpha)} \leq \frac{1+\log(t_n - t_N)}{(t_n - t_N)^2} C\, D_3^{(Z)} \quad (n \geq N+1) \tag{1.5.106}$$

where $C = C(n, m, \alpha, b, \sigma, t_0, \tilde{\Sigma}_0)$ and

$$\begin{aligned} D_3^{(Z)} \doteq \int_{\overline{\Sigma}_{t_0}} \Big(J^{Z,1}(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i \phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i \Omega_j \phi), n \Big) \\ + \int_{\overline{\Sigma}_0} \Big(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i \phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i \Omega_j \phi) + \sum_{i,j,k=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i \Omega_j \Omega_k \phi), n \Big) . \end{aligned} \tag{1.5.107}$$

In other words

$$\int_{\substack{\{t_0 \leq t \leq t_1\} \\ \cap\{r_0 \leq r \leq r_1\}}} t\, K^{(\alpha)} \leq C\, D_3^{(Z)} \tag{1.5.108}$$

because by summing over $n \in \mathbb{N}$

$$\int_{\substack{\{t_0 \leq t \leq t_1'\} \\ \cap\{r_0 \leq r \leq r_1\}}} t\, K^{(\alpha)} \leq t_{N+1} \int_{\substack{\{t_0 \leq t \leq t_{N+1}\} \\ \cap\{r_0 \leq r \leq r_1\}}} K^{(\alpha)} + 2 \sum_{i=N+1}^{\infty} \frac{1+\log(t_n - t_N)}{(t_n - t_N)} C\, D_3$$

with the last sum being finite

$$\sum_{N+1}^{\infty} \frac{1+\log(t_n - t_N)}{t_n - t_N} = \sum_{k=1}^{\infty} \frac{1+\log(2^k - 1) + (N-1)\log 2 + \log t_1}{(2^k - 1)\, 2^{N-1}\, t_1} < \infty$$

since

$$\sum_{k=1}^{\infty} \frac{\log 2^k}{2^k} = (\log 2) \sum_{k=1}^{\infty} \Big(\frac{\sqrt[k]{k}}{2}\Big)^k < \infty$$

as $\sqrt[k]{k} \to 1$ $(k \to \infty)$. This immediatly improves (1.5.83) to

$$-\int_{\mathcal{R}(t_0, t_1)} K^{Z,1} \leq C\, D_3^{(Z)} . \tag{1.5.109}$$

Therefore

$$\int_{\overline{\Sigma}_t \cap \{\tilde{r}_1 \leq r \leq \tilde{r}_2\}} (J^T, n) \leq \frac{1}{t^2}\, C D_3^{(Z)} ; \tag{1.5.110}$$

moreover no power of $(t_0 + \tau + \overline{\tau})$ is lost in (1.5.92) anymore and (1.5.91) is replaced by

$$\int_{\tilde{\Sigma}_\tau \cap \{r \geq \tilde{r}_2\}} (J^T, n) \leq \frac{1}{(t_0+\tau)^2}\, C D_3^{(Z)} \tag{1.5.111}$$

as we let $\overline{\tau} \to \infty$. We arrive when combined with (1.5.104) at the final result:

$$\int_{\tilde{\Sigma}_\tau} (J^N, n) \leq \frac{1}{\tau^2}\, C D_3^{(Z)} \tag{1.5.112}$$

where $C, D_3^{(Z)}$ are as above given by (1.5.107). Note that again by (1.5.83) and (1.5.81)

$$D_3^{(Z)} \leq C(n,m,\alpha,t_0)\times \times \int_{\overline{\Sigma}_0}\Big(J^N(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i\phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^N(\Omega_i\Omega_j\phi) + \sum_{i,j,k=1}^{\frac{n(n-1)}{2}} J^T(\Omega_i\Omega_j\Omega_k\phi), n\Big) + \int_{\overline{\Sigma}_0}\Big(J^{Z,1}(\phi) + \sum_{i=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i\phi) + \sum_{i,j=1}^{\frac{n(n-1)}{2}} J^{Z,1}(\Omega_i\Omega_j\phi), n\Big).$$

□

1.6 Pointwise bounds

In this Section we first prove pointwise estimates on $|\phi|$ and $|\partial_t\phi|$ separately based on the energy decay results Prop. 1.37 and Prop. 1.39 in Section 1.5. Then we give the interpolation argument to improve the pointwise decay on $|\phi|$. As we shall see in view of the nondegenerate energy estimates of Section 1.5 we may restrict ourselves in the first place to a radial region away from the horizon. Recall the definition (1.4.3) of Σ_τ, $(r_1 \doteq R > \sqrt[n-2]{8nm})$.

Proposition 1.53 (Pointwise decay). *(i) Let ϕ be a solution of the wave equation* (1.1.1), *with initial data on Σ_{τ_0} ($\tau_0 > 0$) such that*

$$D \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \sum_{k=0}^{[\frac{n}{2}]+1} r^2\Big(\frac{\partial T^k\cdot\psi}{\partial v^*}\Big)^2\big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}}\Big(\sum_{k=0}^{[\frac{n}{2}]+2} J^N(T^k\cdot\phi), n\Big) < \infty\,. \quad (1.6.1)$$

Then there is a constant $C(n,m)$ such that for $r_0 < r < R$,

$$|\phi(t,r)| \leq \frac{C(n,m)\sqrt{D}}{\tau} \qquad \Big(\tau = \frac{1}{2}(t-R^*) > \tau_0\Big)\,. \quad (1.6.2)$$

(ii) If moreover, the initial data satisfies

$$D \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Big\{\sum_{k=0}^{[\frac{n}{2}]+1} r^{4-\delta}\Big(\frac{\partial(T^k\cdot\chi)}{\partial v^*}\Big)^2 + \sum_{k=0}^{[\frac{n}{2}]+4} r^2\Big(\frac{\partial(T^k\cdot\psi)}{\partial v^*}\Big)^2 + \sum_{k=0}^{[\frac{n}{2}]+3}\sum_{i=1}^{\frac{n(n-1)}{2}} r^2\Big(\frac{\partial T^k\Omega_i\psi}{\partial v^*}\Big)^2\Big\}\Big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}}\Big(\sum_{k=0}^{[\frac{n}{2}]+5} J^N(T^k\cdot\phi) + \sum_{k=0}^{[\frac{n}{2}]+4}\sum_{i=1}^{\frac{n(n-1)}{2}} J^N(T^k\Omega_i\phi), n\Big) < \infty \quad (1.6.3)$$

for some $0 < \delta < \frac{1}{4}$, *and* $R > \sqrt[n-2]{\frac{8nm}{\delta}}$, *then there is a constant* $C(n, m, \delta, R)$ *such that for* $r_0 < r < R$,

$$|\partial_t \phi(t, r)| \leq \frac{C\sqrt{D}}{\tau^{2-2\delta}} \qquad (\tau = \frac{1}{2}(t - R^*) > \tau_0)\,. \tag{1.6.4}$$

The pointwise bounds are obtained from the energy estimates of Section 1.5 using Sobolev inequalties and elliptic estimates; the former provide the link between pointwise and integral quantities, and the latter allow for the expression of these integral quantities in terms of higher order energies.

Sobolev embedding. By the extension theorem applied to the Sobolev embedding $\mathrm{H}^s(\mathbb{R}^n) \subset \mathrm{L}^\infty(\mathbb{R}^n)$ $(s > \frac{n}{2})$ [27] we have, for $r_0 < \overline{r} < R$,

$$|\phi(\overline{t}, \overline{r})|^2 \leq C(n) \int_{r_0^*}^{R^*} \mathrm{d}r^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \Big\{ \phi^2 + \sum_{|\alpha| \geq 1}^{|\alpha| \leq [\frac{n}{2}]+1} \left|\overline{\nabla}^\alpha \phi\right|^2 \Big\} r^{n-1} \Big|_{t=\overline{t}} \tag{1.6.5}$$

where $\overline{\nabla}$ denote the tangential derivatives to the hypersurface $\overline{\Sigma}_t$, and α denotes a multi-ndex of order n.

Elliptic estimates. Note that for any solution ϕ of the wave equation

$$T^2 \cdot \phi = \frac{\partial^2 \phi}{\partial r^{*2}} + \big(1 - \frac{2m}{r^{n-2}}\big) \frac{n-1}{r} \frac{\partial \phi}{\partial r^*} + \big(1 - \frac{2m}{r^{n-2}}\big) \triangle\!\!\!\!/_{\,r^2 \overset{\circ}{\gamma}_{n-1}} \phi \doteq L \cdot \phi \tag{1.6.6}$$

where the operator

$$L = \big(1 - \frac{2m}{r^{n-2}}\big) \overline{g}^{ij} \overline{\nabla}_i \partial_j \tag{1.6.7}$$

is clearly elliptic, (here $\overline{g}_t = g|_{\overline{\Sigma}_t}$ denotes the restriction of g to the spacelike hypersurfaces $\overline{\Sigma}_t$, a Riemannian metric on $\overline{\Sigma}_t$, and $i, j = 1, \ldots, n$). In view of the standard higher order interior elliptic regularity estimate (c.f. [27]),

$$\|\phi\|_{\mathrm{H}^{m+2}(\widehat{\Sigma}_t)} \leq C\Big(\|L \cdot \phi\|_{\mathrm{H}^m(\widehat{\Sigma}_t)} + \|\phi\|_{\mathrm{L}^2(\widehat{\Sigma}_t)} \Big) \qquad \widehat{\Sigma}_t \doteq \overline{\Sigma}_t \cap \{r_0 < r < R\}\,, \tag{1.6.8}$$

we conclude with (1.6.5) that in the case where $[\frac{n}{2}] + 1$ is even,

$$|\phi|^2 \leq C(n, m) \int_{r_0^*}^{R^*} \mathrm{d}r^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \sum_{l=0}^{[\frac{n}{2}]+1} \Big(T^l \cdot \phi \Big)^2 r^{n-1}\,; \tag{1.6.9}$$

in general we have:

Lemma 1.54 (Pointwise estimate in terms of higher order energies)**.** *Let* ϕ *be a solution of the wave equation* (1.1.1), *and* $n \geq 3$. *Then there exists a constant* $C(n, m)$ *such that for all* $r_0 < r < R$:

$$|\phi(t, r)|^2 \leq C(n, m) \Big[\|\phi\|^2_{\mathrm{L}^2(\widehat{\Sigma}_t)} + \int_{\widehat{\Sigma}_t} \sum_{l=0}^{[\frac{n}{2}]} \Big(J^T(T^l \cdot \phi), n \Big) \Big] \tag{1.6.10}$$

Proof of Prop. 1.53. In view of the Lemma 1.54 and the energy decay estimates of Section 1.5 it remains to control the zeroth order term $\|\phi\|_{\mathrm{L}^2(\widehat{\Sigma}_t)}$; we multiply the integrand by $(\frac{R}{r})^2 \geq 1$ and extend the integral to $u^* = \tau = \frac{1}{2}(t - R^*)$, $v^* \geq \frac{1}{2}(t + R^*)$.
(i) By Lemma B.6 we can then estimate $\|\phi\|^2_{\mathrm{L}^2(\widehat{\Sigma}_t)}$ by the energy flux through $\Sigma_{\tau=\frac{1}{2}(t-R^*)}$, and apply Prop. 1.37 to the higher order energies of Lemma 1.54.
(ii) Here we extend the integral only to $\tau + R^* \leq v^* \leq \tau + R^* + \tau^3$ and apply Lemma B.8 to obtain

$$\int_{r_0^*}^{R^*} \mathrm{d}r^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} (\partial_t\phi)^2 r^{n-1} \leq C(n,m)\, R^2 \int_{\Sigma_\tau \cap \{r^* \leq R^* + \tau^3\}} \Big(J^T(\partial_t\phi), n \Big) \\ + C(n,m) \frac{R^2}{r} \int_{\mathbb{S}^{n-1}} r^{n-1} (\partial_t\phi)^2|_{(u^*=\tau, v^*=\tau+R^*+\tau^3)} \,. \tag{1.6.11}$$

As in the proof of Lemma 1.42 we obtain by integrating from infinity and Cauchy's inequality that

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-2} (\partial_t\phi)^2(\tau, \tau + R^*) \leq \frac{C(n,m)}{1 - \frac{2m}{R^{n-2}}} \int_{\Sigma_\tau} \Big(J^T(\partial_t\phi), n \Big) \tag{1.6.12}$$

which decays by Prop. 1.37 with a rate τ^{-2}. Moreover, as in the proof of Lemma 1.42,

$$\int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1} (\partial_t\phi)^2|_{(u^*=\tau, v^*=\tau+R^*+\tau^3)} = \\ = \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-1} (\partial_t\phi)^2|_{(u^*=\tau, v^*=\tau+R^*)} + \int_{\tau+R^*}^{\tau+R^*+\tau^3} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} 2\, \partial_t\psi \, \frac{\partial \partial_t \psi}{\partial v^*}|_{u^*=\tau} \tag{1.6.13}$$

and

$$\int_{\tau+R^*}^{\tau+R^*+\tau^3} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \partial_t\psi \, \frac{\partial \partial_t \psi}{\partial v^*}|_{u^*=\tau} \leq \\ \leq \sqrt{\int_{\tau+R^*}^{\infty} \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \frac{1}{r^2} (\partial_t\phi)^2 r^{n-1}} \times \sqrt{\int_{\tau+R^*}^{\infty} \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^2 \Big(\frac{\partial r^{\frac{n-1}{2}} \partial_t\phi}{\partial v^*} \Big)^2} \,, \tag{1.6.14}$$

the first factor decaying with a rate τ^{-1} by Lemma B.6 and Prop. 1.37, and the second factor bounded by the weighted energy inequality for $r^{\frac{n-1}{2}} \partial_t\phi$ in place of ψ with $p = 2$. Therefore

$$\int_{\mathbb{S}^{n-1}} r^{n-1} (\partial_t\phi)^2|_{(u^*=\tau, v^*=\tau+R^*+\tau^3)} \leq \frac{C(n,m)}{1 - \frac{2m}{R^{n-2}}} \frac{D}{\tau} \,. \tag{1.6.15}$$

By virtue of Prop. 1.39, compare in particular Remark 1.44 on page 98, the first term on the right hand side of (1.6.11) decays with a rate of $\tau^{4-4\delta}$, and this is matched by the second term in view of the prefactor $r^{-1} = (R^* + \tau^3)^{-1}$, which is the result of our choice of powers of τ in the extension of the integral (1.44). Lemma 1.54 applied to the solution $\partial_t\phi$ of (1.1.1) then yields the pointwise decay result (1.6.4) after having applied Prop. 1.39 to the higher order energies on the right hand side of (1.6.10). □

Interpolation. We shall now interpolate between the results Prop. 1.53 (i) and (ii) to improve the pointwise estimate for $|\phi|$. Our argument can in some sense be compared to the proof of improved decay in [36]. The basic observation underlying this argument is that for $r_0 < r < R$ and $t_1 > t_0$

$$\begin{aligned} r^{n-2}\phi^2(r,t_1) &= r^{n-2}\phi^2(r,t_0) + \int_{t_0}^{t_1} 2\,\phi(t,r)\,\frac{\partial\phi}{\partial t}(t,r)\,r^{n-2}\,\mathrm{d}t \\ &\leq r^{n-2}\phi^2(r,t_0) + \frac{1}{t_0^{1-2\delta}}\int_{t_0}^{t_1}\phi^2(t,r)\,r^{n-2}\,\mathrm{d}t + t_0^{1-2\delta}\int_{t_0}^{t_1}\Bigl(\frac{\partial\phi}{\partial t}\Bigr)^2(t,r)\,r^{n-2}\,\mathrm{d}t\,. \end{aligned} \tag{1.6.16}$$

Moreover, as a consequence of Lemma 1.55,

$$r^{n-2}\phi^2(t,r) \leq R^{n-2}\phi^2(t,R) + \bigl(1-\frac{2m}{r_0^{n-2}}\bigr)^{-1}\int_{r^*}^{R^*}\Bigl(\frac{\partial\phi}{\partial r^*}\Bigr)^2 r^{n-1}\,\mathrm{d}r^*\,, \tag{1.6.17}$$

we obtain an estimate for the timelike integrals in terms of the corresponding integrals at $r = R$ and spacetime integrals, using the Sobolev inequality on the sphere:

$$\begin{aligned} \int_{t_0}^{t_1} r^{n-2}\phi^2(t,r)\,\mathrm{d}t &\leq \int_{t_0}^{t_1}\mathrm{d}t\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\sum_{|\alpha|\leq[\frac{n}{2}]+1} R^{n-2}\Bigl(\Omega^\alpha\phi\Bigr)^2(t,R) \\ &+ \bigl(1-\frac{2m}{r_0^{n-2}}\bigr)^{-1}\int_{t_0}^{t_1}\mathrm{d}t\int_{r^*}^{R^*}\mathrm{d}r^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}r^{n-1}\sum_{|\alpha|\leq[\frac{n}{2}]+1}\Bigl(\frac{\partial\Omega^\alpha\phi}{\partial r^*}\Bigr)^2(t,r) \end{aligned} \tag{1.6.18}$$

Lemma 1.55. *Let $a < b \in \mathbb{R}$ and $\phi \in \mathrm{C}^1([a,b])$ then*

$$a^{n-2}\phi^2(a) \leq b^{n-2}\phi^2(b) + \int_a^b\Bigl(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Bigr)^2 x^{n-1}\,\mathrm{d}x \tag{1.6.19}$$

for all $n \geq 3$.

Proof. Since, by integration by parts,

$$\begin{aligned} \int_a^b 2\phi(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x)x^{n-2}\,\mathrm{d}x &= 2\phi^2(x)x^{n-2}|_a^b \\ &- \int_a^b 2\phi(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x)x^{n-2}\,\mathrm{d}x - \int_a^b 2\phi^2(x)(n-2)x^{n-3}\,\mathrm{d}x\,, \end{aligned}$$

it clearly follows, with Cauchy's inequality,

$$\begin{aligned} a^{n-2}\phi^2(a) &\leq b^{n-2}\phi^2(b) + \int_a^b\Bigl(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Bigr)^2 x^{n-1}\,\mathrm{d}x \\ &+ \bigl[1-(n-2)\bigr]\int_a^b\frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x\,. \quad \square \end{aligned}$$

Proposition 1.56 (Improved interior pointwise decay)**.** *Let ϕ be a solution of the wave equation* (1.1.1)*, with initial data on Σ_{τ_0} ($\tau_0 > 1$) satisfying*

$$D \doteq \int_{\tau_0+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \times \Big\{ \sum_{k=0}^{2} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^{4-\delta} \Big(\frac{\partial (T^k\cdot\Omega^\alpha \chi)}{\partial v^*}\Big)^2 + \sum_{k=0}^{5} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^{2} \Big(\frac{\partial T^k\Omega^\alpha \psi}{\partial v^*}\Big)^2 + \sum_{k=0}^{4} \sum_{|\alpha|\leq[\frac{n}{2}]+2} r^{2} \Big(\frac{\partial T^k\Omega^\alpha \psi}{\partial v^*}\Big)^2 \Big\}\Big|_{u^*=\tau_0} + \int_{\Sigma_{\tau_0}} \Big(\sum_{k=0}^{6} \sum_{|\alpha|\leq[\frac{n}{2}]+1} J^N(T^k\Omega^\alpha\phi) + \sum_{k=0}^{5} \sum_{|\alpha|\leq[\frac{n}{2}]+2} J^N(T^k\Omega^\alpha\phi), n \Big) < \infty \,. \tag{1.6.20}$$

for some $0<\delta<\frac{1}{4}$, where $R > \sqrt[n-2]{\frac{8nm}{\delta}}$, $n\geq 3$. Then there exists a constant $C(n,m,\delta,R)$ such that for $\sqrt[n-2]{2m} < r_0 < r < R$,

$$r^{\frac{n-2}{2}}|\phi|(t,r) \leq \frac{C\,D}{t^{\frac{3}{2}-\delta}}\,. \tag{1.6.21}$$

Proof. Let $\bar{t}_0 = 2(\tau_0+\tau_0)+R^*$ and $\bar{t}_1 = \bar{t}_0+2\tau_0$ then by (1.6.18), Prop. 1.14 and Prop. 1.11

$$\int_{\bar{t}_0}^{\bar{t}_1} \phi^2(t,r) r^{n-2}\,\mathrm{d}t \leq C(n,m,R) \int_{\Sigma_{2\tau_0}} \Big(\sum_{k=0}^{1} \sum_{|\alpha|\leq[\frac{n}{2}]+1} J^T[T^k\Omega^\alpha\phi], n\Big)\,; \tag{1.6.22}$$

hence by Prop. 1.37 there exists $t_0' \in (\bar{t}_0, \bar{t}_1)$ such that

$$r^{n-2}\phi^2(t_0',r) \leq \frac{C(n,m,R)\,D}{t_0^3}\,. \tag{1.6.23}$$

Now set $\tau_0' = \frac{1}{2}(t_0'-R^*)$ and $\tau_j' = 2\tau_{j-1}'$ ($j\in\mathbb{N}$), and $t_j' = 2\tau_j' + R^*$ ($j\in\mathbb{N}$); note that $t_{j+1}'-t_j' = \frac{1}{2}(t_j'-R^*)$. Now consider (1.6.16) with $t_1 = t_{j+1}'$, $t_0 = t_j'$; since by (1.6.18), together with Prop. 1.11 and Prop. 1.14,

$$\int_{t_j'}^{t_{j+1}'} r^{n-2}\phi^2(t,r)\,\mathrm{d}t \leq C(n,m,R) \int_{\Sigma_{\tau_j'}} \Big(\sum_{k=0}^{1} \sum_{|\alpha|\leq[\frac{n}{2}]+1} J^T[T^k\Omega^\alpha\phi], n\Big)\,, \tag{1.6.24}$$

and by Prop. 1.32 and Prop. 1.33,

$$\int_{t_j'}^{t_{j+1}'} r^{n-2}(\partial_t\phi)^2(t,r)\,\mathrm{d}t \leq \leq C(n,m,R)\Big\{ \int_{\Sigma_{\tau_j'}\cap\{r^*\leq R^*+(\tau_j')^3\}} \Big(\sum_{k=1}^{2} \sum_{|\alpha|\leq[\frac{n}{2}]+1} J^T[T^k\Omega^\alpha\phi], n\Big) + \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \sum_{|\alpha|\leq[\frac{n}{2}]+1} r^{n-2}(\Omega^\alpha\partial_t\phi)^2|_{(u^*=\tau_j', v^*=R^*+\tau_j'+(\tau_j')^3)} \Big\}\,, \tag{1.6.25}$$

which decays with the rate $\tau^{4-4\delta}$ as is shown in the proof of Prop. 1.53 (ii), we obtain

$$
\begin{aligned}
&r^{n-2}\phi^2(r,t'_{j+1}) \leq \\
&\qquad \leq r^{n-2}\phi^2(r,t'_j) + \frac{C(n,m,R)}{(t'_j)^{1-2\delta}}\frac{D}{(\tau'_j)^2} + C(n,m,\delta,R)(t'_j)^{1-2\delta}\frac{D}{(\tau'_j)^{4-4\delta}} \leq \\
&\qquad\qquad \leq r^{n-2}\phi^2(r,t'_j) + \frac{C(n,m,\delta,R)\,D}{(t'_j)^{3-2\delta}}\,. \qquad (1.6.26)
\end{aligned}
$$

In fact, by induction on $j \in \mathbb{N}$ using (1.6.23) for $j=0$, we have shown

$$r^{n-2}\phi^2(r,t'_j) \leq \frac{C(n,m,\delta,R)\,D}{(t'_j)^{3-2\delta}} \qquad (j \in \mathbb{N}\cup\{0\})\,. \qquad (1.6.27)$$

Finally for any $t \geq t'_0$ we may choose $j \in \mathbb{N}\cup\{0\}$ such that $t \in (t'_j, t'_{j+1})$ and conclude the proof by applying (1.6.27) and (1.6.26) which holds with t in place of t'_{j+1}. □

Extension to the horizon. Note that for $\sqrt[n-2]{2m} \leq r < r_0$ the same interpolation (1.6.16) by integration along lines of constant radius $r < r_0$ can be carried out. However, on the right hand sides of (1.6.17) and (1.6.18) a new term results from the integration on $v^* = \frac{1}{2}(t_0 + r_0^*)$ from the radius $r < r_0$ to $r = r_0$; but we infer from the explicit construction (1.3.18) that the resulting integrand

$$\Bigl(\frac{2}{1-\frac{2m}{r^{n-2}}}\frac{\partial\phi}{\partial u^*}\Bigr)^2 \leq T[\phi](Y,Y) \leq \Bigl(J^N[\phi], N\Bigr) \qquad (1.6.28)$$

is controlled by Cor. 1.13 and the proof of Prop. 1.56 above extends to that of Thm. 2 by replacing J^T by J^N on the right hand sides of (1.6.22), (1.6.24) and (1.6.25).

Chapter 2

Linear waves on expanding Schwarzschild de Sitter spacetimes

2.1 Overview

It is the purpose of the work presented in this Chapter to initiate the global study of linear waves on cosmological spacetimes.

A common feature of expanding spacetimes is suggested by the global causal geometry of the simplest explicitly known solutions to the vacuum Einstein equations with positive cosmological constant. The Schwarzschild de Sitter family exhibits a region of spacetime which is bounded in the past by two cosmological horizons and in the future by a *spacelike* hypersurface of unbounded area.

We provide a suitably robust approach to the analysis of linear waves in these regions. More precisely, we establish uniform energy estimates for general solutions to the linear wave equation in the expanding regions of de Sitter and Schwarzschild de Sitter spacetimes which extend by a stable redshift mechanism to a global estimate.

Statement of the Main Result. Let Σ^+ be the timelike future boundary of a chosen expanding region in a subextremal Schwarzschild de Sitter spacetime $(\mathcal{M}, g)$. Σ^+ is a spacelike hypersurface with topology $\mathbb{R} \times \mathbb{S}^2$ endowed with the standard metric $\mathring{g}$ of the cylinder. Let $\Sigma \subset J^-(\Sigma^+)$ be a spacelike hypersurface in the past of Σ^+ such that Σ^+ is in the domain of dependence of Σ and such that Σ crosses the cosmological and event horizons to the future of the bifurcation spheres (see figure 2.1). We consider the Cauchy problem

$$\Box_g \psi = 0 \tag{2.1.1}$$

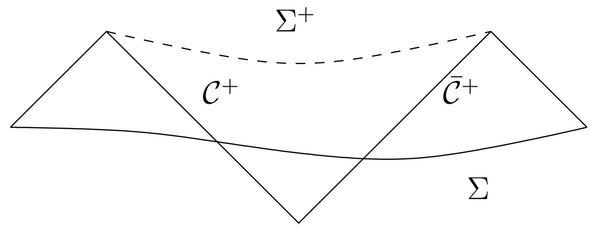

Figure 2.1: Cauchy problem (2.1.1) with initial data on Σ. The expanding region is bounded in the past by the cosmological horizons $\bar{\mathcal{C}}^+ \cup \mathcal{C}^+$ and in the future by the spacelike hypersurface Σ^+.

with initial data prescribed on Σ. Let T be the energy momentum tensor of (2.1.1) and n the normal to Σ. We show if the energy of a solution to (2.1.1) is initially finite

$$D[\psi] \doteq \int_\Sigma T(n,n) < \infty\,, \tag{2.1.2}$$

then it is globally bounded in the expanding region and has a limit on Σ^+. Furthermore, the limit of ψ on Σ^+ *as a function on* $\mathbb{R}\times\mathbb{S}^2$ satisfies

$$\int_{\Sigma^+} \big|\overset{\circ}{\nabla}\psi\big|^2_{\overset{\circ}{g}}\,\mathrm{d}\mu_{\overset{\circ}{g}} \leq C(\mathcal{M},\Sigma)\,D[\psi] \tag{2.1.3}$$

where C is a constant that only depends on the given manifolds $\mathcal{M}$ and Σ, and $\overset{\circ}{\nabla}$ denotes the gradient on the standard cylinder $\mathbb{R}\times\mathbb{S}^2$.

Remark 2.1. The expanding region can be foliated by spacelike hypersurfaces $(\Sigma_r, \overline{g}_r)$ which are *conformal* to the standard cylinder; in fact $\overline{g}_r = r^2\,\overset{\circ}{g} + \mathcal{O}(\frac{1}{r})$. Since the future boundary can be viewed as the set

$$\Sigma^+ = \bigcap_r J^+(\Sigma_r)\,, \tag{2.1.4}$$

the statement (2.1.3) is a decay result for the induced derivatives on Σ_r as they approach Σ^+.

Remark 2.2. The wave equation on “asymptotically de Sitter-like spaces” was previously studied by [47] however with results that are *local* in nature.

The precise statement of the main result and an overview of our proof is given in Section 2.3. The reader is advised however to first familiarize herself with our treatment of linear waves on de Sitter spacetimes in Section 2.2 where many of the ideas for our approach originate and their application is seen more clearly. Moreover, Section 2.2 offers a self-contained global analysis of linear waves on de Sitter spacetimes including results for the Klein-Gordon equation.

2.2 Linear Waves on de Sitter

In the following we consider solutions to the linear wave equation on de Sitter spacetime. The de Sitter spacetime is the simplest solution to the vacuum Einstein equations with positive cosmological constant and as such the primary example of an expanding spacetime.

In the context of the stability problems for the Einstein equations it is generally expected that a positive cosmological constant introduces a stability mechanism. This may in particular be true for the dynamics of black hole exteriors. Accordingly my treatment aims at an understanding of linear waves on *Schwarzschild*-de Sitter spacetimes which should be viewed as a first model problem for the nonlinear stability problem of black hole exteriors in expanding cosmological spacetimes.

Remark 2.3. The favorable role of the positive cosmological constant for a stability problem was first recognized by Friedrich [28] who was able to reduce the global stability problem for initial data close to de Sitter to a local problem for a quasi-linear symmetric hyperbolic system. Ringström has extended these ideas to scalar field models in [40], and the stabilizing effect of the cosmological constant has independently been confirmed for perfect fluids in [41, 43] (the case of pure radiation perfect fluids which is not in the scope of this work has been resolved using the conformal method in [33]). However, it is not in the scope of these approaches to accomodate for initial data close to *Schwarzschild*-de Sitter for which the *global* geometry is very different; (in particular the spacetime is not foliated by *compact* Cauchy hypersurfaces). It is for this reason that I emphasize the global aspect of my approach in this Section which at no point makes use of the homogeneity (or even local homogeneity) of the spacetime; (which is lost in Schwarzschild de Sitter spacetimes). It would be possible to *localize* the linear problem near the future boundary of (Schwarzschild-)de Sitter but this approach would not yield a global result in the absence of homogeneity.

Problem. We are interested in a full understanding of the global behaviour of solutions to the linear wave equation

$$\Box_g \psi = m_\Lambda \psi \tag{2.2.1}$$

on de Sitter spacetime $(\mathcal{M}_\Lambda, g)$ where $m_\Lambda \geq 0$.

We take the point of view on de Sitter as a member of the Schwarzschild de Sitter family with mass parameter $m = 0$. This is equivalent to an a priori choice of a timelike geodesic Γ in $(\mathcal{M}_\Lambda, g)$ as the center of symmetry. (De Sitter spacetime is homogeneous, in particular spherically symmetric with respect to any chosen timelike geodesic.) In Section 2.2.1 I describe the global causal geometry of de Sitter in terms of the past, the past boundary, and its complement of any chosen timelike geodesic Γ.

It is shown that the intersection of the past and future of Γ is in fact a static region of spacetime. Its boundaries are described in analogy to black hole event horizons as cosmological horizons with positive surface gravity. The complement of these domains is the expanding region of spacetime which exhibits a global redshift effect. In Section 2.2.2 it is shown how these crucial geometric properties are used to establish energy estimates for solutions to (2.2.1).

My approach in particular yields explicit estimates on the spacelike future boundary of the spacetime in terms of initial data prescribed on any given spacelike hypersurface.

Main Results. Let Γ be a timelike geodesic in de Sitter $(\mathcal{M}_\Lambda, g)$, where $\Lambda > 0$ is the cosmological constant. We denote by r the area radius of the round spheres defined by the orbits of the SO(3) subgroup of the isometry group that leaves Γ invariant, and by $\Omega_{(i)}$ the generators of this group action.

Consider the level sets Σ_r of the area radius function r on $(\mathcal{M}_\Lambda, g)$. These are timelike hypersurfaces for $r < \sqrt{\frac{3}{\Lambda}}$, and null hypersurfaces for $r = \sqrt{\frac{3}{\Lambda}}$. The domain $r < \sqrt{\frac{3}{\Lambda}}$ is the static region of spacetime in the intersection of the timelike future and past of Γ, and as such endowed with a timelike Killing vectorfield T, which is tangential to Σ_r. We denote by $\mathcal{C}^+$ the future boundary of the past of Γ, and by $\mathcal{C}^-$ the past boundary of the future of Γ; these are the cosmological horizons of Γ.

The level sets Σ_r are spacelike hypersurfaces for $r > \sqrt{\frac{3}{\Lambda}}$. More precisely, the hypersurfaces Σ_r are topologically cylinders that foliate the expanding region $r > \sqrt{\frac{3}{\Lambda}}$, and T extends to a global Killing vectorfield tangential to Σ_r. In this domain the metric then takes the form

$$g = -\phi^2 \,\mathrm{d}r^2 + \overline{g}_r = -\frac{1}{\frac{\Lambda}{3}r^2 - 1}\,\mathrm{d}r^2 + \overline{g}_r\,, \tag{2.2.2}$$

where $\overline{g}_r$ is the induced metric on Σ_r, a Riemannian metric conformal to the standard cylinder; in fact

$$\lim_{r\to\infty} \frac{1}{r^2}\,\overline{g}_r = \frac{1}{4}\,\mathrm{d}\lambda^2 + \mathring{\gamma} \doteq \mathring{g}\,, \tag{2.2.3}$$

where $\lambda \in (-\infty, \infty)$ and $\mathring{\gamma}$ denotes the standard metric on the unit sphere $\mathbb{S}^2$. We denote by Σ^+ the timelike future boundary of the expanding region $\bigcup_{r>\sqrt{\frac{3}{\Lambda}}} \Sigma_r$ endowed with the rescaled metric $\mathring{g}$.

Theorem 4. *Let Σ be a spacelike hypersurface with normal n in de Sitter $(\mathcal{M}_\Lambda, g)$, and Γ a timelike geodesic with cosmological horizons $\mathcal{C}^+$ and $\mathcal{C}^-$, and assume that Σ crosses the horizons to the future of $\mathcal{C}^+ \cap \mathcal{C}^-$ (see figure 2.2). Let moreover ψ be a solution to the linear wave equation* (2.2.1) *for either $m_\Lambda = 0$ or $m_\Lambda \geq 2\frac{\Lambda}{3}$ with initial data prescribed on Σ such that*

$$D[\psi] \doteq \int_\Sigma \Big\{ J^n[\psi] \cdot n + m_\Lambda \psi^2 \Big\} < \infty\,. \tag{2.2.4}$$

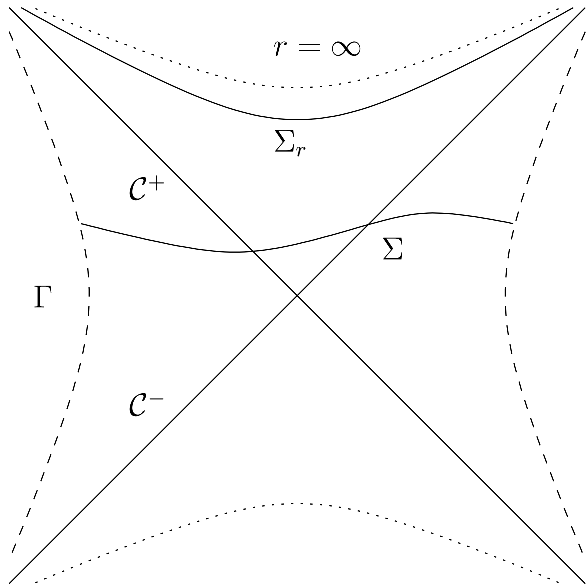

Figure 2.2: Penrose diagram of de Sitter depicted as a spherically symmetric spacetime with respect to any chosen timelike geodesic Γ.

Then the energy of ψ is globally bounded, and satisfies

$$\int_{\Sigma^+}\Big\{\big(T\cdot\psi\big)^2+\sum_{i=1}^{3}\big(\Omega_{(i)}\psi\big)^2+m_\Lambda\big(r\psi\big)^2\Big\}\,\mathrm{d}\mu_{\mathring{g}}\leq C(\Lambda,\Sigma)\,D[\psi] \tag{2.2.5}$$

on the future boundary Σ^+, where C is a constant that only depends on Λ and the initial hypersurface Σ.

Note that the tangent space to Σ^+ is spanned by Killing vectorfields and that those tangential derivatives are controlled in L^2 by (2.2.5). By the classical Sobolev inequalities this immediately implies the following pointwise estimates.

Theorem 5. *Let Σ and Γ be as in Theorem 4, and let ψ be a solution to* (2.2.1) *for either $m_\Lambda=0$ or $m_\Lambda\geq 2\frac{\Lambda}{3}$ with initial data prescribed on Σ such that in addition to* (2.2.4) *we have*

$$D\doteq D[\psi]+\sum_{i=1}^{3}D[\Omega_{(i)}\psi]+\sum_{i,j=1}^{3}D[\Omega_{(i)}\Omega_{(j)}\psi] \\ +D[T\psi]+\sum_{i=1}^{3}D[\Omega_{(i)}T\psi]+\sum_{i,j=1}^{3}D[\Omega_{(i)}\Omega_{(j)}T\psi]<\infty\,. \tag{2.2.6}$$

Then

$$\sup_{p,q\in\Sigma^+}\Big\{\Big|\big(\frac{\partial\psi}{\partial\lambda}\big)^2(p)-\big(\frac{\partial\psi}{\partial\lambda}\big)^2(q)\Big|+\Big|\big|\mathring{\nabla}\,\psi\big|^2_{\mathring{\gamma}}(p)-\big|\mathring{\nabla}\,\psi\big|^2_{\mathring{\gamma}}(q)\Big| \\ +m_\Lambda\Big|\big(r\psi\big)^2(p)-\big(r\psi\big)^2(q)\Big|\Big\}\leq C(\Lambda)\,D\,, \tag{2.2.7}$$

where ∂_λ denotes a coordinate derivative on Σ^+ (c.f. (2.2.3)*) and $\overset{\circ}{\nabla}$ the covariant derivative on the standard sphere $(\mathbb{S}^2, \overset{\circ}{\gamma})$, and C is a constant that only depends on Λ.*

The result in particular states that solutions to the linear wave equation (2.2.1) have a limit on Σ^+ which can be viewed as a function on $\mathbb{R}\times\mathbb{S}^2$ satisfying (2.2.5) in the case $m_\Lambda = 0$ or vanishing identically in the case $m_\Lambda \geq 2\frac{\Lambda}{3}$.

Proof and Overview. In our approach the study of solutions to the wave equation on de Sitter spacetime is carried out in three separate domains: the expanding region, the cosmological horizon and the static region.

In Section 2.2.2.1 we construct a *global* redshift vectorfield that captures the *expansion* of the spacetime in the region $r > \sqrt{\frac{3}{\Lambda}}$. In fact, in Proposition 2.13 we establish that for any solution of (2.2.1) with $m_\Lambda \geq 0$ we have

$$\int_{\Sigma_{r_2}} \phi\Big\{\phi^2\big(T\cdot\psi\big)^2 + |\nabla\!\!\!/\psi|^2 + m_\Lambda\psi^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_2}} \leq$$
$$\leq \int_{\Sigma_{r_1}} \phi\Big\{\phi^2\big(T\cdot\psi\big)^2 + \frac{1}{\phi^2}\big(\frac{\partial\psi}{\partial r}\big)^2 + |\nabla\!\!\!/\psi|^2 + m_\Lambda\psi^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_1}} \qquad (2.2.8)$$

for all $r_2 > r_1 > \sqrt{\frac{3}{\Lambda}}$, where ϕ is the lapse function of the foliation (Σ_r) given by (2.2.38).

In Section 2.2.2.5 we explicitly exploit the positive surface gravity of the cosmological horizons to construct a *local* redshift vectorfield. Given a spacelike hypersurface Σ that crosses the cosmological horizon $\mathcal{C}^+$ to the future of $\mathcal{C}^+\cap\mathcal{C}^-$, we prove in Proposition 2.28 that the right hand side of (2.2.8) is bounded by

$$C(r_1,\Lambda)\int_{\mathcal{C}_0^+}\Big\{\frac{\Lambda}{3}u\big(\frac{\partial}{\partial u}\psi\big)^2 + \frac{1}{u}\big|\nabla\!\!\!/\psi\big|^2 + \frac{m_\Lambda}{u}\psi^2\Big\} + C(r_1)\int_{\Sigma'} J^{n;m_\Lambda}\cdot n\,, \qquad (2.2.9)$$

where Σ' is the segment of Σ truncated by Σ_{r_1} and $\mathcal{C}^+$, and $\mathcal{C}_0^+$ denotes the segment of $\mathcal{C}^+$ truncated by Σ, *provided* $r > \sqrt{\frac{3}{\Lambda}}$ is chosen small enough. Note that Σ_{r_1} is an asymptote hypersurface to the the null hypersurface $\mathcal{C}^+$. The bound (2.2.9) holds for all solutions of (2.2.1) with $m_\Lambda \geq 0$, where $C(r_1,\Lambda)$ is a constant that only depends on the fixed value r_1 and Λ.

In Section 2.2.2.3 we finally obtain control on the nondegenerate energy on the cosmological horizon as the boundary of the static region. More precisely we establish in Propositions 2.23 and 2.26 that *if* $m_\Lambda = 0$ or $m_\Lambda \geq 2\frac{\Lambda}{3}$ then there exists a timelike vectorfield N such that

$$\int_{\mathcal{C}_0^+}\frac{1}{u}\Big\{\big|\nabla\!\!\!/\psi\big|^2 + m_\Lambda\psi^2\Big\} \leq C(\Lambda)\int_\Sigma\Big\{J^N[\psi]\cdot n + m_\Lambda\psi^2\Big\}\,; \qquad (2.2.10)$$

here and in the above (u,v) are the double null coordinates introduced in Section 2.2.1 and $|\nabla\!\!\!/\psi|$ denotes the angular derivatives on the sphere in the induced norm on the spheres of constant area radius r, and . The estimate (2.2.10) relies on an *integrated local energy* estimate in the static region which is established in Proposition 2.20 using Morawetz type vectorfields. It is in this part of our proof that the condition $m_\Lambda \geq 2\frac{\Lambda}{3}$ is introduced; ($m_\Lambda = 2\frac{\Lambda}{3}$ is the *conformal* value of the mass, see also Remark 2.16). It arises because our proof in the homogeneous case $m_\Lambda = 0$ relies on a Poincaré inequality, and we expect that an alternative proof can avoid a lower bound on m_Λ (and establish the required estimate for all $m_\Lambda \geq 0$), which in view of (2.2.8) should not be essential to our result.

Given initial data to the Cauchy problem of (2.2.1) of finite energy on a spacelike hypersurface Σ crossing the cosmological horizons to the future of the sphere $\mathcal{C}^+ \cap \mathcal{C}^-$ we have thus established that also the energy flux through Σ_{r_1} on the right hand side of (2.2.8) is finite and bounded by that initial energy, (provided $r_1 > \sqrt{\frac{3}{\Lambda}}$ is chosen small enough and $m_\Lambda = 0$ or $m_\Lambda \geq 2\frac{\Lambda}{3}$). Since

$$\phi \ \mathrm{d}\mu_{\overline{g}_r} = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\, r^2 \ \mathrm{d}\lambda \wedge \mathrm{d}\mu_{\overset{\circ}{\gamma}} \,, \tag{2.2.11}$$

and by the coercivity equality on the sphere (see Appendix C.2)

$$r^2 \big|\nabla\!\!\!\!/ \psi\big|^2 = \sum_{i=1}^{3} \big(\Omega_{(i)}\psi\big)^2 \,, \tag{2.2.12}$$

we are now allowed to take the limit in (2.2.8) as r_2 tends to infinity, to conclude on the finiteness of the following energy:

$$\lim_{r\to\infty} \int_{\Sigma_r} \phi \left\{ \phi^2 \big(T\cdot\psi\big)^2 + |\nabla\!\!\!\!/ \psi|^2 + m_\Lambda \psi^2 \right\} \mathrm{d}\mu_{\overline{g}_{r_2}} =$$
$$= \frac{1}{2}\sqrt{\frac{3}{\Lambda}} \int_{\Sigma^+} \left\{ \frac{3}{\Lambda}\big(T\cdot\psi\big)^2 + \sum_{i=1}^{3}\big(\Omega_{(i)}\psi\big)^2 + m_\Lambda \big(r\psi\big)^2 \right\} \mathrm{d}\lambda \wedge \mathrm{d}\mu_{\overset{\circ}{\gamma}} \,. \tag{2.2.13}$$

Here Σ^+ is the future boundary of the expanding region $\bigcup_{r>\sqrt{\frac{3}{\Lambda}}} \Sigma_r$, a spacelike hypersurface with topology $\mathbb{R} \times \mathbb{S}^2$ endowed with the standard volume form of the cylinder $\mathrm{d}\lambda \wedge \mathrm{d}\mu_{\overset{\circ}{\gamma}}$.

2.2.1 Global geometry of de Sitter

The de Sitter spacetime is the simplest solution to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \,, \tag{2.2.14}$$

with positive cosmological constant $\Lambda > 0$.

We take the point of view of de Sitter as the member of the Schwarzschild-de Sitter family with $m = 0$. Let us make this more precise, with a few comments on the mass function in spherically symmetric cosmological spacetimes. See also [26].

Spherical Symmetry in cosmological spacetimes. Similarly to the expositions in [13], and Section 1.2 it is useful to decompose (2.2.14) into its spherical part and $\mathcal{Q} \doteq \mathcal{M}/\mathrm{SO}(3)$; here the spherical orbits are centred at a fixed timelike geodesic Γ.

The metric takes the form

$$g = -\Omega^2 \ \mathrm{d}u \ \mathrm{d}v + r^2 \ \overset{\circ}{\gamma}\,, \tag{2.2.15}$$

where g is the standard metric of the unit sphere $\mathbb{S}^2$, and we can think of (2.2.14) as partial differential equations for the area radius r (and the conformal factor Ω) in the double null coordinates u, v. Indeed, it is easily deduced from (2.2.14) that the area radius r satisfies the Hessian equations

$$\frac{\partial^2 r}{\partial u^2} - \frac{2}{\Omega}\frac{\partial \Omega}{\partial u}\frac{\partial r}{\partial u} = 0 \tag{2.2.16a}$$

$$\frac{\partial^2 r}{\partial u\,\partial v} + \frac{1}{r}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} = -\frac{\Omega^2}{4r} + \frac{\Omega^2}{4}\Lambda r \tag{2.2.16b}$$

$$\frac{\partial^2 r}{\partial v^2} - \frac{2}{\Omega}\frac{\partial \Omega}{\partial v}\frac{\partial r}{\partial v} = 0\,. \tag{2.2.16c}$$

It is a key insight that by virtue of these equations the mass function m defined by

$$1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} = -\frac{4}{\Omega^2}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} \tag{2.2.17}$$

is constant (c.f. [13] in the case $\Lambda = 0$), and is precisely the quantity that parametrizes the Schwarzschild-de Sitter family for any fixed $\Lambda > 0$. Here of course,

$$m = 0\,. \tag{2.2.18}$$

It is also useful to introduce the "tortoise coordinate"

$$r^* = \int \frac{1}{1 - \frac{\Lambda r^2}{3}}\,\mathrm{d}r = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\log\Bigl|\frac{1 + \sqrt{\frac{\Lambda}{3}}r}{1 - \sqrt{\frac{\Lambda}{3}}r}\Bigr| \tag{2.2.19}$$

which satisfies – again as a consequence of (2.2.16) – the simple partial differential equation

$$\frac{\partial^2 r^*}{\partial u\,\partial v} = 0\,. \tag{2.2.20}$$

In the coordinate system that covers the entire manifold (with boundary) $\mathcal{Q}$ we have

$$r^* = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\log\frac{1}{|uv|}\,, \tag{2.2.21}$$

and we can express r explicitly as a function of u, v:

$$r = \sqrt{\frac{3}{\Lambda}}\frac{1 + uv}{1 - uv}\,. \tag{2.2.22}$$

Finally, using (2.2.17),

$$\Omega^2 = \frac{3}{\Lambda}\Bigl(1 + \sqrt{\frac{\Lambda}{3}}r\Bigr)^2\,, \tag{2.2.23}$$

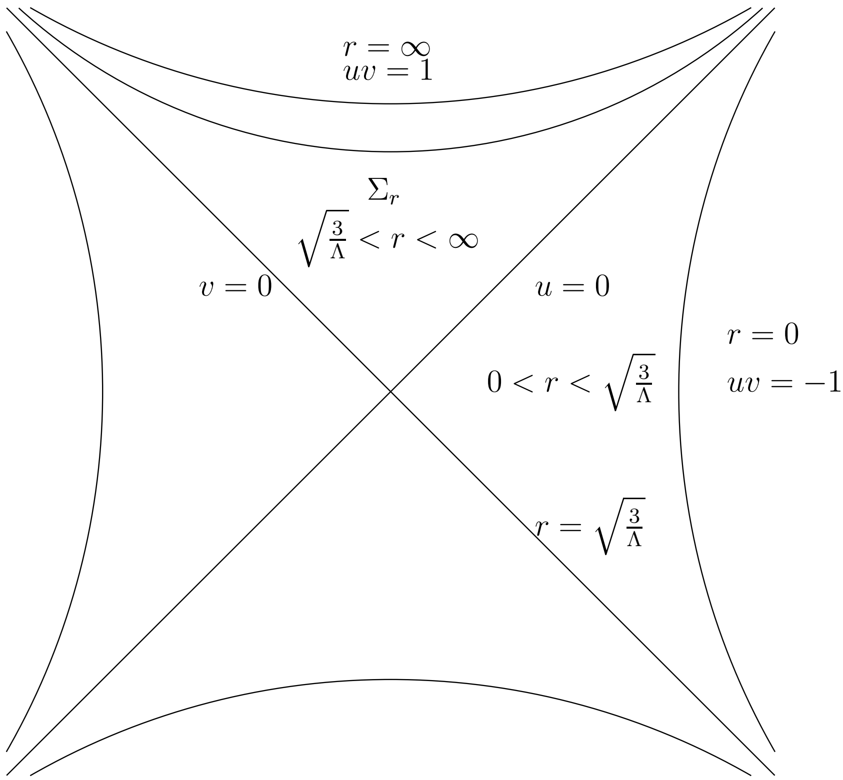

Figure 2.3: Global causal geometry of de Sitter.

and the de Sitter metric takes the form:

$$g = -\frac{3}{\Lambda}\Big(1 + \sqrt{\frac{\Lambda}{3}}r\Big)^2 \,\mathrm{d}u\,\mathrm{d}v + r^2 \,\mathring{\gamma} \ . \tag{2.2.24}$$

The causal geometry of the spacetime can thus be depicted as in figure 2.3.

For future reference, note also

$$\frac{\partial r}{\partial u} = \sqrt{\frac{3}{\Lambda}}\frac{2v}{(1-uv)^2} \qquad \frac{\partial r}{\partial v} = \sqrt{\frac{3}{\Lambda}}\frac{2u}{(1-uv)^2} \tag{2.2.25}$$

and

$$\frac{1}{1+\sqrt{\frac{\Lambda}{3}}r} = \frac{1-uv}{2}\,. \tag{2.2.26}$$

2.2.1.1 Static region

In the patch $\mathcal{S} \doteq \{(u,v) : u > 0, v < 0, uv > -1\}$ we may complement (2.2.22) by

$$t = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\log\frac{u}{-v}\,. \tag{2.2.27}$$

(And similarly in the region $\mathcal{S}' = \{(u,v) : u < 0, v > 0, uv > -1\}$.) Then the metric takes the familiar static form

$$g = -\Big(1 - \frac{\Lambda r^2}{3}\Big)\,\mathrm{d}t^2 + \Big(1 - \frac{\Lambda r^2}{3}\Big)^{-1}\mathrm{d}r^2 + r^2\,\mathring{\gamma} \qquad \Big(0 < r < \sqrt{\frac{3}{\Lambda}}\Big)\,. \tag{2.2.28}$$

We shall also make use of “Eddington-Finkelstein”-coordinates

$$u^* = \frac{1}{2}\bigl(t - r^*\bigr) \tag{2.2.29a}$$

$$v^* = \frac{1}{2}\bigl(t + r^*\bigr) \tag{2.2.29b}$$

which yields as well

$$g = -4\Bigl(1 - \frac{\Lambda r^2}{3}\Bigr)\,\mathrm{d}u^*\,\mathrm{d}v^* + r^2\,\overset{\circ}{\gamma}\ . \tag{2.2.30}$$

Remark 2.4. It is worth noting that this region $\mathcal{S}$ is “small”: it is the intersection of the past of a *point* with the future of a point. To see this we calculate the arclength of the spacelike curve $\gamma_\lambda = \{(u,v) : u^2 - v^2 = \lambda^2\} \cap \mathcal{S}$ to find $\lim_{\lambda\to\infty} \mathrm{L}[\gamma_\lambda] = 0$. In other words the point $(u = \infty, v = 0)$ is indeed a point on the future boundary of $\mathcal{M}$. This should be compared to the past of a point (u, v) with $uv = 1$; as discussed below it turns out that this region of $\mathcal{M}$ is in fact the past of a sphere, and in this sense “large”.

2.2.1.2 Cosmological horizon

This is the null hypersurface $r = \sqrt{\frac{3}{\Lambda}}$, but we will refer more specifically to the future boundary of $\mathcal{S}$ as the cosmological horizon, namely $\mathcal{C} \doteq \{(u,v) : v = 0, u \geq 0\}$. It has positive surface gravity

$$\kappa_{\mathcal{C}} = \sqrt{\frac{\Lambda}{3}}\,, \tag{2.2.31}$$

and thus plays an analogeous role to the event horizon in black hole spacetimes.

The globally defined vectorfield

$$T = \sqrt{\frac{\Lambda}{3}}\Bigl(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\Bigr) \tag{2.2.32}$$

coincides with ∂_t in $\mathcal{S}$, and has the following crucial properties:

Lemma 2.5 (Properties of the vectorfield T). *(i) T is future directed timelike in $\mathcal{S}$, null on $\mathcal{C}$, and spacelike for $r > \sqrt{\frac{3}{\Lambda}}$.*

(ii) T is globally Killing,

$$^{(T)}\pi \doteq \mathcal{L}_T g = 0\,. \tag{2.2.33}$$

(iii) On $\mathcal{C}$,

$$\nabla_T T = \sqrt{\frac{\Lambda}{3}}\,T\,. \tag{2.2.34}$$

Note that *(iii)* is the defining equation for (2.2.31).

2.2.1.3 Expanding region

Let us denote the "expanding" or "cosmological" region by $\mathcal{R} \doteq \{(u,v) : u > 0, v > 0, uv < 1\}$. Note that it is foliated by the level sets of the area radius

$$\mathcal{R} = \bigcup_{r>\sqrt{\frac{3}{\Lambda}}} \Sigma_r\,; \qquad \Sigma_r \doteq \Big\{(u,v) : \sqrt{\frac{3}{\Lambda}}\frac{1+uv}{1-uv} = r\Big\}\,. \tag{2.2.35}$$

Time function. We can use the radial function as a time function in $\mathcal{R}$, because in the expanding region the area radius is increasing towards the future. We denote the gradient vectorfield by

$$V^\mu = -(g^{-1})^{\mu\nu}\partial_\nu r \tag{2.2.36}$$

which is given by

$$V = \sqrt{\frac{\Lambda}{3}}\Big(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\Big)\,. \tag{2.2.37}$$

Therefore we obtain for the lapse function of the foliation (Σ_r)

$$\phi = \frac{1}{\sqrt{-g(V,V)}} = \frac{1}{\sqrt{\frac{\Lambda}{3}r^2 - 1}}\,, \tag{2.2.38}$$

and the decomposition of the metric in $\mathcal{R}$ takes the form

$$g = -\frac{1}{\frac{\Lambda}{3}r^2 - 1}\,\mathrm{d}r^2 + \overline{g}_r\,, \tag{2.2.39}$$

where $\overline{g}_r$ denotes the induced metric on Σ_r. Also note that while the normal to Σ_r is $n = \phi V$, we have for $\partial_r = \phi^2 V$.

Coarea formula.

$$\int_{\mathcal{R}} f\,\mathrm{d}\mu_g = \int_{\sqrt{\frac{3}{\Lambda}}}^{\infty}\Big[\int_{\Sigma_r}\phi\, f\;\mathrm{d}\mu_{\overline{g}_r}\Big]\,\mathrm{d}r \tag{2.2.40}$$

Induced metric on Σ_r. It is useful, for future reference, to give here an explicit expression for $\overline{g}_r$. For this purpose, we also introduce in $\mathcal{R}$ the coordinate

$$\sigma \doteq \sqrt{\frac{3}{\Lambda}}\frac{u}{v}\,. \tag{2.2.41}$$

A short calculation of $\mathrm{d}r$, and $\mathrm{d}\sigma$ in terms of $\mathrm{d}u$ and $\mathrm{d}v$ then reveals that

$$\overline{g}_r = \frac{1}{4}\frac{3}{\Lambda}\Big(\frac{\Lambda}{3}r^2 - 1\Big)\frac{1}{\sigma^2}\,\mathrm{d}\sigma^2 + r^2\,\mathring{\gamma}\,. \tag{2.2.42}$$

The coordinate vectorfield

$$\frac{\partial}{\partial\sigma} = \frac{1}{2}\sqrt{\frac{\Lambda}{3}}v^2\frac{1}{v}\frac{\partial}{\partial u} - \frac{1}{2}\sqrt{\frac{\Lambda}{3}}v^2\frac{1}{u}\frac{\partial}{\partial v} \tag{2.2.43}$$

does have the advantage that it is Lie transported by

$$\frac{\partial}{\partial r} = \frac{1}{2}\sqrt{\frac{\Lambda}{3}}\frac{(1-uv)^2}{2v}\frac{\partial}{\partial u} + \frac{1}{2}\sqrt{\frac{\Lambda}{3}}\frac{(1-uv)^2}{2u}\frac{\partial}{\partial v}\,; \tag{2.2.44}$$

indeed

$$[\frac{\partial}{\partial r}, \frac{\partial}{\partial \sigma}] = 0\,. \tag{2.2.45}$$

This is a convenient frame for the decomposition of the Einstein equations and the wave equation relative to the foliation (Σ_r), as is discussed in Appendix C.1. Alternatively, we can introduce

$$\lambda \doteq \log\sigma\,, \tag{2.2.46}$$

and since then

$$\mathrm{d}\lambda = \frac{1}{\sigma}\,\mathrm{d}\sigma \tag{2.2.47}$$

the volume form on Σ_r takes the form

$$\mathrm{d}\mu_{\overline{g}_r} = \frac{1}{2}\sqrt{\frac{3}{\Lambda}\Big(\frac{\Lambda}{3}r^2 - 1\Big)}\;\mathrm{d}\lambda\wedge\mathrm{d}\mu_{r^2\mathring{\gamma}}\,. \tag{2.2.48}$$

It is important to note that

$$\frac{\partial f}{\partial\lambda}\Big|_r = \sigma\frac{\partial f}{\partial\sigma}\Big|_r = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\,T\cdot f\,. \tag{2.2.49}$$

Together with the vectorfields $\Omega_{(i)}$ (discussed in Appendix C.2) this implies that *the tangent space to Σ_r is spanned by vectorfields that generate isometries of the spacetime.* This is relevant for the Sobolev inequality on Σ_r, see Section 2.2.2.6.

Remark 2.6. We call the region $\mathcal{R}$ (to the future of the cosmological horizons) "expanding" because it is the past of spheres of infinite radius rather than the past of a point. Indeed, a curve $\gamma_r = \{(u,v) : uv = \frac{\sqrt{\frac{\Lambda}{3}}r-1}{\sqrt{\frac{\Lambda}{3}}r+1}, u\leq 1, v\leq 1\}$ has arclength

$$\lim_{r\to\infty}\mathrm{L}[\gamma_r] = 2\sqrt{\frac{3}{\Lambda}} \tag{2.2.50}$$

in $\mathcal{Q}$, and thus defines an ideal point on the boundary representing a sphere of infinite radius.

2.2.2 Energy estimates

In this section I will discuss energy estimates for solutions to the linear homogeneous wave equation

$$\Box_g\psi = 0 \tag{2.2.51}$$

as well as for the inhomogeneous wave equation (or simply the "Klein-Gordon equation")

$$\Box_g\psi = m_\Lambda\psi \qquad (m_\Lambda > 0)\,. \tag{2.2.52}$$

Our main result concerns the expanding region and it applies as is discussed in Section 2.2.2.1 to the general case $m_\Lambda \geq 0$; the construction relies on a "global redshift" vectorfield, and lends itself to generalization in other expanding spacetimes. In Section 2.2.2.3 we prove a version of integrated local energy decay in the static region with an argument that we learn from [16] that dealt with the corresponding region in the Schwarzschild-de Sitter case. In Section 2.2.2.5 we will finally discuss the redshift effect on the cosmological horizon that shall allow us to turn our estimate in Section 2.2.2.1 into a global energy estimate.

2.2.2.1 Energy estimates and global redshift in the expanding region

In this section I present an argument to prove energy estimates in the expanding region, which is complemented by Section 2.2.2.2 where those calculations are to be found which are omitted here in favour of a clear exposition.

Recall the area radius r; its level sets Σ_r are spacelike hypersurfaces in $\mathcal{R}$ and define a foliation with lapse (2.2.38), and the metric takes the form (2.2.39).

Homogeneous wave equation. Consider the energy current J^M associated to the multiplier

$$M = \bar{Y} + Y\,, \tag{2.2.53}$$

where

$$\bar{Y} = \frac{1}{\frac{\partial r}{\partial u}}\frac{\partial}{\partial u}\,, \qquad Y = \frac{1}{\frac{\partial r}{\partial v}}\frac{\partial}{\partial v}\,. \tag{2.2.54}$$

We can show that for any solutions of (2.2.51), this is a positive current, i.e.

$$\nabla \cdot J^M \geq 0\,. \tag{2.2.55}$$

Therefore, by the energy identity for J^M,

$$\int_{\Sigma_{r_2}} J^M \cdot n \,\mathrm{d}\mu_{\overline{g}_{r_2}} \leq \int_{\Sigma_{r_1}} J^M \cdot n \,\mathrm{d}\mu_{\overline{g}_{r_1}} \tag{2.2.56}$$

which yields our energy estimate for solutions of (2.2.51).

Proposition 2.7 (Global Energy Boundedness, Homogeneous Case)**.** *Let ψ be a solution of the homogeneous wave equation* (2.2.51)*, then*

$$\int_{\Sigma_{r_2}} \phi\Big\{\phi^2\big(T\cdot\psi\big)^2 + |\overline{\nabla}\psi|^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_2}} \leq \int_{\Sigma_{r_1}} J^M \cdot n \,\mathrm{d}\mu_{\overline{g}_{r_1}} \tag{2.2.57}$$

for any $r_2 > r_1 > \sqrt{\frac{3}{\Lambda}}$.

Note that $\phi \,\mathrm{d}\mu_{\overline{g}_r} = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\, r^2 \,\mathrm{d}\lambda \wedge \mathrm{d}\mu_{\mathring{\gamma}}$.

It is equivalent to consider the multiplier

$$M = \frac{1}{1+\sqrt{\frac{\Lambda}{3}}r}\left(\bar{Y}+Y\right). \tag{2.2.58}$$

The associated energy current has the "redshift" property

$$\phi\,\nabla\cdot J^M \geq \frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r} J^M\cdot n \tag{2.2.59}$$

for any solution of (2.2.51); (here n denotes the unit normal to Σ_r). The energy identity for J^M then implies, in view of the coarea formula,

$$\int_{\Sigma_{r_2}} J^M\cdot n \,\mathrm{d}\mu_{\overline{g}_{r_2}} + \int_{r_1}^{r_2}\mathrm{d}r \frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r}\int_{\Sigma_r} J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_r} \leq$$

$$\leq \int_{\Sigma_{r_1}} J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}. \tag{2.2.60}$$

By a Gronwall-type inequality we obtain

$$\big(1+\sqrt{\frac{\Lambda}{3}}r_2\big)\int_{\Sigma_{r_2}} J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}} \leq \big(1+\sqrt{\frac{\Lambda}{3}}r_1\big)\int_{\Sigma_{r_1}} J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}, \tag{2.2.61}$$

which precisely reproduces (2.2.57). In fact, as we shall see below, the argument applies to any multiplier

$$M = \big(1+\sqrt{\frac{\Lambda}{3}}r\big)^{\alpha}\left(\bar{Y}+Y\right) \qquad (\alpha<0)\,; \tag{2.2.62}$$

while for all $\alpha < 0$ we can establish the redshift property, we "only" have the positivity property for $\alpha = 0$. In that sense, Prop. 2.7 is sharp.

Inhomogeneous wave equation. Let us now consider the modified energy current

$$J^{M;m_\Lambda} \doteq J^M - \frac{m_\Lambda}{2}\psi^2\, M^\flat \tag{2.2.63}$$

where $M^\flat$ is simply the 1-form corresponding to the vectorfield M, i.e. $M^\flat\cdot X = g(M,X)$, and

$$M = \big(1+\sqrt{\frac{\Lambda}{3}}r\big)^{\alpha}\big(\bar{Y}+Y\big)\,, \qquad (\alpha<0)\,. \tag{2.2.64}$$

For any solution of the *inhomogeneous* wave equation (2.2.52) with $m_\Lambda > 0$, this current then has the "redshift" property, in analogy to (2.2.59),

$$\phi\,\nabla\cdot J^{M;m_\Lambda} \geq |\alpha|\frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r} J^{M;m_\Lambda}\cdot n\,, \tag{2.2.65}$$

which then similarly to the above leads to

$$\big(1+\sqrt{\frac{\Lambda}{3}}r_2\big)^{|\alpha|}\int_{\Sigma_{r_2}} J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}} \leq \big(1+\sqrt{\frac{\Lambda}{3}}r_1\big)^{|\alpha|}\int_{\Sigma_{r_1}} J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}\,. \tag{2.2.66}$$

Proposition 2.8 (Global Energy Boundedness, Inhomogeneous Case)**.** *Let ψ be a solution of the inhomogeneous wave equation* (2.2.52)*, then*

$$\int_{\Sigma_{r_2}}\phi\Big\{\phi^2\big(T\cdot\psi\big)^2+|\nabla\!\!\!/\psi|^2+m_\Lambda\psi^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_2}}\leq$$

$$\leq\Big(1+\sqrt{\frac{\Lambda}{3}}r_1\Big)^{|\alpha|}\int_{\Sigma_{r_1}} J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}} \tag{2.2.67}$$

for any $r_2 > r_1 > \sqrt{\frac{3}{\Lambda}}$.

2.2.2.2 Proof of the energy estimates in the expanding region

In the following I shall use the double null coordinates (2.2.24) to prove the crucial properties of the currents (2.2.63). However, the basic properties of (2.2.63) that are needed for the argument presented above in Section 2.2.2.1, that lead to Prop. 2.7 and 2.8, do not rely on the specific coordinate system used. In fact, in Section 2.3 the multiplier (2.2.58) will find a suitable geometric generalization for the expanding region of the Schwarzschild de Sitter spacetime and the properties of the corresponding current will be demonstrated in an entirely different coordinate system.

Moreover, the following serves as a reference for the argument already presented, and shall not repeat it.

Homogeneous wave equation. Recall the definition of the standard energy current associated to the multiplier M,

$$J^M_\mu[\psi]=T_{\mu\nu}[\psi]M^\nu \tag{2.2.68}$$

where $T_{\mu\nu}$ denotes the standard energy momentum tensor associated to (1.1.1),

$$T_{\mu\nu}[\psi]=\partial_\mu\psi\,\partial_\nu\psi-\frac{1}{2}\,g_{\mu\nu}\,\partial^\alpha\psi\partial_\alpha\psi\,. \tag{2.2.69}$$

For any solution of (2.2.51), (2.2.69) is conserved, and therefore

$$\nabla\cdot J^M\doteq\nabla^\mu J^M_\mu={}^{(M)}\pi^{\mu\nu}T_{\mu\nu}[\psi]\doteq K^M[\psi] \tag{2.2.70}$$

where ${}^{(M)}\pi\doteq\frac{1}{2}\mathcal{L}_M g$ is the deformation tensor. It is the content of the divergence theorem combined with the coarea formula that we then have

$$\int_{\Sigma_{r_2}} J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}}+\int_{r_1}^{r_2}\mathrm{d}r\int_{\Sigma_r}\phi K^M\,\mathrm{d}\mu_{\overline{g}_r}=\int_{\Sigma_{r_1}} J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}\,, \tag{2.2.71}$$

for any vectorfield M, (all $r_2 > r_1 > \sqrt{\frac{3}{\Lambda}}$); we also refer to (2.2.71) as the energy identity for J^M. In order to establish (2.2.56) it is thus enough to show $K^M \geq 0$.

For a general vectorfield

$$M = M^u \frac{\partial}{\partial u} + M^v \frac{\partial}{\partial v} \tag{2.2.72}$$

we calculate

$$\begin{aligned} K^M = &- \frac{\Lambda}{3} \frac{2}{(1+\sqrt{\frac{\Lambda}{3}}r)^2} \frac{\partial M^u}{\partial v} \Big(\frac{\partial \psi}{\partial u}\Big)^2 - \frac{\Lambda}{3} \frac{2}{(1+\sqrt{\frac{\Lambda}{3}}r)^2} \frac{\partial M^v}{\partial u} \Big(\frac{\partial \psi}{\partial v}\Big)^2 \\ &- \frac{1}{2} \partial_\mu M^\mu \, |\nabla\!\!\!/ \psi|^2 - \frac{1}{2} \frac{2}{1+\sqrt{\frac{\Lambda}{3}}r} \sqrt{\frac{\Lambda}{3}} \, (M \cdot r) \, |\nabla\!\!\!/ \psi|^2 \\ &+ \frac{1}{r} \, (M \cdot r) \, \frac{\Lambda}{3} \Big(\frac{2}{1+\sqrt{\frac{\Lambda}{3}}r}\Big)^2 \frac{\partial \psi}{\partial u} \frac{\partial \psi}{\partial v} \,. \end{aligned} \tag{2.2.73}$$

Here we first set

$$M \doteq \bar{Y} + Y = \frac{1}{\frac{\partial r}{\partial u}} \frac{\partial}{\partial u} + \frac{1}{\frac{\partial r}{\partial v}} \frac{\partial}{\partial v} \,. \tag{2.2.74}$$

This choice should be viewed as a global redshift vectorfield. Indeed, $\bar{Y}$ and Y separately exploit the redshift effect locally at the cosmological horizons; ($\bar{Y}$ at $u = 0$ and Y at $v = 0$). While it is clear that $\bar{Y}$ and Y (without a suitable extension) in and by themselves do not give rise to a positive current, the following result shows that the estimate can be *symmetrized*.

Proposition 2.9 (Positivity Property). *Let M be defined by* (2.2.74), *then*

$$K^M \geq 0 \,. \tag{2.2.75}$$

Proof. Here

$$M^u = \sqrt{\frac{\Lambda}{3}} \frac{(1-uv)^2}{2v} \qquad M^v = \sqrt{\frac{\Lambda}{3}} \frac{(1-uv)^2}{2u} \,. \tag{2.2.76}$$

Thus

$$\begin{aligned} &- \frac{\partial M^u}{\partial u} - \frac{\partial M^v}{\partial v} - \frac{2}{1+\sqrt{\frac{\Lambda}{3}}r} \sqrt{\frac{\Lambda}{3}} \, M \cdot r = \\ &\qquad = \sqrt{\frac{\Lambda}{3}}(1-uv) + \sqrt{\frac{\Lambda}{3}}(1-uv) - 2\sqrt{\frac{\Lambda}{3}}(1-uv) = 0 \,, \end{aligned} \tag{2.2.77}$$

and

$$\frac{\partial M^u}{\partial v} = -\sqrt{\frac{\Lambda}{3}} \frac{(1+uv)(1-uv)}{2v^2} \,, \tag{2.2.78}$$

$$\frac{\partial M^v}{\partial u} = -\sqrt{\frac{\Lambda}{3}} \frac{(1+uv)(1-uv)}{2u^2} \,. \tag{2.2.79}$$

Now,

$$\frac{2}{r}\left(M\cdot r\right)\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v}\geq-\frac{2}{r}\frac{u}{v}\left(\frac{\partial\psi}{\partial u}\right)^2-\frac{2}{r}\frac{v}{u}\left(\frac{\partial\psi}{\partial v}\right)^2, \tag{2.2.80}$$

and therefore

$$\begin{aligned}K^M\geq\frac{\Lambda}{3}\frac{1}{\left(1+\sqrt{\frac{\Lambda}{3}}r\right)^2}&\Bigg\{\sqrt{\frac{\Lambda}{3}}\left[\frac{(1+uv)(1-uv)}{v^2}-4\frac{u}{v}\frac{1-uv}{1+uv}\right]\left(\frac{\partial\psi}{\partial u}\right)^2\\&+\sqrt{\frac{\Lambda}{3}}\left[\frac{(1+uv)(1-uv)}{u^2}-4\frac{v}{u}\frac{1-uv}{1+uv}\right]\left(\frac{\partial\psi}{\partial u}\right)^2\Bigg\}\\&=\frac{\Lambda}{3}\frac{1}{\left(1+\sqrt{\frac{\Lambda}{3}}r\right)^2}\sqrt{\frac{\Lambda}{3}}\frac{(1-uv)^3}{(1+uv)}\left[\frac{1}{v^2}\left(\frac{\partial\psi}{\partial u}\right)^2+\frac{1}{u^2}\left(\frac{\partial\psi}{\partial v}\right)^2\right]\geq 0\,. \end{aligned}\tag{2.2.81}$$

□

The energy identity (2.2.71) immediately implies:

Corollary 2.10 (Boundary Terms)**.** *For any* $r_2>r_1>\sqrt{\frac{3}{\Lambda}}$ *and all solutions* ψ *of* (1.1.1) *we have*

$$\begin{aligned}\int_{\Sigma_{r_2}}\phi\left\{\phi^2\left(T\cdot\psi\right)^2+|\nabla\!\!\!/\psi|^2\right\}\mathrm{d}\mu_{\overline{g}_{r_2}}\leq\\ \leq\int_{\Sigma_{r_1}}\phi\left\{\phi^2\left(T\cdot\psi\right)^2+\frac{1}{\phi^2}\left(\frac{\partial\psi}{\partial r}\right)^2+|\nabla\!\!\!/\psi|^2\right\}\mathrm{d}\mu_{\overline{g}_{r_1}}\,.\end{aligned}\tag{2.2.82}$$

It is useful to note that the lapse function may be rewritten as

$$\phi=\frac{1}{(uv)^{\frac{1}{2}}}\frac{1}{1+\sqrt{\frac{\Lambda}{3}}r}\,. \tag{2.2.83}$$

Proof. To verify (2.2.82) note that $J^M\cdot n=T(M,\phi V)$, and

$$T(M,V)=\frac{1}{2}\frac{\Lambda}{3}\left(1-uv\right)^2\left[\frac{u}{v}\left(\frac{\partial\psi}{\partial u}\right)^2+\frac{v}{u}\left(\frac{\partial\psi}{\partial v}\right)^2\right]+\left|\nabla\!\!\!/\psi\right|^2. \tag{2.2.84}$$

Moreover,

$$\left(T\cdot\psi\right)^2+\left(V\cdot\psi\right)^2=2\frac{\Lambda}{3}u^2\left(\frac{\partial\psi}{\partial u}\right)^2+2\frac{\Lambda}{3}v^2\left(\frac{\partial\psi}{\partial v}\right)^2, \tag{2.2.85}$$

and $V=\phi^{-2}\partial_r$. □

We have remarked that it is equivalent to consider the vectorfield

$$M=\frac{1}{1+\sqrt{\frac{\Lambda}{3}}r}\left(\bar{Y}+Y\right). \tag{2.2.86}$$

The crucial property (2.2.59) should here be seen as obtained by prescribing a derivative near the cosmological horizons. Indeed we can think of (2.2.86) as obtained by multiplying $\bar{Y}+Y$ by $(-uv)$ and then adding $\bar{Y}+Y$ to ensure that the vectorfield remains timelike.

We shall give a proof of the "redshift" property (2.2.59) of the multiplier (2.2.86) and its consequences directly in the more general inhomogeneous case.

Inhomogeneous wave equation. Let

$$J^{M;m_\Lambda}_\mu[\psi] \doteq J^M_\mu[\psi] - \frac{m_\Lambda}{2} M_\mu \, \psi^2 \,, \tag{2.2.87}$$

where

$$M \doteq \bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^\alpha \bigl(\bar{Y}+Y\bigr) \qquad (\alpha<0)\,. \tag{2.2.88}$$

Then we have for any solution ψ of (2.2.52),

$$\begin{aligned}\nabla\cdot J^{M;m_\Lambda}[\psi] &= \bigl(\Box\psi\bigr)\, M\cdot\psi + T_{\mu\nu}[\psi]^{(M)}\pi^{\mu\nu} - \frac{m_\Lambda}{2}\,\nabla_\mu M^\mu\,\psi^2 - m_\Lambda\psi\, M\cdot\psi \\ &= K^M[\psi] - \frac{m_\Lambda}{2}\,\nabla\cdot M\,\psi^2\,.\end{aligned} \tag{2.2.89}$$

Proposition 2.11 (Global Redshift Property)**.** *Let ψ be a solution of* (2.2.52)*, and let M be defined by* (2.2.88)*, then*

$$\phi\,\nabla\cdot J^{M;m_\Lambda} \geq |\alpha|\frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r}\, J^{M;m_\Lambda}\cdot n \qquad \bigl(r>\sqrt{\frac{3}{\Lambda}}\bigr)\,, \tag{2.2.90}$$

where n denotes the normal to Σ_r.

Note here,

$$J^{M;m_\Lambda}\cdot n = J^M\cdot n + \frac{m_\Lambda}{2}\phi[-g(M,V)]\psi^2\,. \tag{2.2.91}$$

Since with (2.2.88)

$$-g(M,V) = 2\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^\alpha\,, \tag{2.2.92}$$

we obtain control on the energy density

$$\begin{aligned}J^{M;m_\Lambda}\cdot n = \phi\Bigl\{ & 2\frac{\Lambda}{3}\frac{1}{\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^{2-\alpha}}\Bigl[\frac{u}{v}\bigl(\frac{\partial\psi}{\partial u}\bigr)^2 + \frac{v}{u}\bigl(\frac{\partial\psi}{\partial v}\bigr)^2\Bigr] \\ & + \bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^\alpha|\nabla\!\!\!/\psi|^2 + m_\Lambda\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^\alpha\psi^2\Bigr\}\,.\end{aligned} \tag{2.2.93}$$

Proof. Here

$$M^u = \sqrt{\frac{\Lambda}{3}}\frac{2}{v}\Bigl(\frac{1-uv}{2}\Bigr)^{2-\alpha} \tag{2.2.94}$$

$$M^v = \sqrt{\frac{\Lambda}{3}}\frac{2}{u}\Bigl(\frac{1-uv}{2}\Bigr)^{2-\alpha}\,. \tag{2.2.95}$$

Firstly,

$$-\frac{\partial M^u}{\partial u} - \frac{\partial M^v}{\partial v} - \frac{2}{1+\sqrt{\frac{\Lambda}{3}}r}\sqrt{\frac{\Lambda}{3}}M\cdot r = (-2\alpha)\sqrt{\frac{\Lambda}{3}}\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^{\alpha-1}\,, \tag{2.2.96}$$

and secondly, by symmetrizing as in (2.2.80),

$$-\frac{2}{\Omega^2}\frac{\partial M^u}{\partial v}\Big(\frac{\partial\psi}{\partial u}\Big)^2-\frac{2}{\Omega^2}\frac{\partial M^v}{\partial u}\Big(\frac{\partial\psi}{\partial v}\Big)^2+\frac{1}{r}\frac{4}{\Omega^2}\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v}\geq$$
$$\geq 2\sqrt{\frac{\Lambda}{3}}^3\Big(\frac{1-uv}{2}\Big)^{3-\alpha}\Big[\frac{(1-uv)^2-\alpha(1+uv)uv}{(1+uv)}\Big]\Big\{\frac{1}{v^2}\Big(\frac{\partial\psi}{\partial u}\Big)^2+\frac{1}{u^2}\Big(\frac{\partial\psi}{\partial v}\Big)^2\Big\}$$
$$\geq(-2\alpha)\sqrt{\frac{\Lambda}{3}}^3\frac{1}{\big(1+\sqrt{\frac{\Lambda}{3}}r\big)^{3-\alpha}}\Big[\frac{u}{v}\Big(\frac{\partial\psi}{\partial u}\Big)^2+\frac{v}{u}\Big(\frac{\partial\psi}{\partial v}\Big)^2\Big]. \quad (2.2.97)$$

In accordance with (2.2.96) we also find

$$-\nabla\cdot M=\sqrt{\frac{\Lambda}{3}}\frac{(-2\alpha)}{\big(1+\sqrt{\frac{\Lambda}{3}}r\big)^{1-\alpha}}\,, \quad (2.2.98)$$

and thus

$$\phi\nabla^\mu J^{M;m_\Lambda}_\mu=\phi\,T_{\mu\nu}[\psi]^{(M)}\pi^{\mu\nu}-\frac{m_\Lambda}{2}\,\phi\,\nabla\cdot M\,\psi^2$$
$$=-\alpha\phi\frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r}2\frac{\Lambda}{3}\frac{1}{\big(1+\sqrt{\frac{\Lambda}{3}}r\big)^{2-\alpha}}\Big[\frac{u}{v}\Big(\frac{\partial\psi}{\partial u}\Big)^2+\frac{v}{u}\Big(\frac{\partial\psi}{\partial v}\Big)^2\Big]$$
$$-\alpha\phi\frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r}\Big(1+\sqrt{\frac{\Lambda}{3}}r\Big)^\alpha\big|\nabla\!\!\!/\psi\big|^2-\alpha\phi\,m_\Lambda\frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r}\Big(1+\sqrt{\frac{\Lambda}{3}}r\Big)^\alpha\psi^2\,, \quad (2.2.99)$$

which yields (2.2.90) by comparison with (2.2.93). □

By virtue of the energy identity

$$\int_{\Sigma_{r_2}}J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}}+|\alpha|\int_{r_1}^{r_2}\mathrm{d}r\frac{\sqrt{\frac{\Lambda}{3}}}{1+\sqrt{\frac{\Lambda}{3}}r}\int_{\Sigma_r}J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_r}\leq$$
$$\leq\int_{\Sigma_{r_1}}J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}\,, \quad (2.2.100)$$

and the following Gronwall inequality, we can conclude that

$$\Big(1+\sqrt{\frac{\Lambda}{3}}r_2\Big)^{|\alpha|}\int_{\Sigma_{r_2}}J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}}\leq\Big(1+\sqrt{\frac{\Lambda}{3}}r_1\Big)^{|\alpha|}\int_{\Sigma_{r_1}}J^{M;m_\Lambda}\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}\,. \quad (2.2.101)$$

Lemma 2.12 (Gronwall Inequality for decreasing functions)**.** *Let $\alpha<0$, and $f,g\in \mathrm{C}^1([r_1,r_2])$ with $g\neq 0$, satisfying the inequality*

$$f'\leq\alpha\frac{g'}{g}f \quad (2.2.102)$$

on the interval $[r_1,r_2]$. Then

$$f(r_2)\leq\frac{|g(r_1)|^{|\alpha|}}{|g(r_2)|^{|\alpha|}}f(r_1)\,. \quad (2.2.103)$$

Proof. By (2.2.102),

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big[f(r)\exp\Big[-\alpha\int_{r_1}^{r}\frac{g'(r')}{g(r')}\,\mathrm{d}r'\Big]\Big]\leq 0\,,\tag{2.2.104}$$

which yields (2.2.103) upon integration on the interval $[r_1, r_2]$. □

As an immediate consequence of (2.2.101) we obtain – in view of (2.2.93) – the following estimate, which is precisely the content of Prop. 2.8.

Proposition 2.13. *Let* ψ *be a solution of* (2.2.52)*, then*

$$\int_{\Sigma_{r_2}}\phi\Big\{\phi^2\big(T\cdot\psi\big)^2+\frac{1}{\phi^2}\big(\frac{\partial\psi}{\partial r}\big)^2+|\nabla\!\!\!/\psi|^2+m_\Lambda\psi^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_2}}\leq$$
$$\leq\int_{\Sigma_{r_1}}\phi\Big\{\phi^2\big(T\cdot\psi\big)^2+\frac{1}{\phi^2}\big(\frac{\partial\psi}{\partial r}\big)^2+|\nabla\!\!\!/\psi|^2+m_\Lambda\psi^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_1}}\tag{2.2.105}$$

for all $r_2 > r_1 > \sqrt{\frac{3}{\Lambda}}$.

2.2.2.3 Integrated local energy decay and local redshift effect in the static region

It is the principal purpose of this Section to establish *integrated* local energy estimates in the static region. They are required for our proof because the energy flux in the vicinity of the cosmological horizon that we are presented with in Section 2.2.2.1 can only be controlled with a redshift vectorfield that generates error terms away from the horizon. In the following we construct currents that control these error terms using "Morawetz"-type vectorfields.

Homogeneous wave equation. It useful to approach the problem first with the aim of contructing an integrated energy estimate in the static region without a redshift component. We present a slightly simplified argument of [16] for the corresponding region in Schwarzschild de Sitter. Recall that in this region we can use Eddington-Finkelstein coordinates (2.2.30); let us denote by

$$\mu\doteq\frac{\Lambda r^2}{3}\,.\tag{2.2.106}$$

We shall construct a current based on a multiplier of the form

$$X=f(r)\frac{\partial}{\partial r^*}\,.\tag{2.2.107}$$

Let us illustrate the strategy (which may be referred to as the "Morawetz vectorfield method") by first choosing

$$f(r)=r^*(r)\,.\tag{2.2.108}$$

The current we then wish to consider is the following modification of the standard energy current associated to X:

$$J_\mu^{X,1} = J_\mu^X + \frac{1}{4}\Big[1 + \frac{2}{r} r^*(1-\mu)\Big]\partial_\mu\big[\psi^2\big] - \frac{1}{4}\partial_\mu\Big[1 + \frac{2}{r}r^*(1-\mu)\Big]\psi^2 \tag{2.2.109}$$

Note that

$$\lim_{r\to\sqrt{\frac{3}{\Lambda}}} r^*(r)\Big(1-\mu(r)\Big) = 0\,, \tag{2.2.110}$$

because

$$r^*(r) = \int_0^r \frac{1}{1-\mu(r)}\,\mathrm{d}r = \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\log\Big|\frac{1+\sqrt{\frac{\Lambda}{3}}r}{1-\sqrt{\frac{\Lambda}{3}}r}\Big|\,. \tag{2.2.111}$$

The divergence of the current (2.2.109) satisfies for any solution ψ of the wave equation (2.2.51) of vanishing spherical mean

$$\overline{\psi} = \frac{1}{4\pi r^2}\int_{\mathrm{S}} \psi\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} = 0 \tag{2.2.112}$$

the integral inequality

$$\begin{aligned}\int_{\mathrm{S}} \nabla^\mu J_\mu^{X,1}[\psi]\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \geq \\ \geq \int_{\mathrm{S}}\Big\{(1-\mu)\big(\frac{\partial\psi}{\partial r}\big)^2 + \frac{2}{r^2}\frac{r^*(1-\mu)}{r}(1+\mu)\psi^2 + \frac{2\Lambda}{3}\psi^2\Big\}\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}}\,.\end{aligned} \tag{2.2.113}$$

By Stokes theorem the energy density on the right-hand side is thus controlled in the static region by the corresponding boundary integrals of (2.2.109) which on the cosmological horizon $\mathcal{C}$ take the form

$$\int \mathrm{d}u^* \int_{\mathrm{S}}\Big\{\Big(J^X, \frac{\partial}{\partial u^*}\Big) + \frac{1}{4}\frac{\partial\psi^2}{\partial u^*}\Big\}\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}}\,. \tag{2.2.114}$$

While the terms arising from the modification are finite on the boundary, the main contribution to the energy density from $T(X, \partial_{u^*})$ is of course not finite, because

$$f = r^* \to \infty \qquad \text{as } r \to \sqrt{\frac{3}{\Lambda}}\,. \tag{2.2.115}$$

It is for this reason that we have to introduce a cut-off in (2.2.108), which in turn necessitates the construction of a red-shift vectorfield.

For the multiplier (2.2.107) we will construct more generally in Section 2.2.2.4 a current for which the divergence reads

$$\begin{aligned}\nabla^\mu J_\mu^{X,1}[\psi] = \frac{f^{(1)}}{1-\mu}\big(\frac{\partial\psi}{\partial r^*}\big)^2 + \frac{f}{r}\big|\nabla\!\!\!/\psi\big|^2 \\ + \Big\{-\frac{1}{4}\frac{1}{1-\mu}f^{(3)} - \frac{1}{r}f^{(2)} + \frac{2\mu}{r^2}f^{(1)} - 2\frac{\mu^2}{r^3}f\Big\}\psi^2\,,\end{aligned} \tag{2.2.116}$$

where $f^{(i)} = \frac{\mathrm{d}^i f}{\mathrm{d}r^{*i}}$; note that for $f \geq 0$, $f^{(1)} \geq 0$, and $f^{(2)} \leq 0$ (2.2.113) is replaced by

$$\int_{\mathrm{S}} \nabla^\mu J^{X,1}_\mu[\psi]\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \geq \\ \geq \int_{\mathrm{S}} \Big\{ \frac{f^{(1)}}{1-\mu}\Big(\frac{\partial\psi}{\partial r^*}\Big)^2 + \frac{2f}{r^3}(1-\mu^2)\psi^2 - \frac{1}{4}\frac{1}{1-\mu}\frac{\mathrm{d}^3 f}{\mathrm{d}r^{*3}}\psi^2 \Big\}\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}}\,, \quad (2.2.117)$$

which holds as a consequence of Poincaré's inequality if ψ is assumed to verify (2.2.112). As a result the error that is introduced by the cut-off is precisely of the form of the last term in (2.2.117).

Now, on very general grounds we know that in view of $\sqrt{\frac{\Lambda}{3}} > 0$ (2.2.31) the cosmological horizon exhibits a redshift effect and allows for the construction of a multiplier that gives rise to a positive current on the horizon [17]. However, while in the case of black hole horizons it suffices to provide a construction on the event horizon and then to extend the multiplier suitably into the neighborhood by a continuity argument, here an explicit redshift vectorfield is needed which extends beyond the immediate vicinity of the cosmological horizon, namely into the region where the cut-off is introduced (and the last term in the above inequality becomes indefinite). In analogy to [20] we choose as a starting point for the construction of a vectorfield that captures the red-shift effect

$$Y = \big(1 + \sigma(1-\mu)\big)\hat{Y} + \sigma(1-\mu)T \quad (2.2.118)$$

where

$$\hat{Y} = \frac{2}{1-\mu}\frac{\partial}{\partial v^*}\,, \quad (2.2.119)$$

note that $\sigma > 0$ is precisely the parameter that appears in the general construction [17]. (There we construct Y transversal to the horizon and require $\nabla_Y Y = -\sigma(Y+T)$ for the extension to the vicinity of the null hypersurface). This multiplier gives rise to a positive current near the cosmological horizon, in the sense that

$$K^Y \geq 0 \qquad (1-\mu \ll 1) \quad (2.2.120)$$

and controls all derivatives. However, it does not yield control on the error term unless σ is chosen very large; (note that (2.2.120) only holds for $1-\mu \ll 1$ while the error term may be of order $(1-\mu)^{-1}$). But we cannot add K^Y to $K^{X,1}$ if the magnitude of Y is large, for the latter is required to control the former in the region where the positivity of (2.2.120) fails. We obtain a vectorfield with the desired properties by adding yet another term

$$\overline{Y} = Y + \kappa\frac{\eta}{r^{*\delta}}\big(\hat{Y} + T\big)\,, \quad (2.2.121)$$

where $\kappa > 0$, $\delta > 0$ and η is a suitable cut-off function supported away from the horizon. Let us also denote by $\overline{Y}_c = \chi\overline{Y}$ where χ is a cut-off function such that $\chi = 0$ for $\mu \ll 1$ and $\chi = 1$ for $1-\mu \ll 1$. We are then able to show that only a "small" contribution of the redshift effect is required in this construction.

A precise version of this statement is given in Proposition 2.20 on page 148.

We have seen that the proof of an integrated energy estimate compels the construction of a redshift vectorfield; alternatively we may say that the boundary terms on the cosmological horizon arising from a redshift vectorfield can only be controlled in conjuction with a positive current arising from a Morawetz vectorfield.

The construction which is described above and carried out in more detail in the following Section 2.2.2.4 leads us to the result:

Proposition 2.14. *Let Σ_0 be a spacelike hypersurface with normal n in the static region of de Sitter crossing the cosmological horizon $\mathcal{C}^+$ to the future of the sphere $\mathcal{C}^+\cap\mathcal{C}^-$ (in (u,v) coordinates this is the sphere $(0,0)$), and denote by $\mathcal{C}_0^+ = J^+(\Sigma_0)\cap\mathcal{C}^+$ the segment of the cosmological horizon to the future of Σ_0. Then there exist a strictly timelike vectorfield N (normalized in the sense that $g(T,N)$ is constant on $\mathcal{C}_0^+$) and a constant $C(\Lambda,\Sigma_0)$ such that for all solutions of the homogeneous wave equation* (2.2.51) *we have*

$$\int_{\mathcal{C}_0^+} {}^*J^N \leq C(\Lambda,\Sigma_0)\int_{\Sigma_0}\left(J^n,n\right). \tag{2.2.122}$$

Inhomogeneous wave equation. In the inhomogeneous case we can rely on the same vectorfields but our identitites are based on modified currents. Namely, the current $J^{X,1}$ is replaced by

$$J^{X,1;m_\Lambda} = J^{X,1} - \frac{m_\Lambda}{2}X^\flat\psi^2\,, \tag{2.2.123}$$

where $X^\flat\cdot Y = g(X,Y)$, and instead of (2.2.116) we have

$$\begin{aligned}\nabla^\mu J^{X,1;m_\Lambda}_\mu[\psi] = \frac{f^{(1)}}{1-\mu}\Big(\frac{\partial\psi}{\partial r^*}\Big)^2 + \frac{f}{r}\big|\nabla\!\!\!/\psi\big|^2 + m_\Lambda\frac{f}{r}\psi^2 \\ + \left\{-\frac{1}{4}\frac{1}{1-\mu}f^{(3)} - \frac{1}{r}f^{(2)} + \frac{2\mu}{r^2}f^{(1)} - 2\frac{\mu^2}{r^3}f\right\}\psi^2\,.\end{aligned} \tag{2.2.124}$$

In our analysis of the inhomogeneous case the zeroth order term steps into the role of the angular derivatives term of the homogeneous case. Indeed, while in (2.2.116) we have conluded by Poincaré's inequality that the last terms of each line taken together are always nonnegative after integration over the spheres, c.f. (2.2.117), we argue here that

$$m_\Lambda\frac{f}{r}\psi^2 - 2\frac{\mu^2}{r^3}f\psi^2 = \Big[m_\Lambda - 2\mu\frac{\Lambda}{3}\Big]\frac{f}{r}\psi^2 \geq 0 \quad \text{if } m_\Lambda \geq 2\frac{\Lambda}{3}\,. \tag{2.2.125}$$

This leads us to the following result.

Proposition 2.15. *Let Σ_0 and $\mathcal{C}^+$ be as in Prop. 2.14. Consider solutions ψ to the inhomogeneous wave equation* (2.2.52) *with $m_\Lambda \geq 2\frac{\Lambda}{3}$. There exists a strictly timelike vectorfield N (as in Prop. 2.14) and a constant $C(\Lambda,\Sigma_0)$ such that for all solutions ψ we have*

$$\int_{\mathcal{C}_0^+}\left\{{}^*J^N[\psi] + \psi^2\right\} \leq C(\Lambda,\Sigma_0)\int_{\Sigma_0}\left\{\left(J^n[\psi],n\right) + \psi^2\right\}. \tag{2.2.126}$$

Remark 2.16 (Conformal Invariance of the wave equation). We expect that the lower bound on the mass m_Λ is a shortcoming of the proof of Prop. 2.15, in particular we expect that (2.2.126) holds for all $m_\Lambda > 0$. However, the value $m_\Lambda = 2\frac{\Lambda}{3}$ does have mathematical meaning, for in this case (2.2.52) is conformal to the wave equation on Minkowski space. More precisely, recall that we can cast the de Sitter spacetime as the subset

$$\mathcal{M} = \left\{ x \in \mathbb{R}^{3+1} \,:\, \langle x, x \rangle = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 > -\frac{4}{\kappa} \right\}, \tag{2.2.127}$$

endowed with the metric

$$g = \Omega^2 \, \langle \, \mathrm{d}x, \, \mathrm{d}x \rangle = \Omega^2 \, \eta \, , \tag{2.2.128}$$

where

$$\Omega = \frac{1}{1 + \frac{\kappa}{4} \langle x, x \rangle} \, , \quad \text{and } \kappa = \frac{\Lambda}{3} \, . \tag{2.2.129}$$

Then $(\mathcal{M}, g)$ is a solution to (2.2.14), namely

$$R_{\mu\nu} = \Lambda \, g_{\mu\nu} \, . \tag{2.2.130}$$

Lemma 2.17. *Let ψ be a solution to $\Box_g \psi = 2\kappa \, \psi = 2\frac{\Lambda}{3} \, \psi$, then $\tilde{\psi} = \Omega \, \psi$ is a solution to $\Box_{\tilde{g}} \tilde{\psi} = 0$ where $\tilde{g} = \Omega^{-2} g = \eta$ is the Minkowski metric.*

This well known result demonstrates the significance of the specific value $m_\Lambda = 2\frac{\Lambda}{3}$, and allows us to read off qualitatively our main result in this special case. For a solution of the classical wave equation $\Box_\eta \tilde{\psi} = 0$ will assume nonvanishing values on the boundary $\langle x, x \rangle = -4/\kappa$. The decay of ψ is then entirely due to the conformal factor Ω, i.e. the "expansion" of the spacetime.

2.2.2.4 Proof of integrated local energy decay in the static region

In this section the argument that was motivated in Section 2.2.2.3 will be carried out in more detail.

Homogeneous wave equation. We start with the construction of a suitable current for the multiplier (2.2.107).

Morawetz current. Consider the vectorfield

$$X = f(r) \frac{\partial}{\partial r^*} \, . \tag{2.2.131}$$

Note that

$$\frac{\partial}{\partial r^*} = \frac{1}{2} \frac{\partial}{\partial v^*} - \frac{1}{2} \frac{\partial}{\partial u^*} \tag{2.2.132}$$

and

$$\frac{\partial f}{\partial u^*} = -(1-\mu)f' \qquad \frac{\partial f}{\partial v^*} = (1-\mu)f' \,. \tag{2.2.133}$$

For the (non-vanishing) connection coefficients of (2.2.30) we find

$$\Gamma^{u^*}_{u^*u^*} = \frac{2\Lambda\, r}{3} \qquad \Gamma^{v^*}_{v^*v^*} = -\frac{2\Lambda\, r}{3} \tag{2.2.134a}$$

$$\Gamma^{u^*}_{AB} = \frac{r}{2}\,\overset{\circ}{\gamma}_{AB} \qquad \Gamma^{v^*}_{AB} = -\frac{r}{2}\,\overset{\circ}{\gamma}_{AB} \tag{2.2.134b}$$

$$\Gamma^{B}_{u^*C} = -\frac{1}{r}\Big(1-\frac{\Lambda r^2}{3}\Big)\delta^B_C \qquad \Gamma^{B}_{u^*C} = \frac{1}{r}\Big(1-\frac{\Lambda r^2}{3}\Big)\delta^B_C \,. \tag{2.2.134c}$$

And thus we readily calculate the (non-vanishing) components of the deformation tensor

$$^{(X)}\pi = \mathcal{L}_X g \tag{2.2.135}$$

to be

$$^{(X)}\pi_{u^*u^*} = (1-\mu)^2 f' \qquad ^{(X)}\pi_{v^*v^*} = (1-\mu)^2 f' \tag{2.2.136a}$$

$$^{(X)}\pi_{u^*v^*} = -(1-\mu)^2 f' + (1-\mu)\frac{2\mu}{r} f \tag{2.2.136b}$$

$$^{(X)}\pi_{AB} = \frac{f}{r}(1-\mu) g_{AB} \,. \tag{2.2.136c}$$

Therefore

$$\begin{aligned} K^X[\psi] \doteq {}^{(X)}\pi^{\alpha\beta} T_{\alpha\beta}[\psi] = \\ = \frac{1}{4} f' \Big[\big(\frac{\partial\psi}{\partial u^*}\big)^2 + \big(\frac{\partial\psi}{\partial v^*}\big)^2 \Big]^2 - \frac{1}{2} f'(1-\mu)\big|\nabla\!\!\!/\,\psi\big|^2 \\ + \frac{1}{2}\frac{2\mu}{r} f \big|\nabla\!\!\!/\,\psi\big|^2 + \frac{f}{r}\frac{\partial\psi}{\partial u^*}\frac{\partial\psi}{\partial v^*} \,, \end{aligned} \tag{2.2.137}$$

because

$$T_{u^*u^*}[\psi] = \big(\frac{\partial\psi}{\partial u^*}\big)^2 \qquad T_{v^*v^*}[\psi] = \big(\frac{\partial\psi}{\partial v^*}\big)^2 \tag{2.2.138a}$$

$$T_{u^*v^*}[\psi] = (1-\mu)\big|\nabla\!\!\!/\,\psi\big|^2 \tag{2.2.138b}$$

$$(g^{-1})^{AB} T_{AB} = \frac{1}{1-\mu}\frac{\partial\psi}{\partial u^*}\frac{\partial\psi}{\partial v^*} \,. \tag{2.2.138c}$$

Next, consider the modified current

$$\begin{aligned} J^{X,1}_\mu[\psi] = J^X_\mu + \frac{1}{4}\Big(f' + \frac{2}{r} f\Big)\big(1-\mu\big)\partial_\mu\big[\psi^2\big] \\ - \frac{1}{4}\partial_\mu\Big[\big(f' + \frac{2}{r} f\big)\big(1-\mu\big)\Big]\psi^2 \end{aligned} \tag{2.2.139}$$

which clearly satisfies

$$\begin{aligned} K^{X,1} \doteq \nabla^\mu J^{X,1}_\mu = K^X + \frac{1}{2}\big(f' + \frac{2}{r} f\big)\big(1-\mu\big)\partial^\mu\psi\partial_\mu\psi \\ - \frac{1}{4}\Box_g\Big[\big(f' + \frac{2}{r} f\big)\big(1-\mu\big)\Big]\psi^2 \end{aligned} \tag{2.2.140}$$

for any solution of the homogeneous wave equation (2.2.51). Thus

$$K^{X,1}[\psi] = f'\bigl(\frac{\partial\psi}{\partial r^*}\bigr)^2 + \frac{f}{r}\bigl|\nabla\!\!\!/\psi\bigr|^2 - \frac{1}{4}\Box_g\Bigl[\bigl(f' + \frac{2}{r}f\bigr)\bigl(1-\mu\bigr)\Bigr]\psi^2\,. \tag{2.2.141}$$

While the modification (2.2.139) is chosen such that the indefinite term in (2.2.137) is cancelled, its usefulness only reveals itself in the precise form that the last term in (2.2.141) takes; here find[1]

$$\begin{aligned} K^{X,1}[\psi] = f'\bigl(\frac{\partial\psi}{\partial r^*}\bigr)^2 + \frac{f}{r}\bigl|\nabla\!\!\!/\psi\bigr|^2 - \frac{1}{4}\bigl(1-\mu\bigr)^2 f'''\psi^2 \\ - \frac{1}{2}\bigl(2-5\mu\bigr)\bigl(1-\mu\bigr)\frac{f''}{r}\psi^2 + \bigl(9-11\mu\bigr)\frac{\mu}{2}\frac{f'}{r^2}\psi^2 - 2\mu^2\frac{f}{r^3}\psi^2\,. \end{aligned} \tag{2.2.142}$$

This follows from an elementary albeit lengthy calculation using

$$\Box_g\psi = -\frac{1}{1-\mu}\frac{\partial^2\psi}{\partial t^2} + \frac{\partial}{\partial r}\Bigl[\bigl(1-\mu\bigr)\frac{\partial\psi}{\partial r}\Bigr] + \frac{2}{r}\bigl(1-\mu\bigr)\frac{\partial\psi}{\partial r} + \Delta\!\!\!\!/\,\psi\,. \tag{2.2.143}$$

Note that

$$K^{X,1} = \frac{2}{r^3}\bigl(\frac{\partial\psi}{\partial r^*}\bigr)^2 \geq 0 \qquad \text{for } f = -\frac{1}{r^2} \text{ if } \bigl|\nabla\!\!\!/\psi\bigr|^2 = 0\,. \tag{2.2.144}$$

The above formula (2.2.142) becomes more transparent if we instead express it in terms of derivatives

$$f^{(i)} \doteq \frac{\mathrm{d}^i f}{\mathrm{d}r^{*i}}: \qquad i = 1,\ldots,3 \tag{2.2.145}$$

Indeed, we then have

$$\begin{aligned} K^{X,1}[\psi] = \frac{f^{(1)}}{1-\mu}\bigl(\frac{\partial\psi}{\partial r^*}\bigr)^2 + \frac{f}{r}\bigl|\nabla\!\!\!/\psi\bigr|^2 \\ + \Bigl\{-\frac{1}{4}\frac{1}{1-\mu}f^{(3)} - \frac{1}{r}f^{(2)} + \frac{2\mu}{r^2}f^{(1)} - 2\frac{\mu^2}{r^3}f\Bigr\}\psi^2\,. \end{aligned} \tag{2.2.146}$$

As explained in Section 2.2.2.3 we have to choose the function f such that it remains bounded as $\mu \to 1$, as a consequence of which its derivatives cannot vanish identically. For this reason we discuss the following current which will allow us to control the error terms.

Redshift current. The redshift vectorfield is based on

$$\hat{Y} = \frac{2}{1-\mu}\frac{\partial}{\partial v^*}\,. \tag{2.2.147}$$

Since

$${}^{(\hat{Y})}\pi_{u^*u^*} = 4\frac{2\mu}{r} \qquad {}^{(\hat{Y})}\pi_{v^*v^*} = 0 \qquad {}^{(\hat{Y})}\pi_{u^*v^*} = 0 \tag{2.2.148a}$$

$${}^{(\hat{Y})}\pi_{AB} = \frac{2}{r}g_{AB} \tag{2.2.148b}$$

[1]Note that no boundary terms at the origin $r = 0$ are generated if $\lim_{r\to 0}\frac{f}{r} = 1$.

we immediately obtain

$$K^{\hat{Y}}[\psi] \doteq {}^{(\hat{Y})}\pi^{\mu\nu}T_{\mu\nu}[\psi] = \frac{1}{(1-\mu)^2}\frac{2\mu}{r}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 + \frac{2}{r}\frac{1}{1-\mu}\frac{\partial\psi}{\partial u^*}\frac{\partial\psi}{\partial v^*}\,. \tag{2.2.149}$$

Next let us introduce

$$Y = \big(1+\sigma(1-\mu)\big)\hat{Y} + \sigma(1-\mu)T \tag{2.2.150}$$

with a parameter $\sigma > 0$ to be chosen below. This has the effect that in contrast to (2.2.148) we have

$$ {}^{(Y)}\pi_{u^*u^*} = 4\frac{2\mu}{r} - \sigma\frac{2\mu}{r}(1-\mu)^2 \qquad {}^{(Y)}\pi_{v^*v^*} = \sigma\frac{2\mu}{r}(1-\mu)^2 \tag{2.2.151a}$$

$$ {}^{(Y)}\pi_{u^*v^*} = 2\sigma\frac{2\mu}{r}(1-\mu) \tag{2.2.151b}$$

$$ {}^{(Y)}\pi_{AB} = \frac{2}{r}\big(1+\sigma(1-\mu)\big)g_{AB} \tag{2.2.151c}$$

which in particular recovers the missing derivatives in (2.2.149):

$$\begin{aligned} K^Y = \frac{1}{4}\sigma\frac{2\mu}{r}\Big(\frac{\partial\psi}{\partial u^*}\Big)^2 + \frac{1}{(1-\mu)^2}\frac{2\mu}{r}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 \\ - \frac{1}{4}\sigma\frac{2\mu}{r}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 + \sigma\frac{2\mu}{r}\big|\nabla\!\!\!/\psi\big|^2 \\ + \frac{2}{r}\big(1+\sigma(1-\mu)\big)\frac{1}{1-\mu}\frac{\partial\psi}{\partial u^*}\frac{\partial\psi}{\partial v^*} \end{aligned} \tag{2.2.152}$$

Finally, let us also define

$$\overline{Y} = \Big(1+\sigma(1-\mu)+\kappa\frac{\eta}{r^{*\delta}}\Big)\hat{Y} + \Big(\sigma(1-\mu)+\kappa\frac{\eta}{r^{*\delta}}\Big)T \tag{2.2.153}$$

and

$$\overline{Y}_c = \chi\overline{Y}\,, \tag{2.2.154}$$

where $\eta, \chi \in \mathrm{C}^\infty(\mathbb{R})$ are cut-off functions which will be specified below, and $\kappa > 0$.

Lemma 2.18. *Let $\sigma \geq 8$, then*

$$K^Y \geq \frac{1}{4}\frac{2}{r}\Big(\frac{\partial\psi}{\partial u^*}\Big)^2 + \frac{2}{r}\frac{1}{4}\frac{3}{8}\frac{1}{(1-\mu)^2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 + \sigma\frac{2\mu}{r}\big|\nabla\!\!\!/\psi\big|^2 \tag{2.2.155}$$

for $\frac{7}{8} \leq \mu \leq 1$.

Proof. By Cauchy's inequality, we have from (2.2.152) for any $\mu_0 > 0$

$$\begin{aligned} K^Y \geq \frac{1}{4}\frac{2}{r}\Big(\sigma(\mu-\mu_0) - \frac{1}{\mu_0}\Big)\Big(\frac{\partial\psi}{\partial u^*}\Big)^2 \\ + \frac{2}{r}\Big(\mu-\mu_0-\sigma\big(\frac{1}{\mu_0}+\frac{\mu}{4}\big)(1-\mu)^2\Big)\frac{1}{(1-\mu)^2}\Big(\frac{\partial\psi}{\partial v^*}\Big)^2 + \sigma\frac{2\mu}{r}\big|\nabla\!\!\!/\psi\big|^2. \end{aligned} \tag{2.2.156}$$

For $1-\mu \leq \frac{1}{8}$, all coefficients are bounded below as given in (2.2.155), if we choose $\mu_0 = \frac{1}{2}$ and $\sigma = 8$. □

Auxiliary current. Choose $f = r^2$ then

$$K^X = 2r\Big(\frac{\partial\psi}{\partial t}\Big)^2 + \big(2\mu - 1\big)r\big|\nabla\!\!\!/\psi\big|^2 \tag{2.2.157}$$

by (2.2.137).

Cut-off parameters. Recall the choice (2.2.108) which corresponds to $f^{(1)} = 1$. Here we will have instead

$$f^{(1)} = \begin{cases} 1, & r \leq R_1 \\ 0, & r \geq R_2 \end{cases} \tag{2.2.158}$$

with suitable values $R_1 < R_2 < \infty$.

Lemma 2.19. *Let $\varepsilon > 0$, and $R_1 < \infty$. Then there exists a finite $R_2 > R_1$ and $g \in \mathrm{C}^\infty([0,\infty))$ such that $g(x) = 1$ for $x \leq R_1$, $g(x) = 0$ for $x \geq R_2$ and*

$$|g'(x)| \leq \frac{\varepsilon}{x}\,, \qquad |g''(x)| \leq \frac{\varepsilon}{x^2}\,, \quad \text{for } x \in [R_1, R_2]\,. \tag{2.2.159}$$

Given $R_1 < R_2$ we choose $R_0 < R_1$, $R_3 > R_2$ and $\eta \in \mathrm{C}^\infty_c\big((R_0, R_3)\big)$ such that $\eta = 1$ on $[R_1, R_2]$; moreover let $\chi \in \mathrm{C}^\infty\big([0,\infty)\big)$ such that $\chi = 0$ for $r \leq R_0$ and $\chi = 1$ for $r \geq R_1$.

Proposition 2.20. *Let J be a current of the form*

$$J = J^{X,1} + J^{Y_c} + J^{X_a} \tag{2.2.160}$$

where $X = f(r)\,\partial_{r^*}$, $X_a = f_a(r)\,\partial_{r^*}$ and $J^{X,1}$ is given by (2.2.139)*, and denote the divergence of J by $K = \nabla^\mu J_\mu$. There exist* bounded *functions*

$$f, f_a : [0, \sqrt{\frac{3}{\Lambda}}] \to \mathbb{R} \tag{2.2.161}$$

of the area radius r and a future-directed causal *vectorfield Y_c defined in the static region (with a* regular *extension to the cosmological horizon) such that for any solution ψ of the homogeneous wave equation* (2.2.51) *with the vanishing mean property* (2.2.112) *we have*

$$\int_{\mathrm{S}_r} K \,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \geq 0 \tag{2.2.162}$$

on all spheres $(\mathrm{S}_r, r^2\,\overset{\circ}{\gamma})$ in the static region with $0 < r < \sqrt{\frac{3}{\Lambda}}$.

Proof. For any given $R_1 > 0$, $\varepsilon > 0$ let $f^{(1)}$ be chosen according to Lemma 2.19 such with $R_2(R_1,\varepsilon) > R_1$

$$f^{(1)} = \begin{cases} 1, & r \leq R_1 \\ 0, & r \geq R_2 \end{cases} \tag{2.2.163}$$

and $f^{(1)} \geq 0$, $f^{(2)} \leq 0$ on $r \in [R_1, R_2]$ with

$$|f^{(2)}| \leq \begin{cases} \frac{\varepsilon}{|r^*|}, & R_1 \leq r \leq R_2 \\ 0, & r \leq R_1 \text{ or } r \geq R_2 \end{cases} \tag{2.2.164}$$

$$|f^{(3)}| \leq \begin{cases} \frac{\varepsilon}{|r^*|^2}, & R_1 \leq r \leq R_2 \\ 0, & r \leq R_1 \text{ or } r \geq R_2 \end{cases} . \tag{2.2.165}$$

We then define

$$f(r) = \int_0^{r^*(r)} f^{(1)} \, \mathrm{d}r^* \, ; \tag{2.2.166}$$

note that $f \leq R_2^* < \infty$. In the following it will be shown that we can choose $R_1 > 0$ large enough and $\varepsilon > 0$ small enough for (2.2.162) to hold with $Y_c = \varepsilon \overline{Y}_c$.

By Poincaré's inequality

$$\int_{\mathrm{S}_r} \frac{1}{r} \left|\nabla\!\!\!/ \psi\right|^2 \, \mathrm{d}\mu_{r^2 \mathring{\gamma}} \geq \int_{\mathrm{S}_r} \frac{2}{r^3} \psi^2 \, \mathrm{d}\mu_{r^2 \mathring{\gamma}} \tag{2.2.167}$$

and thus

$$\int_{\mathrm{S}} K^{X,1} \, \mathrm{d}\mu_{r^2 \mathring{\gamma}} \geq 0 \qquad \Big(r \in (0, R_1] \cup [R_2, \sqrt{\frac{3}{\Lambda}}) \Big) \, . \tag{2.2.168}$$

$\overline{\mathbf{Y}}$. Recall $\eta \in \mathrm{C}_c^\infty\big((R_0, R_3)\big)$ such that $\eta = 1$ on $[R_1, R_2]$. Since

$$^{(\overline{Y})}\pi_{u^* u^*} = {}^{(Y)}\pi_{u^* u^*} + \frac{\kappa}{r^{*\delta}} \big(4 + (1-\mu)\big) \Big(\eta^{(1)} - \delta \frac{\eta}{r^*} \Big) + 4\kappa \frac{\eta}{r^{*\delta}} \frac{2\mu}{r} \tag{2.2.169a}$$

$$^{(\overline{Y})}\pi_{v^* v^*} = {}^{(Y)}\pi_{v^* v^*} - \frac{\kappa}{r^{*\delta}} \Big(\eta^{(1)} - \delta \frac{\eta}{r^*} \Big) (1-\mu) \tag{2.2.169b}$$

$$^{(\overline{Y})}\pi_{u^* v^*} = {}^{(Y)}\pi_{u^* v^*} - 2 \frac{\kappa}{r^{*\delta}} \Big(\eta^{(1)} - \delta \frac{\eta}{r^*} \Big) \tag{2.2.169c}$$

$$^{(\overline{Y})}\pi_{AB} = {}^{(Y)}\pi_{AB} + \frac{2}{r} \kappa \frac{\eta}{r^{*\delta}} g_{AB} \tag{2.2.169d}$$

we find (cf. Lemma 2.18)

$$\begin{aligned} K^{\overline{Y}} = {}^{(\overline{Y})}\pi^{\mu\nu} T_{\mu\nu}[\psi] \geq & \\ \geq \frac{1}{4} \Big[\Big(\sigma(\mu - \mu_0) - \frac{1}{\mu_0} \big(1 + \kappa \frac{\eta}{r^{*\delta}}\big) \Big) \frac{2}{r} + \big(\delta \frac{\eta}{r^*} - \eta^{(1)} \big) \frac{\kappa}{r^{*\delta}} \frac{1}{1-\mu} \Big] \big(\frac{\partial \psi}{\partial u^*} \big)^2 & \\ + \Big\{ \Big[(\mu - \mu_0) \big(1 + \kappa \frac{\eta}{r^{*\delta}}\big) - \sigma \big(\frac{1}{\mu_0} + \frac{\mu}{4} \big) (1-\mu)^2 \Big] \frac{2}{r} & \\ - \big(1 + \frac{1}{4}(1-\mu)\big) \Big(\delta \frac{\eta}{r^*} - \eta^{(1)} \Big) \frac{\kappa}{r^{*\delta}} \Big\} \frac{1}{(1-\mu)^2} \big(\frac{\partial \psi}{\partial v^*} \big)^2 & \\ + \Big[\sigma \frac{2\mu}{r} + \big(\delta \frac{\eta}{r^*} - \eta^{(1)} \big) \frac{\kappa}{r^{*\delta}} \frac{1}{1-\mu} \Big] \left|\nabla\!\!\!/ \psi\right|^2 , & \end{aligned} \tag{2.2.170}$$

where we choose in fact $\mu_0 = \frac{1}{2}$.

$\mathbf{R_1 \leq r \leq R_2}$: Observe that on $[R_1, R_2]$ we have gained better control for $\left|\nabla\!\!\!/ \psi\right|^2$ in

(2.2.170) than in our previous (2.2.155). Here $\eta = 1$ and $\eta^{(1)} = 0$. In view of (2.2.167) the final term in (2.2.170) is precisely the positive quantity that allows us to control the error introduced by the cut-off of f. For here

$$\int_{\mathrm{S}} K^{X,1}[\psi]\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \geq -\frac{1}{4}\int_{\mathrm{S}}\frac{1}{1-\mu}f^{(3)}\psi^2\,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}}\,, \tag{2.2.171}$$

and

$$-\frac{1}{4}\frac{1}{1-\mu}f^{(3)} \geq -\frac{1}{4}\frac{1}{1-\mu}\frac{\varepsilon}{|r^*|^2} = \\ = -\varepsilon\frac{1}{4}\frac{1}{|r^*|^{1+\delta}}\frac{1}{|r^*|^{1-\delta}}\frac{1}{1-\mu} \geq -\varepsilon\frac{\kappa\delta}{|r^*|^{1+\delta}}\frac{1}{1-\mu}\frac{2}{r^2} \tag{2.2.172}$$

if

$$\frac{1}{4}\frac{1}{{R_1^*}^{1-\delta}} \leq 2\kappa\delta\frac{\Lambda}{3}\,; \tag{2.2.173}$$

the latter being a condition on the largeness of $R_1(\kappa,\delta,\Lambda) < \infty$. It remains to show that all other coefficients in $K^{\overline{Y}}$ are positive on $[R_1, R_2]$ as well. This is in fact true up to and including the cosmological horizon.

$\mathbf{r^* \geq R_1^*}$: Let $\sigma \geq 12$, and let R_1 in addition to (2.2.173) be chosen large enough so that for all $r \geq R_1(\sigma)$

$$1-\mu(r) \leq \frac{1}{\sigma}\,. \tag{2.2.174}$$

Then

$$\mu - \mu_0 - \sigma\Big(\frac{1}{\mu_0} + \frac{\mu}{4}\Big)(1-\mu)^2 \geq \frac{1}{6}\,, \tag{2.2.175}$$

and also

$$\Big[(\mu-\mu_0)\big(1+\kappa\frac{\eta}{r^{*\delta}}\big) - \sigma\big(\frac{1}{\mu_0}+\frac{\mu}{4}\big)(1-\mu)^2\Big]\frac{2}{r} - \big(1+\frac{1}{4}(1-\mu)\big)\Big(\delta\frac{\eta}{r^*} - \eta^{(1)}\Big)\frac{\kappa}{r^{*\delta}} \\ \geq \frac{1}{3}\frac{1}{r} - \kappa\big(1+\frac{1}{4}(1-\mu)\big)\delta\frac{\eta}{|r^*|^{1+\delta}} \geq \frac{1}{6}\frac{1}{r}\,, \tag{2.2.176}$$

because $\kappa \geq 0$, $\eta^{(1)} \leq 0$, if

$$\frac{1}{|R_1^*|^{1+\delta}} \leq \frac{1}{6}\frac{4}{5}\frac{1}{\delta\kappa}\sqrt{\frac{\Lambda}{3}} \tag{2.2.177}$$

which is satisfied for $R_1(\kappa,\delta,\sigma,\Lambda)$ chosen large enough. Finally, with $\sigma \geq 12$, $\mu_0 = \frac{1}{2}$,

$$\sigma(\mu-\mu_0) - \frac{1}{\mu_0}\big(1+\kappa\frac{\eta}{r^{*\delta}}\big) \geq 2 \tag{2.2.178}$$

if

$$\frac{1}{|R_1^*|^{\delta}} \leq \frac{1}{2\kappa}. \tag{2.2.179}$$

Choose $R_1(\kappa,\delta,\sigma,\Lambda) < \infty$ large enough such that (2.2.173), (2.2.174), (2.2.177), and (2.2.179) are satisfied, then we have shown by (2.2.172), (2.2.178), and (2.2.176) that for

$R_1 \leq r < \sqrt{\frac{3}{\Lambda}}$:

$$\int_{\mathrm{S}} \Big\{ K^{X,1} + \varepsilon K^{\overline{Y}} \Big\} \mathrm{d}\mu_{r^2 \overset{\circ}{\gamma}} \geq$$
$$\geq \int_{\mathrm{S}} \Big\{ \frac{\varepsilon}{r} \big(\frac{\partial\psi}{\partial u^*}\big)^2 + \frac{1}{6}\frac{\varepsilon}{r}\frac{1}{(1-\mu)^2}\big(\frac{\partial\psi}{\partial v^*}\big)^2 + \varepsilon\,\sigma\frac{2\mu}{r}\big|\nabla\!\!\!/\psi\big|^2 \Big\} \mathrm{d}\mu_{r^2 \overset{\circ}{\gamma}} \tag{2.2.180}$$

Note that we may choose $\varepsilon > 0$ arbitrarily small, for in this construction this will only result in a sufficiently large choice of $R_2(\varepsilon) > R_1(\kappa, \delta, \sigma, \Lambda)$ (cf. Lemma 2.19).

$\mathbf{\overline{Y}_c}$. Recall $\chi = 1$ on $r \geq R_1$, and $\chi = 0$ on $r \leq R_0$. We define a cut-off of $\overline{Y}$ by

$$\overline{Y}_c = \chi \overline{Y}\,, \tag{2.2.181}$$

whence

$$\begin{aligned}
{}^{(\overline{Y}_c)}\pi_{u^*u^*} &= 4\chi^{(1)}\big(1 + \sigma(1-\mu)\big) + \kappa\frac{\eta}{r^{*\delta}}\big) \\
&\quad + \chi^{(1)}\big(\sigma(1-\mu) + \kappa\frac{\eta}{r^{*\delta}}\big)(1-\mu) + \chi\,{}^{(\overline{Y})}\pi_{u^*u^*} && (2.2.182a)\\
{}^{(\overline{Y}_c)}\pi_{v^*v^*} &= -\chi^{(1)}\big(\sigma(1-\mu) + \kappa\frac{\eta}{r^{*\delta}}\big)(1-\mu) + \chi\,{}^{(\overline{Y})}\pi_{v^*v^*} && (2.2.182b)\\
{}^{(\overline{Y}_c)}\pi_{u^*v^*} &= -2\chi^{(1)}\big(1 + \sigma(1-\mu) + \kappa\frac{\eta}{r^{*\delta}}\big) + \chi\,{}^{(\overline{Y})}\pi_{u^*v^*} && (2.2.182c)\\
{}^{(\overline{Y}_c)}\pi_{AB} &= \chi\,{}^{(\overline{Y})}\pi_{AB}\,. && (2.2.182d)
\end{aligned}$$

$\mathbf{R_0 \leq r \leq R_1}$: Note that $\overline{Y}_c = \overline{Y}$ on $[R_1, R_2]$. For $r \in (R_0, R_1)$ we only have $\chi^{(1)} \geq 0$ (and $1 \geq \chi \geq 0$) and can infer from (2.2.182) the estimate

$$K^{\overline{Y}_c} \geq \chi\, K^{\overline{Y}} - \frac{1}{4}\Big(\sigma(1-\mu) + \kappa\frac{\eta}{r^{*\delta}}\Big)\frac{\chi^{(1)}}{1-\mu}\big(\frac{\partial\psi}{\partial u^*}\big)^2$$
$$- \Big(1 + \sigma(1-\mu) + \kappa\frac{\eta}{r^{*\delta}}\Big)\frac{\chi^{(1)}}{1-\mu}\big|\nabla\!\!\!/\psi\big|^2\,. \tag{2.2.183}$$

Let in fact R_0 be chosen such that

$$\mu(R_0) = \mu_0 = \frac{1}{2}\,. \tag{2.2.184}$$

Recall also (2.2.157), to see that then

$$K^{X_a} \geq \frac{\lambda}{1-\mu}\big(\frac{\partial\psi}{\partial t}\big)^2 \quad \text{on } [R_0, R_1]\,, \tag{2.2.185}$$

for any $1 \geq \lambda > 0$ where

$$f_a = \frac{\lambda}{1-\mu(R_1)}\frac{1}{2R_0}\,r^2\,, \qquad X_a = f_a\frac{\partial}{\partial r^*}\,. \tag{2.2.186}$$

Thus on $[R_0, R_1]$ (where $f^{(1)} = 1$, $f^{(2)} = f^{(3)} = 0$, and also $\eta^{(1)} \geq 0$) we have using (2.2.146) as well as (2.2.170) and (2.2.183) that

$$\int_{\mathrm{S}} \Big\{ K^{X,1} + \varepsilon K^{\overline{Y}_c} + K^{X_a} \Big\} \mathrm{d}\mu_{r^2 \overset{\circ}{\gamma}} \geq$$

$$\geq \int_{\mathrm{S}}\Big\{ \frac{\lambda}{1-\mu}\Big[\big(\frac{\partial\psi}{\partial r^*}\big)^2+\big(\frac{\partial\psi}{\partial t}\big)^2\Big]+\frac{2r^*}{r^3}(1-\mu^2)\psi^2$$
$$-\frac{1}{4}\,\varepsilon\,\chi\Big[\frac{1}{\mu_0}\big(1+\kappa\frac{\eta}{R_0^*}\big)\frac{2}{R_0}+\eta^{(1)}\frac{\kappa}{R_0^*}\frac{1}{1-\mu(R_1)}\Big]\big(\frac{\partial\psi}{\partial u^*}\big)^2$$
$$-\frac{1}{4}\,\varepsilon\,\chi^{(1)}\Big(\frac{\sigma}{2}+\kappa\frac{\eta}{R_0^*}\Big)\frac{1}{1-\mu(R_1)}\big(\frac{\partial\psi}{\partial u^*}\big)^2$$
$$-\ \varepsilon\,\chi\Big[\sigma\frac{5}{4}\frac{1}{4}\frac{2}{R_0}+\frac{5}{4}\delta\frac{\eta\,\kappa}{R_0^{*\,1+\delta}}\Big]\frac{1}{1-\mu(R_1)}\frac{1}{1-\mu}\big(\frac{\partial\psi}{\partial v^*}\big)^2$$
$$-\ \varepsilon\,\chi\,\eta^{(1)}\frac{\kappa}{R_0^{*\delta}}\frac{1}{1-\mu(R_1)}\frac{2}{R_0^2}\psi^2$$
$$-\ \varepsilon\,\chi^{(1)}\Big(1+\frac{\sigma}{2}+\frac{\kappa}{R_0^{*\,1+\delta}}\Big)\frac{1}{1-\mu(R_1)}\frac{2}{R_0^2}\psi^2\Big\}\,\mathrm{d}\mu_{r^2\mathring{\gamma}}\geq$$
$$\geq \int_{\mathrm{S}}\Big\{ \frac{1}{4}\frac{\lambda}{1-\mu}\Big[\big(\frac{\partial\psi}{\partial u^*}\big)^2+\big(\frac{\partial\psi}{\partial v^*}\big)^2\Big]+\frac{r^*}{r^3}(1-\mu^2)\psi^2\Big\}\,\mathrm{d}\mu_{r^2\mathring{\gamma}} \tag{2.2.187}$$

provided $\varepsilon=\varepsilon(\sigma,\kappa,\delta,\lambda,R_0,R_1)$ is chosen sufficiently small.

$\mathbf{0<r<R_0}$: Note finally that

$$\int_{\mathrm{S}}\Big\{ K^{X,1}+K^{X_a}\Big\}\,\mathrm{d}\mu_{r^2\mathring{\gamma}}\geq 0 \tag{2.2.188}$$

is ensured by choosing $\lambda>0$ suitable small. □

Timelike Killing vectorfield. In the static region (2.2.32) is timelike and

$$^{(T)}\pi=0\,. \tag{2.2.189}$$

Moreover it will allow us to control all boundary terms of the general current (2.2.160); note here

$$\big(J^T,n\big)=\frac{1}{4}\frac{1}{\sqrt{1-\mu}}\Big[\big(\frac{\partial\psi}{\partial u^*}\big)^2+\big(\frac{\partial\psi}{\partial v^*}\big)^2+2(1-\mu)\big|\nabla\!\!\!/\psi\big|^2\Big] \tag{2.2.190a}$$
$$\big(J^T,\frac{\partial}{\partial v^*}\big)=\frac{1}{2}\big(\frac{\partial\psi}{\partial v^*}\big)^2+\frac{1}{2}(1-\mu)\big|\nabla\!\!\!/\psi\big|^2\,, \tag{2.2.190b}$$

where

$$n=\frac{1}{\sqrt{1-\mu}}T \tag{2.2.191}$$

is the normal to the surfaces of constant t (2.2.27).

Foliation. Recall that the proof of Prop. 2.20 provides us with a function $0\leq f<\infty$ and values $0<R_1<R_2<\sqrt{\frac{3}{\Lambda}}$ such that $f^{(1)}\leq 1$ for $r\leq R_2$ and $f^{(1)}=0$ for $r\geq R_2$. We define hypersurfaces Σ_t by

$$\Sigma_t\doteq\Bigg(\Big\{(u,v):\frac{1}{2}\sqrt{\frac{3}{\Lambda}}\log\frac{u}{-v}=t\Big\}\cap\Big\{(u,v):0\leq\sqrt{\frac{3}{\Lambda}}\frac{1+uv}{1-uv}\leq R_2\Big\}\Bigg)$$
$$\cup\Bigg(\Big\{(u,v):u^*=\frac{1}{2}(t-R_2^*)\Big\}\cap\Big\{(u,v):R_2\leq\sqrt{\frac{3}{\Lambda}}\frac{1+uv}{1-uv}\leq\sqrt{\frac{3}{\Lambda}}\Big\}\Bigg)\,; \tag{2.2.192}$$

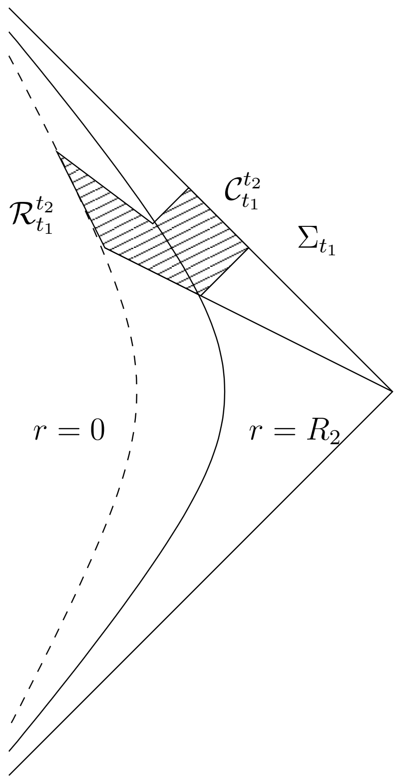

Figure 2.4: Foliation of the static region by hypersurfaces Σ_t.

in other words, Σ_t coincides with the hypersurface of constant t for $r \leq R_2$, and with the outgoing null hypersurfaces from the sphere (t, R_2) for $r \geq R_2$. Also denote by

$$\mathcal{R}_{t_1}^{t_2} = \bigcup_{t_1 \leq t \leq t_2} \Sigma_t \tag{2.2.193}$$

the spacetime region enclosed by Σ_{t_1}, Σ_{t_2}, and the incoming null segment

$$\mathcal{C}_{t_1}^{t_2} = \left\{ (u, 0) : \frac{1}{2}(t_1 - R_2^*) \leq u^* \leq \frac{1}{2}(t_2 - R_2^*) \right\}; \tag{2.2.194}$$

compare also figure 2.4.

In any of these regions $\mathcal{R}$ we can use Stokes' theorem to derive the energy identities for any of the currents J:

$$\int_{\mathcal{R}} K \, \mathrm{d}\mu_g = \int_{\partial\mathcal{R}} {}^*J \tag{2.2.195}$$

Here $K = \nabla^\mu J_\mu$, and

$$\mathrm{d}\mu_g = 2(1-\mu)\, r^2 \, \mathrm{d}u^* \wedge \mathrm{d}v^* \wedge \mathrm{d}\mu_{\overset{\circ}{\gamma}} . \tag{2.2.196}$$

Lemma 2.21. *Let f be as above, and let ψ satisfy $\overline{\psi} = 0$. Then for all $-\infty < t < \infty$*

$$\int_{\Sigma_t} \left(J^{X,1}[\psi], n_\Sigma \right) \leq C(\max|f|) \int_{\Sigma_t} \left(J^T[\psi], n_\Sigma \right), \tag{2.2.197}$$

where $n_\Sigma = n$ on the spacelike segment, and $n_\Sigma = \frac{\partial}{\partial v^*}$ on the null segment of Σ_t.

Proof. Recall (2.2.190).

$\mathbf{r \leq R_2}$. Clearly,

$$\Bigl|\bigl(J^X, n\bigr)\Bigr| \leq 2|f|\Bigl(J^T, n\Bigr)\,, \tag{2.2.198}$$

$$\Bigl|\bigl(J^{X,1}, n\bigr)\Bigr| \leq 2|f|\Bigl(J^T, n\Bigr) + \frac{1}{4}\Bigl(1 + \frac{2}{r}|f|\Bigr)\Bigl[\psi^2 + \bigl(\frac{\partial\psi}{\partial t}\bigr)^2\Bigr]\,. \tag{2.2.199}$$

Thus by Poincaré's inequality (2.2.167)

$$\int_{\mathrm{S}}\Bigl|\bigl(J^{X,1}, n\bigr)\Bigr|\,\mathrm{d}\mu_{r^2\mathring{\gamma}} \leq \\ \leq \Bigl[2|f| + \frac{1}{2}\Bigl(1 + \frac{2}{r}|f|\Bigr)\Bigl(1 + \frac{r^2}{2}\frac{1}{\sqrt{1-\mu(R_2)}}\Bigr)\Bigr]\int_{\mathrm{S}}\bigl(J^T, n\bigr)\,\mathrm{d}\mu_{r^2\mathring{\gamma}}\,. \tag{2.2.200}$$

$\mathbf{r \geq R_2}$. Again,

$$\Bigl|\bigl(J^X, \frac{\partial}{\partial v^*}\bigr)\Bigr| \leq |f|\Bigl(J^T, \frac{\partial}{\partial v^*}\Bigr)\,. \tag{2.2.201}$$

Here however,

$$\Bigl|\bigl(J^{X,1}, \frac{\partial}{\partial v^*}\bigr)\Bigr| \leq \Bigl|\bigl(J^X, \frac{\partial}{\partial v^*}\bigr)\Bigr| \\ + \frac{1}{r}|f|(1-\mu)|\psi||\partial_{v^*}\psi| + \frac{1}{4}\Bigl|\frac{\partial}{\partial v^*}\Bigl[\frac{2}{r}f(1-\mu)\Bigr]\Bigr|\psi^2\,. \tag{2.2.202}$$

Thus

$$\int_{\mathrm{S}}\Bigl|\bigl(J^{X,1}, \frac{\partial}{\partial v^*}\bigr)\Bigr|\,\mathrm{d}\mu_{r^2\mathring{\gamma}} \leq \\ \leq \Bigl[|f| + \frac{1}{2}r|f| + \frac{1}{r}|f| + \frac{1}{2} + \mu|f|\Bigr]\int_{\mathrm{S}}\bigl(J^T, \frac{\partial}{\partial v^*}\bigr)\,\mathrm{d}\mu_{r^2\mathring{\gamma}}\,. \tag{2.2.203}$$

Since by construction f is bounded and $r^{-1}f \to 1$ as $r \to 0$, (2.2.197) follows from the estimates above with a constant only depending on R_2 and $\max|f| < \infty$. □

Note that Lemma 2.21 suffices to estimate the boundary terms arising from the current (2.2.160) on Σ_t; for we have already shown

$$\Bigl|\bigl(J^{X_a}, n_\Sigma\bigr)\Bigr| \leq 2|f_a|\Bigl(J^T, n_\Sigma\Bigr)\,, \tag{2.2.204}$$

and since Y_c and n_Σ are both causal we obtain

$$\bigl(J^{Y_c}, n_\Sigma\bigr) = T(Y_c, n_\Sigma) \geq 0\,. \tag{2.2.205}$$

Lemma 2.22. *On* $\mathcal{C}^+$,

(i) $\Bigl|\bigl(J^X, \frac{\partial}{\partial u}\bigr)\Bigr| \leq |f|\bigl(J^T, \frac{\partial}{\partial u}\bigr)$

(ii) $\Big|\big(J^{X,1},\frac{\partial}{\partial u}\big)\Big| \leq |f|\big(J^T,\frac{\partial}{\partial u}\big)\,.$

Moreover, on the cosmological horizon $\mathcal{C}^+$,

$$\Big(J^{T+\overline{Y}_c},\frac{\partial}{\partial u}\Big) \geq \sqrt{\frac{\Lambda}{3}}\,u\,\big(\frac{\partial\psi}{\partial u}\big)^2 + \sqrt{\frac{3}{\Lambda}}\,\frac{1}{u}\big|\nabla\!\!\!/\psi\big|^2\,. \tag{2.2.206}$$

Recall here the coordinate system (u,v), in particular the relation (2.2.21) which shows that $\frac{\partial}{\partial r^*}$ has a regular extension to the horizon is given by

$$\frac{\partial}{\partial r^*} = -\sqrt{\frac{\Lambda}{3}}\,u\,\frac{\partial}{\partial u} - \sqrt{\frac{\Lambda}{3}}\,v\,\frac{\partial}{\partial v}\,. \tag{2.2.207}$$

We also recall (2.2.32).

Proof. Since $T|_{\mathcal{C}^+} = \sqrt{\frac{\Lambda}{3}}\,u\,\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial r^*}|_{\mathcal{C}^+} = -\sqrt{\frac{\Lambda}{3}}\,u\,\frac{\partial}{\partial u}$ we clearly have (i). And by (2.2.139) we have with f bounded and $f^{(1)}=0$ simply $|(J^{X,1},\partial_u)| = |(J^X,\partial_u)|$ on $\mathcal{C}^+$, and thus (ii). Now,

$$\overline{Y}_c|_{\mathcal{C}^+} = \hat{Y}|_{\mathcal{C}^+} = \frac{2}{\frac{\partial r}{\partial v}}\Big|_{v=0}\frac{\partial}{\partial v} = \sqrt{\frac{\Lambda}{3}}\frac{1}{u}\frac{\partial}{\partial v}\,, \tag{2.2.208}$$

which yields

$$\Big(J^{Y_c},\frac{\partial}{\partial u}\Big)\Big|_{\mathcal{C}^+} = \sqrt{\frac{\Lambda}{3}}\frac{1}{u}T_{uv}\big|_{r=\sqrt{\frac{3}{\Lambda}}} = \sqrt{\frac{3}{\Lambda}}\frac{1}{u}\big|\nabla\!\!\!/\psi\big|^2\,. \qquad \square$$

We are now able to prove the main conclusion of this section.

Proposition 2.23. *There exists a timelike vectorfield N, and a constant C only depending on Λ, such that for all solutions ψ of the wave equation* (2.2.51) *we have*

$$\int_{\mathcal{C}_{t_1}^{t_2}}\frac{1}{u}\big|\nabla\!\!\!/\psi\big|^2 \leq C(\Lambda)\int_{\Sigma_{t_1}}\big(J^N[\psi],n\big) \tag{2.2.209}$$

for all $t_2 > t_1 > 0$.

Proof. Since $\nabla\!\!\!/\overline{\psi}=0$ we may assume without loss of generality that $\overline{\psi}=0$. With the currents used in the proof of Prop. 2.20 we know in particular from (2.2.187) that there exist vectorfields X, X_a, $\overline{Y}_c$ and $\varepsilon>0$, $\lambda>0$ such that

$$0 \leq \int_{\mathcal{R}_{t_1}^{t_2}}\Big\{\frac{\lambda}{2}\Big[\big(\frac{\partial\psi}{\partial u^*}\big)^2+\big(\frac{\partial\psi}{\partial v^*}\big)^2\Big]r^2 + 2\frac{r^*}{r}(1-\mu)^2(1+\mu)\psi^2\Big\}\,\mathrm{d}u^*\wedge\mathrm{d}v^*\wedge\mathrm{d}\mu_{\overset{\circ}{\gamma}} \leq$$
$$\leq \int_{\mathcal{R}_{t_1}^{t_2}}\Big\{K^{X,1}+\varepsilon\,K^{\overline{Y}_c}+K^{X_a}\Big\}\,\mathrm{d}\mu_g\,. \tag{2.2.210}$$

Therefore by (2.2.195)

$$\int_{\Sigma_{t_2}}\Big(J^{X,1}+\varepsilon\,J^{\overline{Y}_c}+J^{X_a},n\Big) + \int_{\mathcal{C}_{t_1}^{t_2}}{}^*\Big(J^{X,1}+\varepsilon\,J^{\overline{Y}_c}+J^{X_a}\Big) \leq$$
$$\leq \int_{\Sigma_{t_1}}\Big(J^{X,1}+\varepsilon\,J^{\overline{Y}_c}+J^{X_a},n\Big)\,. \tag{2.2.211}$$

More precisely, we have with $u_i^* = \frac{1}{2}(t_i - R_2^*)$, $i = 1, 2$, obtained an estimate on:

$$\varepsilon \int_{u_1}^{u_2} \int_{\mathrm{S}} \big(J^{\overline{Y}_c}, \frac{\partial}{\partial u}\big) \,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \,\mathrm{d}u \leq \sum_{i=1}^{2} \Big| \int_{\Sigma_{t_i}} \Big(J^{X,1} + J^{X_a}, n\Big) \Big| + \Big| \int_{\mathcal{C}_{t_1}^{t_2}} {}^*\Big(J^{X,1} + J^{X_a}\Big) \Big| + \varepsilon \int_{\Sigma_{t_1}} \Big(J^{\overline{Y}_c}, n\Big) . \quad (2.2.212)$$

In view of Lemma 2.21 and Lemma 2.22 we immediately obtain

$$\varepsilon \int_{u_1}^{u_2} \int_{\mathrm{S}} \sqrt{\frac{3}{\Lambda}} \frac{1}{u} \big|\nabla\!\!\!/ \psi\big|^2 \,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \,\mathrm{d}u \leq \leq C(\max\{|f|, |f_a|\}, R_2, \Lambda) \int_{\Sigma_{t_1}} \Big(J^T + \varepsilon J^{\overline{Y}_c}, n\Big) \quad (2.2.213)$$

by (2.2.189). The statement of the Proposition thus follows with $N = T + \varepsilon \overline{Y}_c$. □

Remark 2.24. In view of (2.2.206) and (2.2.189) we have in fact proven that there exists a constant C such that

$$\int_{u_1}^{u_2} \int_{\mathrm{S}} \Big\{ \sqrt{\frac{\Lambda}{3}} u \big(\frac{\partial \psi}{\partial u}\big)^2 + \varepsilon \sqrt{\frac{3}{\Lambda}} \frac{1}{u} \big|\nabla\!\!\!/ \psi\big|^2 \Big\} r^2 \,\mathrm{d}\mu_{r^2\overset{\circ}{\gamma}} \,\mathrm{d}u \leq \leq C(\varepsilon, \Lambda) \int_{\Sigma_{t_1}} \Big(J^T + \varepsilon J^{\overline{Y}_c}, n\Big) , \quad (2.2.214)$$

for all $0 < u_1 < u_2 < \infty$.

Inhomogeneous wave equation. As discussed in Section 2.2.2.3 our analysis of the inhomogeneous wave equation is based on the same vectorfields but modified currents. This is motivated by the following insight.

Lemma 2.25. *Let* $(\mathcal{M}, g)$ *be a spherically symmetric spacetime,*

$$g = g_{ab} \,\mathrm{d}x^a \,\mathrm{d}x^b + r^2 \overset{\circ}{\gamma} = -\Omega^2 \,\mathrm{d}u \,\mathrm{d}v + r^2 \overset{\circ}{\gamma} , \quad (2.2.215)$$

and Y *a vectorfield on the quotient plane,* $Y = Y^u \partial_u + Y^v \partial_v$. *Let the modified current of* J^Y *be defined by*

$$J^{Y;m_\Lambda}[\psi] = J^Y[\psi] - \frac{m_\Lambda}{2} Y^\flat \psi^2 , \quad (2.2.216)$$

then for all solutions ψ *of* (2.2.52) *we have*

$$\nabla \cdot J^{Y;m_\Lambda} = \sum_{a,b=1}^{2} K^{ab} \partial_a \psi \,\partial_b \psi + K\!\!\!/ \big|\nabla\!\!\!/ \psi\big|^2 + m_\Lambda \,K \psi^2 , \quad (2.2.217)$$

where

$$K = K\!\!\!/ = -\frac{1}{2} \,\mathrm{tr}\, {}^{(Y)}\pi , \quad (2.2.218)$$

and

$$K^{uu} = -\frac{2}{\Omega^2}\frac{\partial Y^u}{\partial v} \qquad K^{vv} = -\frac{2}{\Omega^2}\frac{\partial Y^v}{\partial u} \tag{2.2.219}$$

$$K^{uv} = K^{vu} = -\frac{2}{\Omega^2}\frac{1}{r}\big(Y\cdot r\big)\,. \tag{2.2.220}$$

Note that $J^{Y;m_\Lambda}$ reduces to J^Y if $m_\Lambda = 0$. The relevance of Lemma 2.25 lies however in the fact that if $m_\Lambda > 0$ then the coefficients to the zeroth order term are *the same* as for the angular derivatives term, as expressed in (2.2.218).

We have already stated an instance of this result for the Morawetz current, namely in (2.2.123) and (2.2.124). But we shall also apply this modification to the currents J^{Y_c} and J^T to obtain:

Proposition 2.26. *There exists a timelike vectorfield N, and a constant C only depending on Λ, such that for all solutions ψ of the inhomogeneous wave equation* (2.2.52) *we have*

$$\int_{\mathcal{C}_{t_1}^{t_2}} \frac{1}{u}\Big\{\big|\nabla\!\!\!/\psi\big|^2 + m_\Lambda\psi^2\Big\} \leq C(\Lambda)\int_{\Sigma_{t_1}}\big(J^{N;m_\Lambda}[\psi],n\big) \tag{2.2.221}$$

for all $t_2 > t_1 > 0$, provided $m_\Lambda \geq 2\frac{\Lambda}{3}$.

Proof. Let $\varepsilon > 0$, $R_1 > 0$, and let f, f_a, X, X_a, and $\overline{Y}_c$ be constructed as in the proof of Prop. 2.20. Let $J^{X,1;m_\Lambda}$ be defined by (2.2.123), then we have on one hand

$$\nabla\cdot J^{X,1;m_\Lambda} \geq \Big[m_\Lambda - 2\mu\frac{\Lambda}{3}\Big]\frac{f}{r}\psi^2 \geq 0\,, \qquad \text{on } r\in\big(0,R_1\big]\cup\Big[R_2,\sqrt{\frac{3}{\Lambda}}\Big]\,, \tag{2.2.222}$$

since $m_\Lambda \geq 2\frac{\Lambda}{3}$, and

$$\nabla\cdot J^{X,1;m_\Lambda}[\psi] \geq -\frac{1}{4}\frac{1}{1-\mu}f^{(3)}\psi^2\,, \qquad \text{on } r\in[R_1,R_2]\,. \tag{2.2.223}$$

On the other hand, by Lemma 2.25 and (2.2.170) we have

$$\nabla\cdot J^{\overline{Y}_c;m_\Lambda} \geq \frac{\delta\kappa}{r^{*1+\delta}}\frac{1}{1-\mu}\psi^2\,, \qquad \text{on } r\in[R_1,R_2]\,, \tag{2.2.224}$$

which shows that the modified redshift current associated to $Y_c = \varepsilon\overline{Y}_c$ cancels the error (2.2.223) similarly to (2.2.172) provided R_1 is chosen large enough. By Lemma 2.25 and the proof of Prop. 2.20 we also have that $\nabla\cdot J^{\overline{Y}_c;m_\Lambda}$ is positive on $R_1 \leq r < \sqrt{\frac{3}{\Lambda}}$, and that

$$\nabla\cdot J^{X,1;m_\Lambda} + \varepsilon\nabla\cdot J^{\overline{Y};m_\Lambda} + \nabla\cdot J^{X_a;m_\Lambda} \geq 0\,, \qquad \text{on } 0 < r < R_1\,, \tag{2.2.225}$$

since ε can be chosen arbitrarily small.

The boundary terms of the currents $J^{X,1:m_\Lambda}$ and $J^{X_a;m_\Lambda}$ are controlled by the current

$$J^{T;m_\Lambda} = J^T - \frac{m_\Lambda}{2}T^\flat\psi^2 \tag{2.2.226}$$

which is conserved by (2.2.52) because T is Killing,

$$\nabla\cdot J^{T;m_\Lambda} = K^T - \frac{m_\Lambda}{2}\nabla\cdot T\,\psi^2\,, \tag{2.2.227}$$

$$K^T = {}^{(T)}\pi^{\mu\nu}T_{\mu\nu} = 0 \qquad \nabla\cdot T = \mathrm{tr}\,{}^{(T)}\pi = 0\,. \tag{2.2.228}$$

Note for example, for the boundary terms on surfaces of constant t,

$$J^{T;m_\Lambda}\cdot T = \frac{1}{4}\Bigl(\frac{\partial\psi}{\partial u^*}\Bigr)^2 + \frac{1}{4}\Bigl(\frac{\partial\psi}{\partial u^*}\Bigr)^2 + \frac{1}{2}(1-\mu)\bigl|\nabla\!\!\!/\psi\bigr|^2 + \frac{m_\Lambda}{2}(1-\mu)\psi^2\,. \tag{2.2.229}$$

Finally, since

$$Y_c\Big|_{\mathcal{C}^+} = \varepsilon\hat{Y}\Big|_{\mathcal{C}^+} = \sqrt{\frac{\Lambda}{3}}\frac{\varepsilon}{u}\frac{\partial}{\partial v}\,, \tag{2.2.230}$$

we obtain for the boundary terms of the redshift vectorfield on the cosmological horizon:

$$J^{Y_c;m_\Lambda}\cdot\frac{\partial}{\partial u}\Big|_{\mathcal{C}^+} = \sqrt{\frac{3}{\Lambda}}\frac{2\varepsilon}{u}\bigl|\nabla\!\!\!/\psi\bigr|^2 + m_\Lambda\sqrt{\frac{3}{\Lambda}}\frac{\varepsilon}{u}\psi^2\,. \tag{2.2.231}$$

The result then follows with $N = T + \varepsilon\overline{Y}_c$. □

Remark 2.27. Taking into account the contribution from the vectorfield T we have in fact shown that if $m_\Lambda \geq 2\frac{\Lambda}{3}$ then there exists a constant C such that

$$\int_{u'}^\infty \mathrm{d}u\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\Bigl\{\frac{\Lambda}{3}u\Bigl(\frac{\partial\psi}{\partial u}\Bigr)^2 + \frac{1}{u}\bigl|\nabla\!\!\!/\psi\bigr|^2 + m_\Lambda\frac{1}{u}\psi^2\Bigr\}\Big|_{v=0} \leq C(\Lambda)\int_{\Sigma_{t'}}\bigl(J^{N;m_\Lambda},n\bigr)\,, \tag{2.2.232}$$

where $2(u')^* = t - R_1^*$.

2.2.2.5 The redshift effect on the cosmological horizon

In this Section we tie our results for the expanding region in Section 2.2.2.1 to our results for the static region in Section 2.2.2.3. The energy estimates of Section 2.2.2.1 provide control on solutions to the linear wave equation up to and including the spacelike future boundary of the spacetime in terms of energies on spacelike hypersurfaces that can be chosen arbitrarily close to the cosmological horizon. In Section 2.2.2.3 we have in particular established that we can control the nondegenerate energy on the cosmological horizon. Here we prove that as a consequence of the redshift effect the latter controls the energies arising in Section 2.2.2.1. It is in particular a consequence of this Section that our results in Thm. 4 and 5 can expressed in terms of initial data prescribed on an arbitrary spacelike hypersurface.

Redshift vectorfield. In Section 2.2.2.1 we have found a redshift vectorfield based on the combination $\bar{Y} + Y$. The positivity property of the associated current depends on a symmetrization argument that is only valid for $r > \sqrt{\frac{3}{\Lambda}}$. Here we construct a redshift vectorfield *on* the cosmological horizon and its vicinity based on the vectorfields $\bar{Y}$ and Y separately. We will address the general case $m_\Lambda \geq 0$ directly.

Let us recall the vectorfield T from (2.2.32) and its properties summarized in Lemma 2.5, and set

$$Y\Big|_{\mathcal{C}^+} = \frac{2}{\frac{\partial r}{\partial v}}\Big|_{v=0}\frac{\partial}{\partial v}\,. \tag{2.2.233}$$

We have

$$g(T,Y)\Big|_{\mathcal{C}^+} = -2\,, \tag{2.2.234}$$

and

$$\big[T,Y\big]\Big|_{\mathcal{C}^+} = 0\,. \tag{2.2.235}$$

Extend, as in the construction for black hole horizons [17], the vectorfield Y into the expanding region by

$$\nabla_Y Y = -\sigma(Y+T)\,, \tag{2.2.236}$$

where $\sigma > 0$. It is then easy to show (cf. [17]) that

$$K^Y = T_{\mu\nu}[\psi]^{(Y)}\pi^{\mu\nu} \geq \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\big(Y\cdot\psi\big)^2 + \Big[\frac{\sigma}{2} - 4\sqrt{\frac{\Lambda}{3}}\Big]\big(T\cdot\psi\big)^2 + \frac{\sigma}{2}\big|\nabla\!\!\!/\psi\big|^2\,, \tag{2.2.237}$$

which is positive for $\sigma > 8\sqrt{\frac{\Lambda}{3}}$.

However, in contrast to the construction for black hole horizons the vectorfield Y defined by the extension (2.2.236) does not remain causal in the expanding region.[2] But, as we shall see next $T+Y$ remains timelike in the vicinity of $\mathcal{C}^+$.

The condition (2.2.236) is easily shown to be equivalent to

$$\frac{\mathrm{d}Y^u}{\mathrm{d}v}\Big|_{v=0} = -\sigma u^2 \tag{2.2.238}$$

$$\frac{\mathrm{d}Y^v}{\mathrm{d}v}\Big|_{v=0} = -2\sqrt{\frac{\Lambda}{3}} - \sigma\,. \tag{2.2.239}$$

Also,

$$Y\Big|_{\mathcal{C}^+} = \sqrt{\frac{\Lambda}{3}}\frac{1}{u}\frac{\partial}{\partial v} \qquad T\Big|_{\mathcal{C}^+} = \sqrt{\frac{\Lambda}{3}}u\frac{\partial}{\partial u}\,. \tag{2.2.240}$$

For $\sigma = 8\sqrt{\frac{\Lambda}{3}}$ we thus obtain

$$Y+T\,\Big|_{(u,v)} = \sqrt{\frac{\Lambda}{3}}\big(1-8\,uv\big)\,u\frac{\partial}{\partial u} + \sqrt{\frac{\Lambda}{3}}\big(\frac{1}{uv} - 11\big)\,v\frac{\partial}{\partial v} + \mathcal{O}(v^2)\,. \tag{2.2.241}$$

Therefore we can choose $\sigma > 8\sqrt{\frac{\Lambda}{3}}$ and $r_0(\sigma) > \sqrt{\frac{3}{\Lambda}}$ sufficiently small such that $K^N \geq 0$ and $T+Y$ is *timelike* on $\sqrt{\frac{3}{\Lambda}} \leq r \leq r_0$ to the future of a given $u_0 > 0$.

[2] On the level of Penrose diagrams the extension of the vectorfield Y from the cosmological horizon the the expanding region corresponds in the black hole case to the extension from the event horizon to the *interior* of the black hole.

Redshift current. We consider in the general case $m_\Lambda \geq 0$ the current

$$J^{N;m_\Lambda} = J^N - \frac{m_\Lambda}{2} N^\flat \psi^2 \,, \tag{2.2.242}$$

associated to the multiplier

$$N\Big|_{(u,v)} = \sqrt{\frac{\Lambda}{3}}\big(1 - 10\,uv\big)\,u\frac{\partial}{\partial u} + \sqrt{\frac{\Lambda}{3}}\big(\frac{1}{uv} - 13\big)\,v\frac{\partial}{\partial v}\,; \tag{2.2.243}$$

(this vectorfield arises in the above construction as $N = T + Y$ with $\sigma = 10\sqrt{\frac{\Lambda}{3}}$).

Proposition 2.28. *Let Σ be a spacelike hypersurface with normal n crossing the cosmological horizon $\mathcal{C}^+$ to the future of the sphere $\mathcal{C}^+ \cap \mathcal{C}^-$. Let ψ be a solution to* (2.2.52) *with $m_\Lambda > 0$ or $m_\Lambda = 0$, and $r_1 > \sqrt{\frac{3}{\Lambda}}$ a fixed radius close enough to the horizon such that*

$$\frac{\sqrt{\frac{\Lambda}{3}}r_1 - 1}{\sqrt{\frac{\Lambda}{3}}r_1 + 1} \leq \frac{1}{15}\,, \tag{2.2.244}$$

and denote by Σ' the segment of Σ truncated by Σ_{r_1} and $\mathcal{C}^+$, and by $\mathcal{C}_0^+$ the segment of $\mathcal{C}^+$ truncated by Σ. Then

$$\int_{\Sigma_{r_1}\cap J^+(\Sigma')} \phi\Big\{\phi^2\big(T\cdot\psi\big)^2 + \frac{1}{\phi^2}\big(\frac{\partial\psi}{\partial r}\big)^2 + \big|\nabla\!\!\!/\psi\big|^2 + m_\Lambda\psi^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_1}} \leq$$
$$\leq C(r_1,\Lambda)\int_{\mathcal{C}_0^+}\Big\{\frac{\Lambda}{3}u\big(\frac{\partial}{\partial u}\psi\big)^2 + \frac{1}{u}\big|\nabla\!\!\!/\psi\big|^2 + \frac{m_\Lambda}{u}\psi^2\Big\} + C(r_1)\int_{\Sigma'} J^{n;m_\Lambda}\cdot n\,, \tag{2.2.245}$$

where C is a constant that only depends on the chosen value of r_1, and Λ.

Remark 2.29. Note that the energy on the right hand side of (2.2.245) is precisely of the form controlled in (2.2.232).

Proof. We have

$$\nabla\cdot J^{N;m_\Lambda} = K^N - \frac{m_\Lambda}{2}\,\nabla\cdot N\,\psi^2\,, \tag{2.2.246}$$

and by construction on $\mathcal{C}^+$:

$$K^N \geq \frac{1}{4}\sqrt{\frac{\Lambda}{3}}\big(Y\cdot\psi\big)^2 + \sqrt{\frac{\Lambda}{3}}\big(T\cdot\psi\big)^2 + 5\sqrt{\frac{\Lambda}{3}}\big|\nabla\!\!\!/\psi\big|^2\,. \tag{2.2.247}$$

By continuity the positivity of $K^N \geq 0$ holds up to a fixed radius $r_1 > \sqrt{\frac{3}{\Lambda}}$. Using the explicit expression (2.2.243) we find

$$\nabla\cdot N = \nabla_\mu N^\mu = -12\sqrt{\frac{\Lambda}{3}} - 2\sqrt{\frac{\Lambda}{3}}\,10\,uv + \frac{2}{1-uv}\sqrt{\frac{\Lambda}{3}}$$
$$- \frac{2\,uv}{1-uv}\sqrt{\frac{\Lambda}{3}}(12 + 10uv)\,, \tag{2.2.248}$$

and thus on $uv \leq \frac{1}{2}$:

$$-\nabla \cdot N \geq 8\sqrt{\frac{\Lambda}{3}}\,. \tag{2.2.249}$$

It remains to calculate the boundary terms of the current $J^{N;m_\Lambda}$ using the expression (2.2.243). Let in addition $uv \leq \frac{1}{15}$ then on Σ_r

$$\begin{aligned} J^{N;m_\Lambda} \cdot n = \phi\, J^{N;m_\Lambda} \cdot V &= \phi\, T(N,V) + \frac{m_\Lambda}{2}\phi\bigl[-g(N,V)\bigr]\psi^2 \\ &\geq \phi\, uv\Bigl[\frac{1}{3}\frac{\Lambda}{3}\frac{u}{v}\bigl(\frac{\partial\psi}{\partial u}\bigr)^2 + 2\frac{\Lambda}{3}\frac{v}{u}\bigl(\frac{\partial\psi}{\partial v}\bigr)^2 + \frac{1}{2}\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^2\bigl|\nabla\!\!\!/\psi\bigr|^2\Bigr] \\ &\qquad + \frac{m_\Lambda}{2}\phi\frac{1}{2}\frac{1}{10}\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^2\psi^2 \\ &\geq \frac{1}{6}\phi^{-1}\Bigl[\phi^2\bigl(T\cdot\psi\bigr)^2 + \frac{1}{\phi^2}\bigl(\frac{\partial\psi}{\partial r}\bigr)^2 + \bigl|\nabla\!\!\!/\psi\bigr|^2\Bigr] + \frac{1}{6}\frac{m_\Lambda}{8}\bigl(1+\sqrt{\frac{\Lambda}{3}}r\bigr)^2\psi^2\,, \end{aligned} \tag{2.2.250}$$

and on $\mathcal{C}^+$

$$\begin{aligned} J^{N;m_\Lambda} \cdot \frac{\partial}{\partial u} &= \sqrt{\frac{3}{\Lambda}}\frac{1}{u}J^{N;m_\Lambda}\cdot T \\ &= \sqrt{\frac{3}{\Lambda}}\frac{1}{u}T(T+Y,T) - \sqrt{\frac{3}{\Lambda}}\frac{1}{u}\frac{m_\Lambda}{2}g(T+Y,T)\psi^2 \\ &= \sqrt{\frac{3}{\Lambda}}\frac{1}{u}\Bigl[\bigl(T\cdot\psi\bigr)^2 + \bigl|\nabla\!\!\!/\psi\bigr|^2 + m_\Lambda\psi^2\Bigr]\,. \end{aligned} \tag{2.2.251}$$

The estimate of the Proposition then follows from the energy identity for $J^{N;m_\Lambda}$ on the domain bounded by $\mathcal{C}^+$, Σ_{r_1} and Σ provided $r_1 > \sqrt{\frac{3}{\Lambda}}$ is chosen small enough (in particular such that $uv \leq 1/15$ on Σ_{r_1}), with

$$C(\Lambda, r_1) = 6\frac{8}{\frac{\Lambda}{3}r_1^2 - 1}\,. \tag{2.2.252}$$

□

2.2.2.6 Pointwise estimates on the timelike future boundary

Pointwise estimates follow from energy estimates by Sobolev inequalities. Here we are interested in the relevant inequality on Σ_r.

Proposition 2.30 (Sobolev inequality on Σ_r)**.** *Let $\psi \in \mathrm{H}^3(\Sigma_r)$, $r > \sqrt{\frac{3}{\Lambda}}$, then*

$$\begin{aligned} \sqrt{\frac{3}{\Lambda}\Bigl(\frac{\Lambda}{3}r^2 - 1\Bigr)}\, r^2 \sup_{p,q\in\Sigma_r}|\psi^2(p) - \psi^2(q)| &\leq \\ \leq C(\Lambda)\int_{\Sigma_r}\Bigl\{\psi^2 + \sum_{i=1}^3\bigl(\Omega_{(i)}\psi\bigr)^2 &+ \sum_{i,j=1}^3\bigl(\Omega_{(i)}\Omega_{(j)}\psi\bigr)^2 \\ + \bigl(T\psi\bigr)^2 + \sum_{i=1}^3\bigl(\Omega_{(i)}T\psi\bigr)^2 &+ \sum_{i,j=1}^3\bigl(\Omega_{(i)}\Omega_{(j)}T\psi\bigr)^2\Bigr\}\,\mathrm{d}\mu_{\overline{g}_r}\,. \end{aligned} \tag{2.2.253}$$

The angular derivatives on the sphere are here estimated using the generators of the spherical isometries $\Omega_{(i)} : i = 1,2,3$. This fact is known as the coercivity inequality on the sphere which is discussed in Appendix C.2, where also the precise definition of the vectorfields $\Omega_{(i)}$ can be found.

Proof. Recall the induced metric on Σ_r derived in Section 2.2.1.3,

$$\overline{g}_r = \frac{1}{4}\frac{3}{\Lambda}\Bigl(\frac{\Lambda}{3}r^2 - 1\Bigr)\,\mathrm{d}\lambda^2 + r^2\,\mathring{\gamma}\ . \tag{2.2.254}$$

Let $\lambda\,,\ \lambda_0 \in (-\infty,\infty)$, and $\xi \in \mathbb{S}^2$, then

$$|\psi^2(\lambda;\xi) - \psi^2(\lambda_0;\xi)| \leq 2\int_{\lambda_0}^{\lambda}|\psi||\frac{\partial\psi}{\partial\lambda}|\,\mathrm{d}\lambda \tag{2.2.255}$$

and by (2.2.49)

$$\int_{\lambda_0}^{\lambda}|\frac{\partial\psi}{\partial\lambda}|^2\,\mathrm{d}\lambda = \int_{\lambda_0}^{\lambda}\frac{1}{4}\frac{3}{\Lambda}\Bigl(T\psi\Bigr)^2\,\mathrm{d}\lambda\,. \tag{2.2.256}$$

Therefore by Corollary C.3

$$\begin{aligned}|r^2\psi^2(\lambda) - r^2\psi^2(\lambda_0)| \leq\\ \leq C\int_{-\infty}^{\infty}\int_{S_r}|\psi|^2 + \sum_{i=1}^{3}\Bigl(\Omega_{(i)}\psi\Bigr)^2 + \sum_{i,j=1}^{3}\Bigl(\Omega_{(i)}\Omega_{(j)}\psi\Bigr)^2\,\mathrm{d}\mu_{\gamma_r}\,\mathrm{d}\lambda\\ + C\int_{-\infty}^{\infty}\frac{1}{4}\frac{3}{\Lambda}\int_{S_r}|T\psi|^2 + \sum_{i=1}^{3}\Bigl(\Omega_{(i)}T\psi\Bigr)^2 + \sum_{i,j=1}^{3}\Bigl(\Omega_{(i)}\Omega_{(j)}T\psi\Bigr)^2\,\mathrm{d}\mu_{\gamma_r}\,\mathrm{d}\lambda\,,\end{aligned} \tag{2.2.257}$$

which proves the stated inequality in view of (2.2.48). □

For a solution ψ to the wave equation Proposition 2.30 applied to the functions $T\psi$ and $\Omega_{(i)}\psi : i = 1,2,3$ yields quantities on the right hand side which are monotone by our results of Section 2.2.2.1. As a consequence we obtain the following pointwise bounds.

Proposition 2.31 (Pointwise estimates)**.** *Let ψ be a solution to the linear wave equation* (1.1.1) *with $m_\Lambda \geq 0$, and assume that*

$$D[\psi;\Sigma_{r_1}] \doteq \int_{\Sigma_{r_1}}\phi\Bigl\{\phi^2\bigl(T\cdot\psi\bigr)^2 + \frac{1}{\phi^2}\bigl(\frac{\partial\psi}{\partial r}\bigr)^2 + |\nabla\!\!\!/\,\psi|^2 + m_\Lambda\psi^2\Bigr\}\,\mathrm{d}\mu_{\overline{g}_{r_1}} < \infty \tag{2.2.258}$$

for some $r_1 > \sqrt{\frac{3}{\Lambda}}$; moreover assume that also

$$D[\Omega_{(i)}\psi,\Sigma_{r_1}] < \infty : i = 1,2,3\,, \qquad D[\Omega_{(i)}\Omega_{(j)}\psi,\Sigma_{r_1}] < \infty : i,j = 1,2,3$$

as well as $D[T\psi,\Sigma_{r_1}] < \infty$, and

$$D[\Omega_{(i)}T\psi,\Sigma_{r_1}] < \infty : i = 1,2,3\,, \qquad D[\Omega_{(i)}\Omega_{(j)}T\psi,\Sigma_{r_1}] < \infty : i,j = 1,2,3\,.$$

Then for all $r > r_1$ also

$$r^2 \sup_{p,q\in\Sigma_r}\Big||\nabla\!\!\!/\psi(p)|^2_{\gamma_r} - |\nabla\!\!\!/\psi(q)|^2_{\gamma_r}\Big| < \infty\,, \tag{2.2.259}$$

and

$$(r\phi)^2 \sup_{p,q\in\Sigma_r}\Big|(T\psi)^2(p) - (T\psi)^2(q)\Big| < \infty\,, \tag{2.2.260}$$

and if $m_\Lambda > 0$ also

$$\sup_{p,q\in\Sigma_r}\Big|(r\psi)^2(p) - (r\psi)^2(q)\Big| < \infty\,, \tag{2.2.261}$$

and there exists a constant C that only depends on Λ such that for all $r > r_1$

$$\begin{aligned}\sup_{p,q\in\Sigma_r}\Big\{(r\phi)^2\Big|(T\psi)^2(p) - (T\psi)^2(q)\Big| \\ + r^2\Big||\nabla\!\!\!/\psi(p)|^2_{\gamma_r} - |\nabla\!\!\!/\psi(q)|^2_{\gamma_r}\Big| + m_\Lambda\Big|(r\psi)^2(p) - (r\psi)^2(q)\Big|\Big\} \leq \\ \leq C(\Lambda)\Big[D[\psi;\Sigma_{r_1}] + \sum_{i=1}^3 D[\Omega_{(i)}\psi;\Sigma_{r_1}] + \sum_{i,j=1}^3 D[\Omega_{(i)}\Omega_{(j)}\psi;\Sigma_{r_1}] \\ + D[T\psi;\Sigma_{r_1}] + \sum_{i=1}^3 D[\Omega_{(i)}T\psi;\Sigma_{r_1}] + \sum_{i,j=1}^3 D[\Omega_{(i)}\Omega_{(j)}T\psi;\Sigma_{r_1}]\Big]\,.\end{aligned} \tag{2.2.262}$$

Proof. Apply Proposition 2.30 to the functions $\phi^{\frac{3}{2}}T\cdot\psi$ and $\phi^{\frac{1}{2}}\frac{1}{r}\Omega_{(i)}\psi : i = 1,2,3$ as well as $(\phi m_\Lambda\psi)^{\frac{1}{2}}$ and use Propositions 2.7 and 2.9. Note that $[T,\Omega_{(i)}] = 0 : i = 1,2,3$. □

We may now proceed as before in Section 2.2.2.5 to estimate the energies on the right hand side of (2.2.262) in terms of energies on a spacelike hypersurface that crosses the cosmological horizon. This yields Theorem 5.

2.3 Linear Waves on Schwarzschild de Sitter

This Section applies the ideas and constructions of Section 2.2 to the global study of solutions to

$$\Box_g\psi = 0 \tag{2.3.1}$$

on Schwarzschild de Sitter spacetimes $(\mathcal{M}^{(m)}_\Lambda, g)$.

We have chosen to view the homogeneous de Sitter spacetime as a member of the spherically symmetric Schwarzschild de Sitter family corresponding to the parameter $m = 0$. This shall allow us to proceed analogously to the analysis in Section 2.2 in the case $m > 0$.

We recall in Section 2.3.1 that the *causal geometry of the expanding region* is qualitatively the same as in the case $m = 0$, namely it is bounded in the timelike future by a spacelike hypersurface along which the area of every sphere is infinitely large, and in the past by the cosmological horizons $\mathcal{C}^+\cup\bar{\mathcal{C}}^+$. Moreover this region is foliated by spacelike hypersurfaces of constant area radius. The adjacent domains beyond the cosmological horizons are as in the case $m=0$ *static* regions of spacetime, which are in turn bounded by *black hole event horizons* and the interior domains of black holes. However, for our analysis the region of interest is the expanding region and its extension across the cosmological horizons; correspondingly we derive in Section 2.3.1 a double null foliation which covers this domain.

For definiteness, the global picture described above is correct for any choice of $\Lambda>0$ and $0<m<(3\sqrt{\Lambda})^{-1}$. The Schwarzschild de Sitter spacetimes $(\mathcal{M}_\Lambda^{(m)},g)$ are spherically symmetric and have topology $\mathcal{Q}_\Lambda^{(m)}\times\mathrm{SO}(3)$ where $\mathcal{Q}_\Lambda^{(m)}$ is a $1+1$-dimensional Lorentzian manifold; as discussed in Section 2.3.1 they are unique as solutions to (2.2.14) with that topology for $\Lambda>0$. We denote as above by r the area radius of the orbits of the SO(3) group action. It turns out that the expanding region corresponds to $r>r_{\mathcal{C}}$ where $r_{\mathcal{C}}(\Lambda,m)$ is a root of a polynomial of degree 3 with coefficients in Λ and m, and that the metric on this domain takes the form

$$g=-\phi^2\,\mathrm{d}r^2+\overline{g}_r=-\frac{1}{\frac{\Lambda r^2}{3}+\frac{2m}{r}-1}\,\mathrm{d}r^2+\overline{g}_r\,, \tag{2.3.2}$$

where $\overline{g}_r$ is the Riemannian metric of a cylinder on the level sets Σ_r of r as a function on $\mathcal{Q}_\Lambda^{(m)}$; (this expression should be compared to (2.2.2)). The cosmological horizons are the null hypersurfaces $r=r_{\mathcal{C}}$, and have positive surface gravity $\kappa_{\mathcal{C}}>0$ as in the case $m=0$.

Global Redshift Vectorfield. The crucial estimate for our analysis is given in Section 2.3.2.1 where we establish the global redshift property of the vectorfield

$$M=\frac{1}{r}\frac{\partial}{\partial r}\,; \tag{2.3.3}$$

here $\partial_r=\phi n$, where n is the normal to Σ_r. Indeed, according to Proposition 2.34 we have for all solutions of (2.3.1) the lower bound

$$\phi\,\nabla\cdot J^M[\psi]\geq\frac{1}{r}\,J^M[\psi]\cdot n\,, \tag{2.3.4}$$

where J^M is the energy current associated to the multiplier M; (recall the discussion in Section 2.2.2.2). By the energy identity for J^M on

$$\mathcal{R}_{r_1}^{r_2}\doteq\bigcup_{r_1\leq r\leq r_2}\Sigma_r\,, \tag{2.3.5}$$

we thus obtain

$$\int_{\Sigma_{r_2}}J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}}+\int_{r_1}^{r_2}\mathrm{d}r\frac{1}{r}\int_{\Sigma_r}J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_r}\leq$$
$$\leq\int_{\Sigma_{r_2}}J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_2}}+\int_{\mathcal{R}_{r_1}^{r_2}}K^M\,\mathrm{d}\mu_g=\int_{\Sigma_{r_1}}J^M\cdot n\,\mathrm{d}\mu_{\overline{g}_{r_1}}\,, \tag{2.3.6}$$

which implies (compare the discussion of the inhomogeneous wave equation in Section 2.2.2.2, in particular Lemma 2.12):

$$r_2 \int_{\Sigma_{r_2}} J^M \cdot n \, \mathrm{d}\mu_{\overline{g}_{r_2}} \leq r_1 \int_{\Sigma_{r_1}} J^M \cdot n \, \mathrm{d}\mu_{\overline{g}_{r_1}} \qquad (r_2 > r_1 > r_{\mathcal{C}}) \,. \tag{2.3.7}$$

This is the content of Corollary 2.35.

Local Redshift Effect. In Lemma 2.33 we state the strict positivity of the surface gravity of the cosmological horizons $\bar{\mathcal{C}}^+ \cup \mathcal{C}^+$ in $(\mathcal{M}_\Lambda^{(m)}, g)$. While the global redshift captured above can be attributed to the expansion of the spacetime, a *local* redshift is due to the stated property of the horizons. In Section 2.3.2.2 we apply the general construction provided by [17] to the cosmological horizons, to estimate the right hand side of (2.3.7).

More precisely, we construct in Proposition 2.36 a vectorfield $N = T + Y$ with

$$Y\Big|_{\mathcal{C}^+} = \frac{2}{\frac{\partial r}{\partial v}}\Big|_{v=0} \frac{\partial}{\partial v} \tag{2.3.8}$$

which is timelike and gives rise to a positive current

$$\nabla \cdot J^N \geq 0 \tag{2.3.9}$$

in a neighborhood of $\mathcal{C}^+$ which is of the form

$$\Big\{ r_{\mathcal{C}} \leq r \leq r_0 \Big\} \cap J^+(\Sigma) \tag{2.3.10}$$

where Σ is an achronal hypersurface that crosses $\bar{\mathcal{C}}^+ \cup \mathcal{C}^+$ to the future of $\bar{\mathcal{C}}^+ \cap \mathcal{C}^+$, and r_0 only depends on Λ, m and our choice of Σ.

Let now $r_1 \leq r_0$ in (2.3.7). It follows in particular from (2.3.91) and the energy identity for J^N on (2.3.10) that for the right hand side of (2.3.7) we have

$$r_1 \int_{\Sigma_{r_1} \cap J^+(\Sigma)} J^M \cdot n \, \mathrm{d}\mu_{\overline{g}_{r_1}} \leq$$
$$\leq C(r_0) \int_{\mathcal{C}^+ \cap J^+(\Sigma)} {}^* J^{T+Y} + C(r_0) \int_{\substack{\Sigma \cap \\ \{r_{\mathcal{C}} \leq r \leq r_0\}}} J^N \cdot n \, \mathrm{d}\mu_{\overline{g}} \;. \tag{2.3.11}$$

Given any spacelike hypersurface Σ the energy on the left hand side of (2.3.7) is thus bounded for all r_2 such that $\Sigma_{r_2} \subset J^+(\Sigma)$ by the energy on Σ and the nondegenerate energy with respect to N on the segments of the cosmological horizons lying to the future of Σ.

Static regions. A large part of our analysis of linear waves on de Sitter is preoccupied with proving an integrated local energy estimate in the static regions which allows us to control the nondegenerate energy with the respect to the redshift vectorfield on the cosmological horizon; see Section 2.2.2.3. The corresponding proof for the static regions of Schwarzschild de Sitter is more involved due to the presence of *trapping*; however the required result to close our estimate has already been obtained in [16].

Proposition 2.32 (Prop. 10.3.2, [16])**.** *Let Σ be a spacelike hypersurface with normal n in the static region $r_{\mathcal{H}} \leq r \leq r_{\mathcal{C}}$ of $(\mathcal{M}_\Lambda^{(m)}, g)$ crossing the horizons to the future of the bifurcation spheres, and let ψ be a solution* (2.3.1)*. Then there exists a constant C (only depending on Λ, m and Σ) such that*

$$\int_{\mathcal{C}^+ \cap J^+(\Sigma)} {}^*J^Y[\psi] \leq C \int_\Sigma J^n[\psi] \cdot n \, \mathrm{d}\mu_{\overline{g}} \,. \tag{2.3.12}$$

While Y is constructed in [16] using Eddington Finkelstein coordinates for the static region, it *coincides* with our definition (2.3.8) *on the cosmological horizon*; in fact the definiton of Y in [16] is in the same spirit as our construction of $\overline{Y}_c$ for the static region of de Sitter (see discussion of the redshift vectorfield in Section 2.2.2.3, in particular (2.2.118), (2.2.119) and (2.2.121)). Hence (also in view of the fact that T is globally a Killing vectorfield) we are able to control the energy flux through the cosmological horizon in (2.3.11) in terms of a naturally prescribed energy on an initial spacelike hypersurface Σ.

Future Boundary. The induced metric on the level sets Σ_r in (2.3.2) is in fact given by

$$\overline{g}_r = \Big(\frac{\Lambda r^2}{3} + \frac{2m}{r} - 1\Big) \, \mathrm{d}t^2 + r^2 \, \mathring{\gamma} \,, \tag{2.3.13}$$

and we find

$$\phi \, \mathrm{d}\mu_{\overline{g}_r} = r^2 \, \mathrm{d}t \wedge \mathrm{d}\mu_{\mathring{\gamma}} \,. \tag{2.3.14}$$

In view of the results stated above we are then allowed to take the limit on the left hand side of (2.3.7) to conclude on the finiteness of the following integral:

$$\lim_{r\to\infty} r \int_{\Sigma_r} J^M \cdot n \, \mathrm{d}\mu_{\overline{g}_r} = \int_{\Sigma^+} \Big\{ \frac{3}{\Lambda} \Big(\frac{\partial \psi}{\partial t}\Big)^2 + r^2 \big|\mathring{\nabla} \psi\big|^2 \Big\} \, \mathrm{d}t \wedge \mathrm{d}\mu_{\mathring{\gamma}} \,. \tag{2.3.15}$$

This is the main result of our work.

Theorem 6. *Let Σ^+ be the future boundary of the expanding region in $(\mathcal{M}_\Lambda^{(m)}, g)$ endowed with the volume form $\mathrm{d}t \wedge \mathrm{d}\mu_{\mathring{\gamma}}$ of the standard cylinder, and let $\Sigma \subset J^-(\Sigma^+)$ be a spacelike hypersurface with normal n in the past of Σ^+ (and Σ^+ in the domain of dependence of Σ) crossing the horizons to the future of the bifurcation spheres (see figure 2.5). Then there*

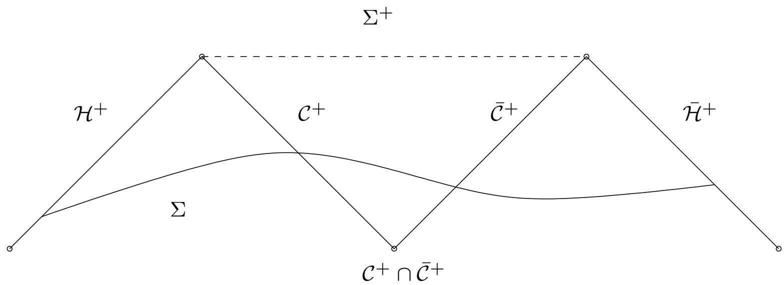

Figure 2.5: Cauchy Problem for Theorem 6.

exists a constant C that only depends on Λ, m and Σ such that for all solutions to (2.3.1) *with initial data prescribed on Σ such that*

$$D[\psi] = \int_\Sigma J^n[\psi] \cdot n \,\mathrm{d}\mu_{\overline{g}} < \infty \,, \tag{2.3.16}$$

we have that the energy is globally bounded in the domain of dependence of Σ and satisfies the bound

$$\int_{\Sigma^+} \Big\{ \big(\frac{\partial\psi}{\partial t}\big)^2 + \big|\overset{\circ}{\nabla}\psi\big|^2 \Big\} \,\mathrm{d}t \wedge \mathrm{d}\mu_{\overset{\circ}{\gamma}} \leq C(\Lambda, m, \Sigma)\, D \tag{2.3.17}$$

on the future boundary Σ^+.

Alternatively we can write the result (2.3.17) using the coercivity equality on the sphere (see Appendix C.2) as

$$\int_{\Sigma^+} \Big\{ \big(T \cdot \psi\big)^2 + \sum_{i=1}^{3} \big(\Omega_{(i)}\psi\big)^2 \Big\} \leq C\, D \,, \tag{2.3.18}$$

and note that the tangent space to Σ_r is spanned by the Killing vectorfields T, and $\Omega_{(i)} : i = 1, 2, 3$. This immediately implies the following pointwise estimates, (similarly to Section 2.2.2.6 in the de Sitter case).

Theorem 7. *Let Σ^+ and Σ be as in Thm. 6. There exists a constant $C(\Lambda, m, \Sigma)$ such that for all solutions to* (2.3.1) *which satisfy in addition to* (2.3.16) *the condition*

$$\begin{aligned} D_c[\psi] = D[\psi] + \sum_{i=1}^{3} D[\Omega_{(i)}\psi] + \sum_{i,j=1}^{3} D[\Omega_{(i)}\Omega_{(j)}\psi] \\ + D[T\psi] + \sum_{i=1}^{3} D[\Omega_{(i)}T\psi] + \sum_{i,j=1}^{3} D[\Omega_{(i)}\Omega_{(j)}T\psi] < \infty \end{aligned} \tag{2.3.19}$$

we have the pointwise estimates

$$\sup_{p,q\in\Sigma^+} \Big| \Big| \big(\frac{\partial\psi}{\partial t}\big)^2(p) - \big(\frac{\partial\psi}{\partial t}\big)^2(q) \Big| + \Big| \big|\overset{\circ}{\nabla}\psi\big|^2(p) - \big|\overset{\circ}{\nabla}\psi\big|^2(q) \Big| \Big| \leq C(\Lambda, m, \Sigma)\, D_c \tag{2.3.20}$$

on the future boundary Σ^+.

We conclude that solutions to the linear wave equation on Schwarzschild de Sitter spacetimes decay in the expanding region to a function on the spacelike future boundary given by a function on $\mathbb{R} \times \mathbb{S}^2$ with bounded derivatives.

2.3.1 Geometry of the expanding region of Schwarzschild de Sitter

In this section we shall discuss the Schwarzschild de Sitter spacetimes as solutions to (2.2.14) with positive $\Lambda > 0$ and $m > 0$. It is here useful to recall our discussion of the global geometry of de Sitter in Section 1.2.

Spherically symmetric cosmological spacetimes with positive mass. Recall that the mass function m on the quotient $\mathcal{Q} = \mathcal{M}/\mathrm{SO}(3)$ of a spherically symmetric solution $(\mathcal{M}, g)$ to (2.2.14) defined by

$$1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} = -\frac{4}{\Omega^2}\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} \tag{2.3.21}$$

is *constant* by virtue of the Hessian equations (2.2.16). The mass m parametrizes precisely the Schwarzschild de Sitter family for any given $\Lambda > 0$ and correspondingly we choose here

$$m > 0\,. \tag{2.3.22}$$

We have also seen that as a consequence of (2.2.16) the function

$$r^* = \int \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}\,\mathrm{d}r \tag{2.3.23}$$

is a solution to (2.2.20). Thus we have that in a given double null coordinate system (u, v) on $\mathcal{Q}$ the "tortoise" coordinate has the form

$$r^* = f(u) + g(v)\,, \tag{2.3.24}$$

where f, g may be any functions on $\mathbb{R}$. The different charts of the manifold $(\mathcal{M}, g)$ are obtained in explicit form for different choices of centering (2.3.23) and suitable choices of f, g in (2.3.24). In particular the charts that cover the *horizons* (where the right hand side of (2.3.21) vanishes) are found in this manner. In contrast however to the case $m = 0$, there is no *single* chart that covers the entire spacetime manifold for $m > 0$. In the following we shall briefly discuss a double null coordinate system that covers the *cosmological horizons* and extends to the adjacent regions, in particular to the future into the *expanding region*.

Extension across the cosmological horizons. The polynomial on the left hand side of (2.3.21) has three distinct real roots in r provided

$$3m < \frac{1}{\sqrt{\Lambda}}\,. \tag{2.3.25}$$

In fact, as discussed in [34], if we set

$$\cos\xi = -3m\sqrt{\Lambda} \qquad (\pi < \xi < \frac{3\pi}{2})\,, \tag{2.3.26}$$

then the three roots are given by

$$r_{\mathcal{H}} = \frac{2}{\sqrt{\Lambda}} \cos \frac{\xi}{3} \tag{2.3.27a}$$

$$r_{\mathcal{C}} = \frac{2}{\sqrt{\Lambda}} \cos \frac{\xi + 4\pi}{3} \tag{2.3.27b}$$

$$\overline{r_{\mathcal{C}}} = \frac{2}{\sqrt{\Lambda}} \cos \frac{\xi + 2\pi}{3}\,, \tag{2.3.27c}$$

and satisfy

$$\overline{r_{\mathcal{C}}} < 0 < 2m < r_{\mathcal{H}} < 3m < r_{\mathcal{C}}\,. \tag{2.3.28}$$

In particular we have

$$r - 2m - \frac{\Lambda r^3}{3} = -\frac{\Lambda}{3}(r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|)\,, \tag{2.3.29}$$

with

$$-|\overline{r_{\mathcal{C}}}| + r_{\mathcal{C}} + r_{\mathcal{H}} = 0 \tag{2.3.30a}$$

$$r_{\mathcal{C}}|\overline{r_{\mathcal{C}}}| + r_{\mathcal{H}}|\overline{r_{\mathcal{C}}}| - r_{\mathcal{H}} r_{\mathcal{C}} = \frac{3}{\Lambda} \tag{2.3.30b}$$

$$r_{\mathcal{H}} r_{\mathcal{C}} |\overline{r_{\mathcal{C}}}| = \frac{6m}{\Lambda}\,, \tag{2.3.30c}$$

and by decomposition into partial fractions

$$\begin{aligned} \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} = \frac{3}{\Lambda} \frac{r_{\mathcal{H}}}{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{H}} + |\overline{r_{\mathcal{C}}}|)} \frac{1}{r - r_{\mathcal{H}}} \\ - \frac{3}{\Lambda} \frac{r_{\mathcal{C}}}{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)} \frac{1}{r - r_{\mathcal{C}}} \\ + \frac{3}{\Lambda} \frac{|\overline{r_{\mathcal{C}}}|}{(|\overline{r_{\mathcal{C}}}| + r_{\mathcal{H}})(|\overline{r_{\mathcal{C}}}| + r_{\mathcal{C}})} \frac{1}{r + |\overline{r_{\mathcal{C}}}|}\,. \end{aligned} \tag{2.3.31}$$

Let (2.3.23) be centred at $r = 3m$,

$$r^*(r) = \int_{3m}^{r} \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}\, \mathrm{d}r\,, \tag{2.3.32}$$

and choose

$$f(u) = -\frac{3}{\Lambda} \frac{r_{\mathcal{C}}}{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)} \log \frac{|u|}{A} \tag{2.3.33a}$$

$$g(v) = -\frac{3}{\Lambda} \frac{r_{\mathcal{C}}}{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)} \log \frac{|v|}{A} \tag{2.3.33b}$$

in (2.3.24) with

$$A^2 = (r_{\mathcal{C}} - 3m)(3m - r_{\mathcal{H}})^{-\frac{r_{\mathcal{H}}}{r_{\mathcal{C}}} \frac{r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|}{r_{\mathcal{H}} + |\overline{r_{\mathcal{C}}}|}} (3m + |\overline{r_{\mathcal{C}}}|)^{-\frac{|\overline{r_{\mathcal{C}}}|}{r_{\mathcal{C}}} \frac{r_{\mathcal{C}} - r_{\mathcal{H}}}{|\overline{r_{\mathcal{C}}}| + r_{\mathcal{H}}}}\,. \tag{2.3.34}$$

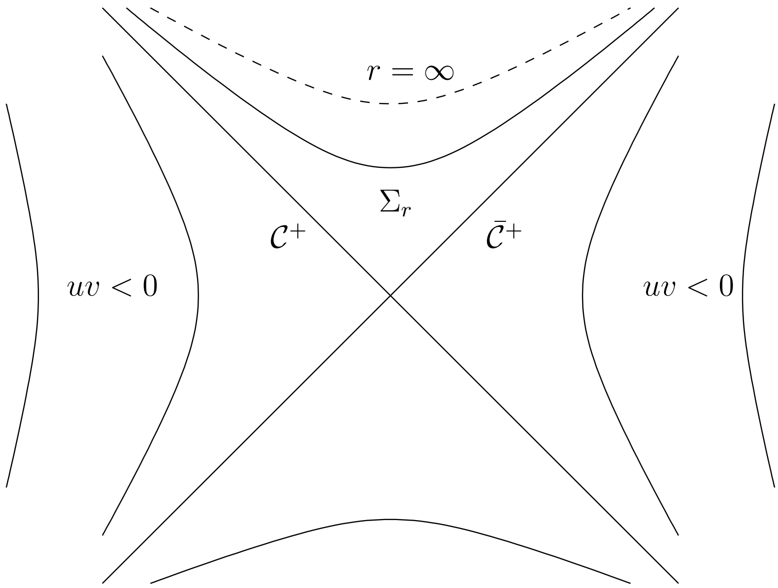

Figure 2.6: Causal geometry of the spacetime $(\mathcal{M}_\Lambda^{(m)}, g)$ for $3m\sqrt{\Lambda} < 1$ in the domain $r_{\mathcal{H}} < r < \infty$.

Then integrating (2.3.32) using (2.3.31) we obtain by (2.3.24) in view of (2.3.33) the following relation between the chosen null coordinates (u, v) and the radius function r:

$$uv = \frac{r - r_{\mathcal{C}}}{(r - r_{\mathcal{H}})^{\frac{r_{\mathcal{H}}}{r_{\mathcal{C}}}\frac{r_{\mathcal{C}}+|\overline{r_{\mathcal{C}}}|}{r_{\mathcal{H}}+|\overline{r_{\mathcal{C}}}|}}(r + |\overline{r_{\mathcal{C}}}|)^{\frac{|\overline{r_{\mathcal{C}}}|}{r_{\mathcal{C}}}\frac{r_{\mathcal{C}}-r_{\mathcal{H}}}{|\overline{r_{\mathcal{C}}}|+r_{\mathcal{H}}}}} \,. \tag{2.3.35}$$

Note that the null hypersurfaces $u = 0$ and $v = 0$ are the cosmological horizons $r = r_{\mathcal{C}}$, while the surfaces of constant $r \neq r_{\mathcal{C}}$ are spacelike hyperbolas in the uv-plane for $r > r_{\mathcal{C}}$ and timelike hyperbolas for $r < r_{\mathcal{C}}$. Moreover the future timelike boundary $r = \infty$ is identified with the spacelike hyperbola $uv = 1$. (See figure 2.6.) Let us also denote by

$$\mathcal{C}^+ = \Big\{(u, 0) : u \geq 0\Big\} \tag{2.3.36a}$$
$$\bar{\mathcal{C}}^+ = \Big\{(0, v) : v \geq 0\Big\} \tag{2.3.36b}$$

the two components of the past boundary of the expanding region $r > r_{\mathcal{C}}$. Since

$$\frac{\partial r^*}{\partial u} = \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}\frac{\partial r}{\partial u} = -\frac{3}{\Lambda}\frac{r_{\mathcal{C}}}{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)}\frac{1}{u} \tag{2.3.37a}$$
$$\frac{\partial r^*}{\partial v} = \frac{1}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}}\frac{\partial r}{\partial v} = -\frac{3}{\Lambda}\frac{r_{\mathcal{C}}}{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)}\frac{1}{v} \tag{2.3.37b}$$

we can solve (2.3.21) for Ω^2 to obtain:

$$\Omega^2 = \frac{4}{r}\frac{3}{\Lambda}\frac{r_{\mathcal{C}}^2}{(r_{\mathcal{C}} - r_{\mathcal{H}})^2(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)^2}\times$$
$$\times \Big(r - r_{\mathcal{H}}\Big)^{1+\frac{r_{\mathcal{H}}}{r_{\mathcal{C}}}\frac{r_{\mathcal{C}}+|\overline{r_{\mathcal{C}}}|}{r_{\mathcal{H}}+|\overline{r_{\mathcal{C}}}|}}\Big(r + |\overline{r_{\mathcal{C}}}|\Big)^{1+\frac{|\overline{r_{\mathcal{C}}}|}{r_{\mathcal{C}}}\frac{r_{\mathcal{C}}-r_{\mathcal{H}}}{|\overline{r_{\mathcal{C}}}|+r_{\mathcal{H}}}} \tag{2.3.38}$$

The metric g on the chart that covers the region $r_{\mathcal{H}} < r < \infty$ and extends across the cosmological horizon $r = r_{\mathcal{C}}$ thus takes in double null coordinates (u, v) the form

$$g = -\frac{4}{r}\frac{3}{\Lambda}\frac{r_{\mathcal{C}}^2}{(r_{\mathcal{C}} - r_{\mathcal{H}})^2 (r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)^2} \times \\ \times \bigl(r - r_{\mathcal{H}}\bigr)^{1 + \frac{r_{\mathcal{H}}}{r_{\mathcal{C}}}\frac{r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|}{r_{\mathcal{H}} + |\overline{r_{\mathcal{C}}}|}} \bigl(r + |\overline{r_{\mathcal{C}}}|\bigr)^{1 + \frac{|\overline{r_{\mathcal{C}}}|}{r_{\mathcal{C}}}\frac{r_{\mathcal{C}} - r_{\mathcal{H}}}{|\overline{r_{\mathcal{C}}}| + r_{\mathcal{H}}}} \mathrm{d}u\,\mathrm{d}v + r^2 \overset{\circ}{\gamma} \tag{2.3.39}$$

where r is a function of (u, v) implicity given by (2.3.35).

Expanding Region. In the domain $uv > 0$ it is convenient to introduce the coordinate

$$t = -\frac{3}{\Lambda}\frac{r_{\mathcal{C}}}{(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)(r_{\mathcal{C}} - r_{\mathcal{H}})} \log\Bigl|\frac{v}{u}\Bigr| \,. \tag{2.3.40}$$

The metric can then alternatively be expressed in (t, r) coordinates as

$$g = -\frac{1}{\frac{\Lambda r^2}{3} + \frac{2m}{r} - 1}\mathrm{d}r^2 + \Bigl(\frac{\Lambda r^2}{3} + \frac{2m}{r} - 1\Bigr)\mathrm{d}t^2 + r^2 \overset{\circ}{\gamma} \,. \tag{2.3.41}$$

From the expressions for the differentials that follow from (2.3.35) and (2.3.40) we can also conclude that

$$\frac{\partial}{\partial t} = \frac{1}{2}\frac{\Lambda}{3}\frac{(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)(r_{\mathcal{C}} - r_{\mathcal{H}})}{r_{\mathcal{C}}}\Bigl(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\Bigr) \,, \tag{2.3.42}$$

and

$$\frac{\partial r}{\partial u} = \frac{1}{4}\frac{\Lambda}{3}\frac{(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)(r_{\mathcal{C}} - r_{\mathcal{H}})}{r_{\mathcal{C}}}\Omega^2\, v \tag{2.3.43a}$$

$$\frac{\partial r}{\partial v} = \frac{1}{4}\frac{\Lambda}{3}\frac{(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)(r_{\mathcal{C}} - r_{\mathcal{H}})}{r_{\mathcal{C}}}\Omega^2\, u \,. \tag{2.3.43b}$$

We shall think of (2.3.41) as the decomposition of the metric g in the expanding region $uv > 0$ with respect to the level sets of the *time function* r. Indeed, if we set

$$V^\mu = -g^{\mu\nu}\partial_\nu r \,, \tag{2.3.44}$$

or in other words

$$V = \frac{1}{2}\frac{\Lambda}{3}\frac{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)}{r_{\mathcal{C}}}\Bigl(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\Bigr) \,, \tag{2.3.45}$$

then we find for the lapse function

$$\phi \doteq \frac{1}{\sqrt{-g(V, V)}} = \frac{1}{\sqrt{\frac{\Lambda r^2}{3} + \frac{2m}{r} - 1}} \tag{2.3.46}$$

and the metric takes the form (2.3.41), namely

$$g = -\phi^2\,\mathrm{d}r^2 + \overline{g}_r \,. \tag{2.3.47}$$

Moreover, the normal to the level sets

$$\Sigma_r = \Bigl\{(u, v) : (2.3.35)\Bigr\} \qquad (r > r_{\mathcal{C}}) \tag{2.3.48}$$

is given by

$$n = \phi V \,, \tag{2.3.49}$$

and $\partial_r = \phi^2 V$.

Cosmological Horizon. The vectorfield (2.3.42) extends to a global vectorfield that characterizes the cosmological horizon as a Killing horizon with positive surface gravity.

Lemma 2.33 (Positive surface gravity of the cosmological horizons)**.** *The vectorfield*

$$T \doteq \kappa_{\mathcal{C}}\Big(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\Big) \tag{2.3.50}$$

is globally Killing, i.e.

$$^{(T)}\pi \doteq \frac{1}{2}\mathcal{L}_T g = 0\,, \tag{2.3.51}$$

and satisfies

$$\nabla_T T = \kappa_{\mathcal{C}} T \qquad \text{on } \mathcal{C}^+\,, \tag{2.3.52}$$

where

$$\kappa_{\mathcal{C}} = \frac{1}{2}\frac{\Lambda}{3}\frac{(r_{\mathcal{C}} - r_{\mathcal{H}})(r_{\mathcal{C}} + |\overline{r_{\mathcal{C}}}|)}{r_{\mathcal{C}}} > 0 \tag{2.3.53}$$

is the surface gravity of the cosmological horizons.

The result is obtained with the help of the vectorfield

$$Y\Big|_{\mathcal{C}^+} = \frac{2}{\frac{\partial r}{\partial v}}\frac{\partial}{\partial v} \tag{2.3.54}$$

which is conjugate to T along the horizon:

$$g(T,Y)\Big|_{\mathcal{C}^+} = -2\,. \tag{2.3.55}$$

Indeed, by (2.3.51),

$$\begin{aligned} g(\nabla_T T, Y)\Big|_{\mathcal{C}^+} = -g(\nabla_Y T, T)\Big|_{\mathcal{C}^+} &= -\frac{1}{2}Y\cdot g(T,T)\Big|_{\mathcal{C}^+} \\ &= \frac{\mathrm{d}}{\mathrm{d}r}\Big(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\Big)\Big|_{r=r_{\mathcal{C}}}\,. \end{aligned} \tag{2.3.56}$$

Alternatively $\kappa_{\mathcal{C}}$ is characterized by

$$\nabla_Y T = \nabla_T Y = -\kappa_{\mathcal{C}} Y \qquad \text{on } \mathcal{C}^+\,; \tag{2.3.57}$$

note that this in particular implies that Y is Lie transported by T along the horizon:

$$[T,Y]\Big|_{\mathcal{C}^+} = 0\,. \tag{2.3.58}$$

2.3.2 Energy estimates

We develop here the energy estimates for the linear wave equation (2.3.1) on Schwarzschild de Sitter. We study the expanding region in Section 2.3.2.1, and the cosmological horizons in Section 2.3.2.2, while the static region is already dealt with in [16].

2.3.2.1 Energy estimate in the expanding region

It is the purpose of this section to show that the vectorfield

$$M = \frac{1}{r}\frac{\partial}{\partial r} \tag{2.3.59}$$

captures the global redshift effect in the expanding region of Schwarzschild de Sitter.

Proposition 2.34. *Let J^M be the current associated to the multiplier* (2.3.59), *and n the normal to Σ_r $(r > r_{\mathcal{C}})$. Then for any solution ψ to the wave equation* (2.3.1) *we have*

$$\phi\, \nabla \cdot J^M[\psi] \geq \frac{1}{r}\, J^M[\psi] \cdot n\,. \tag{2.3.60}$$

Proof. It is equivalent to establish

$$K^M \doteq {}^{(M)}\pi^{\mu\nu} T_{\mu\nu}[\psi] \geq \frac{1}{r^2}\frac{1}{\phi^2} T_{rr}[\psi]\,. \tag{2.3.61}$$

Recall from (2.3.41) that

$$g_{rr} = -\frac{3}{\Lambda}\frac{r}{(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)} \tag{2.3.62a}$$

$$g_{tt} = \frac{1}{r}\frac{\Lambda}{3}(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|) \tag{2.3.62b}$$

$$g_{AB} = r^2\, \mathring{\gamma}_{AB}\,, \tag{2.3.62c}$$

and thus the non-vanishing connection coefficients are:

$$\begin{aligned}\Gamma^r_{rr} &= \frac{1}{2}(g^{-1})^{rr}\partial_r g_{rr}\\ &= \frac{1}{2}\frac{1}{r}\frac{r_{\mathcal{H}}}{(r_{\mathcal{C}}-r_{\mathcal{H}})(r_{\mathcal{H}}+|\overline{r_{\mathcal{C}}}|)}\frac{(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)}{r-r_{\mathcal{H}}}\\ &\quad -\frac{1}{2}\frac{1}{r}\frac{r_{\mathcal{C}}}{(r_{\mathcal{C}}-r_{\mathcal{H}})(r_{\mathcal{C}}+|\overline{r_{\mathcal{C}}}|)}\frac{(r-r_{\mathcal{H}})(r+|\overline{r_{\mathcal{C}}}|)}{r-r_{\mathcal{C}}}\\ &\quad +\frac{1}{2}\frac{1}{r}\frac{|\overline{r_{\mathcal{C}}}|}{(|\overline{r_{\mathcal{C}}}|+r_{\mathcal{H}})(|\overline{r_{\mathcal{C}}}|+r_{\mathcal{C}})}\frac{(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})}{r+r_{\mathcal{H}}}\end{aligned} \tag{2.3.63a}$$

$$\Gamma^t_{rt} = \frac{1}{2}(g^{-1})^{tt}\partial_r g_{tt} = \frac{1}{2}\Big[-\frac{1}{r}+\frac{1}{r-r_{\mathcal{H}}}+\frac{1}{r-r_{\mathcal{C}}}+\frac{1}{r+|\overline{r_{\mathcal{C}}}|}\Big]\,, \tag{2.3.63b}$$

and Γ^r_{tt} as well as

$$\Gamma^r_{AB} = \frac{\Lambda}{3}(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)\,\mathring{\gamma}_{AB} \tag{2.3.64a}$$

$$\Gamma^B_{rA} = \frac{1}{r}\delta^B_A\,. \tag{2.3.64b}$$

Let us first consider

$$M' = \frac{\partial}{\partial r}\,. \tag{2.3.65}$$

We have

$$^{(M')}\pi^{rr} = (g^{-1})^{rr}\Gamma^{r}_{rr} \qquad ^{(M')}\pi^{tt} = (g^{-1})^{tt}\Gamma^{t}_{rt} \tag{2.3.66a}$$

$$^{(M')}\pi^{AB} = \frac{1}{r}(g^{-1})^{AB}\,, \tag{2.3.66b}$$

and

$$K^{M'} = {}^{(M')}\pi^{rr}T_{rr} + {}^{(M')}\pi^{tt}T_{tt} + {}^{(M')}\pi^{AB}T_{AB}\,. \tag{2.3.67}$$

Now,

$$T_{rr} = \frac{1}{2}\big(\frac{\partial\psi}{\partial r}\big)^2 + \frac{1}{2}\big(\frac{3}{\Lambda}\big)^2\frac{r^2}{(r-r_{\mathcal{H}})^2(r-r_{\mathcal{C}})^2(r+|\overline{r_{\mathcal{C}}}|)^2}\big(\frac{\partial\psi}{\partial t}\big)^2 + \frac{1}{2}\frac{3}{\Lambda}\frac{r}{(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)}\big|\nabla\!\!\!/\psi\big|^2, \tag{2.3.68}$$

$$T_{tt} = \frac{1}{2}\big(\frac{\partial\psi}{\partial t}\big)^2 + \frac{1}{2}\frac{1}{r^2}\big(\frac{\Lambda}{3}\big)^2(r-r_{\mathcal{H}})^2(r-r_{\mathcal{C}})^2(r+|\overline{r_{\mathcal{C}}}|)^2\big(\frac{\partial\psi}{\partial r}\big)^2 - \frac{1}{2}\frac{1}{r}\frac{\Lambda}{3}(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)\big|\nabla\!\!\!/\psi\big|^2, \tag{2.3.69}$$

and

$$g^{AB}T_{AB} = \frac{\Lambda}{3}\frac{1}{r}(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)\big(\frac{\partial\psi}{\partial r}\big)^2 - \frac{3}{\Lambda}\frac{r}{(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)}\big(\frac{\partial\psi}{\partial t}\big)^2; \tag{2.3.70}$$

also note that

$$\frac{1}{\phi^2}T_{rr} = \frac{1}{2}\frac{1}{r}\frac{\Lambda}{3}(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)\big(\frac{\partial\psi}{\partial r}\big)^2 + \frac{1}{2}\frac{3}{\Lambda}\frac{r}{(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)}\big(\frac{\partial\psi}{\partial t}\big)^2 + \frac{1}{2}\big|\nabla\!\!\!/\psi\big|^2 \doteq \frac{1}{2}L_r\big(\frac{\partial\psi}{\partial r}\big)^2 + \frac{1}{2}L_t\big(\frac{\partial\psi}{\partial t}\big)^2 + \frac{1}{2}\big|\nabla\!\!\!/\psi\big|^2. \tag{2.3.71}$$

We then find that

$$K^{M'} = \frac{1}{2}\Big[K_0 + K_1 + K_2\Big]L_r\big(\frac{\partial\psi}{\partial r}\big)^2 + \frac{1}{2}\Big[K_0 + K_1 - K_2\Big]L_t\big(\frac{\partial\psi}{\partial t}\big)^2 + \frac{1}{2}\Big[K_0 - K_1\Big]\big|\nabla\!\!\!/\psi\big|^2, \tag{2.3.72}$$

where

$$K_0 = -\frac{1}{2}\frac{1}{r}\frac{r_{\mathcal{H}}}{(r_{\mathcal{C}}-r_{\mathcal{H}})(r_{\mathcal{H}}+|\overline{r_{\mathcal{C}}}|)}\frac{(r-r_{\mathcal{C}})(r+|\overline{r_{\mathcal{C}}}|)}{r-r_{\mathcal{H}}} + \frac{1}{2}\frac{1}{r}\frac{r_{\mathcal{C}}}{(r_{\mathcal{C}}-r_{\mathcal{H}})(r_{\mathcal{C}}+|\overline{r_{\mathcal{C}}}|)}\frac{(r-r_{\mathcal{H}})(r+|\overline{r_{\mathcal{C}}}|)}{r-r_{\mathcal{C}}} - \frac{1}{2}\frac{1}{r}\frac{|\overline{r_{\mathcal{C}}}|}{(|\overline{r_{\mathcal{C}}}|+r_{\mathcal{H}})(|\overline{r_{\mathcal{C}}}|+r_{\mathcal{C}})}\frac{(r-r_{\mathcal{H}})(r-r_{\mathcal{C}})}{r+|\overline{r_{\mathcal{C}}}|}$$

$$= -\frac{1}{2}\Big(\frac{\Lambda r^2}{3} + \frac{2m}{r} - 1\Big)\frac{\partial}{\partial r}\frac{1}{\frac{\Lambda r^2}{3} + \frac{2m}{r} - 1}$$

$$= \frac{1}{2}\frac{3}{\Lambda}\frac{2\frac{\Lambda r^2}{3} - \frac{2m}{r}}{(r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|)}\,, \quad (2.3.73)$$

and

$$K_1 = \frac{1}{2}\frac{1}{r}\Big(-1 + \frac{r}{r - r_{\mathcal{H}}} + \frac{r}{r - r_{\mathcal{C}}} + \frac{r}{r + |\overline{r_{\mathcal{C}}}|}\Big) \quad (2.3.74)$$

and finally

$$K_2 = 2\frac{1}{r}\,. \quad (2.3.75)$$

Next we observe that

$$\begin{aligned} K_0 - K_1 = \frac{1}{2}\frac{1}{r}\frac{1}{(r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|)}\times \\ \times\Big[2r^3 - \frac{6m}{\Lambda} + (r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|) \\ - r\big(r^2 + |\overline{r_{\mathcal{C}}}|r - r_{\mathcal{C}}r - r_{\mathcal{C}}|\overline{r_{\mathcal{C}}}|\big) \\ - r\big(r^2 + |\overline{r_{\mathcal{C}}}|r - r_{\mathcal{H}}r - r_{\mathcal{H}}|\overline{r_{\mathcal{C}}}|\big) \\ - r\big(r^2 - r_{\mathcal{C}}r - r_{\mathcal{H}}r + r_{\mathcal{H}}r_{\mathcal{C}}\big)\Big] = 0 \quad (2.3.76) \end{aligned}$$

by (2.3.30), and thus

$$\begin{aligned} K_0 + K_1 - K_2 = 2K_1 - K_2 = \frac{1}{r}\Big(-3 + \frac{r}{r - r_{\mathcal{H}}} + \frac{r}{r - r_{\mathcal{C}}} + \frac{r}{r + |\overline{r_{\mathcal{C}}}|}\Big) \\ = \frac{1}{r}\frac{1}{(r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|)}\Big[2r_{\mathcal{H}}|\overline{r_{\mathcal{C}}}|r + 2r_{\mathcal{C}}|\overline{r_{\mathcal{C}}}|r - 2r_{\mathcal{H}}r_{\mathcal{C}}r - 3r_{\mathcal{C}}|\overline{r_{\mathcal{C}}}|r_{\mathcal{H}}\Big] \\ = \frac{3}{\Lambda}\frac{2}{r}\frac{1}{(r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|)}\Big[r - 3m\Big] \geq \frac{3}{\Lambda}\frac{2}{r}\frac{1}{(r - r_{\mathcal{H}})(r + |\overline{r_{\mathcal{C}}}|)}\,, \quad (2.3.77) \end{aligned}$$

again using the relations (2.3.30); and finally

$$K_0 + K_1 + K_2 \geq 2K_2 = 4\frac{1}{r}\,. \quad (2.3.78)$$

We have shown in particular

$$K^{M'} \geq 0\,. \quad (2.3.79)$$

Since

$$^{(M)}\pi_{rr} = -\frac{1}{r^2}\,g_{rr} + \frac{1}{r}\,^{(M')}\pi_{rr} \quad (2.3.80a)$$

$$^{(M)}\pi_{tt} = \frac{1}{r}\,^{(M')}\pi_{tt} \quad (2.3.80b)$$

$$^{(M)}\pi_{AB} = \frac{1}{r}\,^{(M')}\pi_{AB}\,, \quad (2.3.80c)$$

we conclude

$$\begin{aligned} K^M = -\frac{1}{r^2}(g^{-1})^{rr}T_{rr} + \frac{1}{r}K^{M'} \\ = \frac{\Lambda}{3}\frac{1}{r^3}(r - r_{\mathcal{H}})(r - r_{\mathcal{C}})(r + |\overline{r_{\mathcal{C}}}|)\,T_{rr} + \frac{1}{r}K^{M'} \geq \frac{1}{r^2}\frac{1}{\phi^2}T_{rr}\,. \quad (2.3.81) \end{aligned}$$

□

As a immediate consequence of the energy identity for J^M we obtain:

Corollary 2.35. *Let ψ be a solution to* (2.3.1) *then for all $r_2 > r_1 > r_{\mathcal{C}}$ we have*

$$\int_{\Sigma_{r_2}} \phi\Big\{\phi^2\big(\frac{\partial\psi}{\partial t}\big)^2 + \big|\nabla\!\!\!/\psi\big|^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_2}} \leq \\ \leq \int_{\Sigma_{r_1}} \phi\Big\{\frac{1}{\phi^2}\big(\frac{\partial\psi}{\partial r}\big)^2 + \phi^2\big(\frac{\partial\psi}{\partial t}\big)^2 + \big|\nabla\!\!\!/\psi\big|^2\Big\}\,\mathrm{d}\mu_{\overline{g}_{r_1}}\,. \tag{2.3.82}$$

Proof. Note here that

$$J^M\cdot n = \frac{1}{r}\frac{1}{\phi}T_{rr} = \frac{1}{2}\frac{1}{r^2}\phi\Big(\frac{\Lambda r^3}{3}+2m-r\Big)\big(\frac{\partial\psi}{\partial r}\big)^2 \\ +\frac{1}{2}\phi\frac{1}{\frac{\Lambda r^3}{3}+2m-r}\big(\frac{\partial\psi}{\partial t}\big)^2 + \frac{1}{2}\frac{1}{r}\phi\big|\nabla\!\!\!/\psi\big|^2\,. \tag{2.3.83}$$

□

2.3.2.2 Redshift vectorfield on the cosmological horizon

In this Section we construct a vectorfield that captures the *local* redshift effect of the cosmological horizon. As in Section 2.2.2.5 we follow [17], however using the double null coordinates introduced in Section 2.3.1.

Extension of the Null Frame. Recall the null frame $(T, Y, E_A : A = 1,2)$ from our discussion of the cosmological horizon in Section 2.3.1. We have seen that, also using (2.3.43),

$$T\Big|_{\mathcal{C}^+} = \kappa_{\mathcal{C}}\,u\frac{\partial}{\partial u}\,, \tag{2.3.84a}$$

$$Y\Big|_{\mathcal{C}^+} = \frac{2}{\frac{\partial r}{\partial v}\big|_{v=0}}\frac{\partial}{\partial v} = \iota_{\mathcal{C}}\frac{1}{u}\frac{\partial}{\partial v}\,, \qquad \iota_{\mathcal{C}} \doteq \frac{4}{\kappa_{\mathcal{C}}}\frac{1}{\Omega^2}\Big|_{\mathcal{C}^+}\,, \tag{2.3.84b}$$

are conjugate null vectors on the cosmological horizon

$$g(T,Y)\Big|_{\mathcal{C}^+} = \iota_{\mathcal{C}}\kappa_{\mathcal{C}}\,g_{uv} = -2\,, \tag{2.3.85a}$$

$$g(T,E_A)\Big|_{\mathcal{C}^+} = 0 \qquad g(Y,E_A)\Big|_{\mathcal{C}^+} = 0\,, \tag{2.3.85b}$$

and satisfy the commutation relation (2.3.58). Here T is globally given by (2.3.50). Let us now *extend* the vectorfield Y away from the horizon by

$$\nabla_Y Y = -\sigma(Y+T)\,, \tag{2.3.86}$$

where $\sigma > 0$. It is well known from [17] that this extension gives rise to a positive current by virtue of Lemma 2.33. Indeed, as in Section 2.2.2.5, we find using (2.3.57) and (2.3.86) that

$$K^Y\Big|_{\mathcal{C}^+} \doteq {}^{(Y)}\pi^{\mu\nu}T_{\mu\nu}\Big|_{\mathcal{C}^+} = \frac{1}{2}\sigma\bigl(T\cdot\psi\bigr)^2 + \frac{1}{2}\kappa_{\mathcal{C}}\bigl(Y\cdot\psi\bigr)^2 + \frac{1}{2}\sigma\bigl|\nabla\!\!\!\!/\,\psi\bigr|^2 + \frac{2}{r}\bigl(T\cdot\psi\bigr)\bigl(Y\cdot\psi\bigr)\,, \quad (2.3.87)$$

and thus

$$K^Y\Big|_{\mathcal{C}^+} \geq \frac{1}{4}\kappa_{\mathcal{C}}\bigl(Y\cdot\psi\bigr)^2 + \Bigl[\frac{1}{2}\sigma - \bigl(\frac{2}{r_{\mathcal{C}}}\bigr)^2\frac{1}{\kappa_{\mathcal{C}}}\Bigr]\bigl(T\cdot\psi\bigr)^2 + \frac{1}{2}\sigma\bigl|\nabla\!\!\!\!/\,\psi\bigr|^2 \geq 0\,, \quad (2.3.88)$$

if $\sigma > (2/r_{\mathcal{C}})^2 2/\kappa_{\mathcal{C}}$. By construction Y is Lie transported by T,

$$[T, Y] = 0\,. \quad (2.3.89)$$

However, it remains to show that $T + Y$ is *timelike* in the vicinity of the cosmological horizon.

Proposition 2.36. *Let $N = T + Y$ with Y as constructed above, $M = \frac{\partial}{\partial r}$ and let ψ be a solution* (2.3.1)*. There exists $r_0 > r_{\mathcal{C}}$ (for any given $u_0 > 0$, only depending on Λ and m) such that*

$$K^N[\psi] \geq 0 \qquad \text{on } r_{\mathcal{C}} \leq r \leq r_0 \text{ (and } u \geq u_0)\,, \quad (2.3.90)$$

and there exists a constant C that only depends on r_0 such that

$$J^M[\psi]\cdot n \leq C(r_0)\, J^N[\psi]\cdot n \qquad \text{on } r = r_0 \text{ (and } u \geq u_0)\,. \quad (2.3.91)$$

Proof. In view of (2.3.88) and (2.3.51), by continuity

$$K^N = K^Y \geq 0 \quad (2.3.92)$$

in a neighborhood of $v = 0$, $u \geq u_0$; since Y is invariant under the flow of T, the positivity property is preserved uniformly in $r_{\mathcal{C}} \leq r \leq r_0$, for $r_0 > r_{\mathcal{C}}$ chosen small enough, as stated in (2.3.90).

Next we shall find an explicit expression for the extension of N to verify that N remains timelike in such a domain. The condition (2.3.86) on $\mathcal{C}^+$ is equivalent to the ordinary differential equations

$$\frac{\partial Y^u}{\partial v}\Big|_{v=0} = -\sigma\frac{\kappa_{\mathcal{C}}}{\iota_{\mathcal{C}}}u^2 \quad (2.3.93a)$$

$$\frac{\partial Y^v}{\partial v}\Big|_{v=0} = -\sigma - \frac{2}{\Omega^2}\frac{\partial\Omega^2}{\partial r}\Big|_{r=r_{\mathcal{C}}}\,. \quad (2.3.93b)$$

Since, by differentiating (2.3.38),

$$\frac{1}{\Omega^2}\frac{\partial\Omega^2}{\partial r}\Big|_{r=r_{\mathcal{C}}} = \frac{\Lambda}{3}\frac{1}{r_{\mathcal{C}}^2}\frac{1}{\kappa_{\mathcal{C}}}\Bigl(r_{\mathcal{C}}^2 + r_{\mathcal{H}}|\overline{r_{\mathcal{C}}}|\Bigr)\,, \quad (2.3.94)$$

we obtain with say

$$\sigma = k\Big(\frac{2}{r_{\mathcal{C}}}\Big)^2\frac{2}{\kappa_{\mathcal{C}}}\,,\quad k\geq 1\,, \tag{2.3.95}$$

furthermore that

$$\frac{\partial Y^u}{\partial v}\Big|_{v=0} = -k\Big(\frac{2}{r_{\mathcal{C}}}\Big)^3 u^2 \tag{2.3.96a}$$

$$\frac{\partial Y^v}{\partial v}\Big|_{v=0} = -\Big(\frac{2}{r_{\mathcal{C}}}\Big)^2\frac{2}{\kappa_{\mathcal{C}}}\Big[k+\frac{1}{4}\frac{\Lambda}{3}\big(r_{\mathcal{C}}^2+r_{\mathcal{H}}|\overline{r_{\mathcal{C}}}|\big)\Big]\,, \tag{2.3.96b}$$

and thus

$$\begin{aligned} N = T + Y = \kappa_{\mathcal{C}}\Big[1-\frac{k}{\kappa_{\mathcal{C}}}\Big(\frac{2}{r_{\mathcal{C}}}\Big)^3 uv\Big]u\frac{\partial}{\partial u} \\ +\iota_{\mathcal{C}}\Big[\frac{1}{uv}-\frac{\kappa_{\mathcal{C}}}{\iota_{\mathcal{C}}}-\frac{1}{\iota_{\mathcal{C}}}\Big(\frac{2}{r_{\mathcal{C}}}\Big)^2\frac{2}{\kappa_{\mathcal{C}}}\Big[k+\frac{1}{4}\frac{\Lambda}{3}\big(r_{\mathcal{C}}^2+r_{\mathcal{H}}|\overline{r_{\mathcal{C}}}|\big)\Big]\Big]v\frac{\partial}{\partial v}+\mathcal{O}(v^2)\,, \end{aligned} \tag{2.3.97}$$

which is *timelike* for uv small enough, i.e. in view of (2.3.35) on $r\leq r_0$ for $r_0(k,\Lambda,m)$ chosen small enough.

Let now r_0 be chosen according to the two smallness conditions above, so that $K^Y\geq 0$ and N timelike on $r_{\mathcal{C}}\leq r\leq r_0$.

Recall (2.3.49), namely that the normal to Σ_r is given by $n=\phi V$ where

$$V = \kappa_{\mathcal{C}}\Big(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\Big)\,. \tag{2.3.98}$$

Since also here

$$M = \frac{\partial}{\partial r} = \phi^2 V\,, \tag{2.3.99}$$

we obtain on one hand

$$\begin{aligned} J^M\cdot n = T(M,n) = \phi^3\,T(V,V) = \\ = \frac{1}{2\kappa_{\mathcal{C}}}\phi\Big\{\phi^2\Big(\frac{\partial\psi}{\partial t}\Big)^2+\frac{1}{\phi^2}\Big(\frac{\partial\psi}{\partial r}\Big)^2+\big|\nabla\!\!\!/\psi\big|^2\Big\}\,. \end{aligned} \tag{2.3.100}$$

On the other hand, we can assume by (2.3.97) that on $r=r_0$ ($u\geq u_0$) N has the form

$$N = \kappa_{\mathcal{C}}\alpha\,u\frac{\partial}{\partial u}+\iota_{\mathcal{C}}\beta\,v\frac{\partial}{\partial v}\,, \tag{2.3.101}$$

where $0<\alpha\leq 1$, $\beta\geq 1$ are constants on Σ_{r_0}. Therefore

$$\begin{aligned} J^N\cdot n = T(N,n) = \alpha\phi\kappa_{\mathcal{C}}^2\,u^2 T_{uu}+\phi\big(\alpha\kappa_{\mathcal{C}}+\beta\iota_{\mathcal{C}}\big)\kappa_{\mathcal{C}}\,uv\,T_{uv}+\beta\phi\iota_{\mathcal{C}}\kappa_{\mathcal{C}}\,v^2 T_{vv} \\ \geq \frac{1}{2\kappa_{\mathcal{C}}}\frac{1}{\phi}\min\big\{\alpha\kappa_{\mathcal{C}},\beta\iota_{\mathcal{C}}\big\}\Big\{\phi^2\Big(\frac{\partial\psi}{\partial t}\Big)^2+\frac{1}{\phi^2}\Big(\frac{\partial\psi}{\partial r}\Big)^2+\big|\nabla\!\!\!/\psi\big|^2\Big\}\,, \end{aligned} \tag{2.3.102}$$

which yields

$$J^M\cdot n\leq C(r_0)\,J^N\cdot n\,, \tag{2.3.103}$$

as desired, with

$$C(r_0) = \frac{1}{\min\big\{\alpha\kappa_{\mathcal{C}},\beta\iota_{\mathcal{C}}\big\}}\phi^2(r_0)\,.$$

□

Appendix A

Improved interior decay of higher order energy for the wave equation on $3+1$-dimensional Minkowski space

In this Appendix we prepare the argument of Section 1.5.3 in the simpler framework of Minkowski space. Here of course, much more is known than is proven in this Appendix. However, the discussion that follows has implications for the study of the wave equation on perturbations of Minkowski space.

In this Appendix we give as an exercise a proof of "improved" interior decay of the first order energy for the wave equation on 3+1 dimensional Minkowski space. More precisely, given a solution ϕ of the wave equation $\Box\phi = 0$ with finite inital higher order energy $D < \infty$, we show that

$$\int_{\Sigma_\tau \cap \{r \leq R\}} (J^T(T \cdot \phi), n) \leq \frac{C(\delta, R)}{\tau^{4-\delta}} D \tag{A.1}$$

for any $\delta > 0$, where C is a constant, and Σ_τ is the hypersurface coinciding with $t = R+2\tau$ for $r \leq R$ and with the corresponding outgoing null hypersurface for $r \geq R$. We follow the new physical-space approach of [22], and introduce in addition a commutation with the vectorfield ∂_v.

The wave equation on Minkowski space

Minkowski space. The background spacetime $(\mathcal{M}, g)$ is here the flat 3+1 dimensional Minkowski space.

$$g = -\,\mathrm{d}t^2 + \,\mathrm{d}r^2 + r^2 \,\overset{\circ}{\gamma} \tag{A.2}$$

In null coordinates

$$u = \frac{1}{2}(t - r) \qquad v = \frac{1}{2}(t + r) \tag{A.3}$$

the metric takes the form

$$g = -4 \, \mathrm{d}u \, \mathrm{d}v + r^2 \overset{\circ}{\gamma} \tag{A.4}$$

where $\overset{\circ}{\gamma}$ is the standard metric on the unit sphere.

Remark A.1 (Connection Coefficients). The connection coefficients of (A.4), which we may write in local coordinates as

$$g = g_{ab} \, \mathrm{d}x^a \, \mathrm{d}x^b + r^2 \overset{\circ}{\gamma}_{AB} \; \mathrm{d}y^A \, \mathrm{d}y^B \,, \tag{A.5}$$

are easily calculated:

$$\begin{gathered}
\Gamma^u_{uu} = \Gamma^v_{vv} = 0 \\
\Gamma^u_{uv} = \Gamma^v_{uv} = \Gamma^A_{uv} = 0 \\
\Gamma^u_{AB} = \frac{r}{2} \overset{\circ}{\gamma}_{AB} \qquad \Gamma^v_{AB} = -\frac{r}{2} \gamma_{AB} \\
\Gamma^A_{ab} = \Gamma^b_{aA} = 0 \\
\Gamma^B_{Av} = \frac{1}{r} \delta^B_A \qquad \Gamma^B_{Au} = -\frac{1}{r} \delta^B_A \\
\Gamma^C_{AB} = \overset{\circ}{\Gamma}{}^C_{AB}
\end{gathered} \tag{A.6}$$

Wave equation. The wave equation on $(\mathcal{M}, g)$ takes the classical form

$$\Box \phi = (g^{-1})^{\alpha\beta} \nabla_\alpha \partial_\beta \phi = -\partial_u \partial_v \phi + \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \overset{\circ}{\Delta} \phi = 0 \,. \tag{A.7}$$

Note that given a solution ϕ the function $\psi = r\phi$ satisfies

$$\partial_u \partial_v \psi = \not{\Delta} \psi \,. \tag{A.8}$$

Energy currents. The Lagrangian theory of (A.7) provides us with a conserved energy momentum tensor

$$T_{\mu\nu}(\phi) = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \, \partial^\alpha \phi \, \partial_\alpha \phi \,, \tag{A.9}$$

which allows us to construct energy currents with multiplier vectorfields V:

$$J^V_\mu(\phi) = T_{\mu\nu}(\phi) V^\nu \tag{A.10}$$

Recall the basic identities for the construction of energy estimates for (A.7):

$$K^V(\phi) \doteq \nabla^\mu J^V_\mu(\phi) = {}^{(V)}\pi^{\mu\nu} T_{\mu\nu}(\phi) \,; \qquad {}^{(V)}\pi = \frac{1}{2} \mathcal{L}_V g \tag{A.11}$$

$$\int_{\mathcal{D}} K^V \, \mathrm{d}\mu_g = \int_{\partial \mathcal{D}} {}^*J^V \tag{A.12}$$

Here $\mathcal{D} \subset \mathcal{M}$, and ${}^*J^V_{\alpha\beta\gamma} = (J^V)^\mu \, \epsilon_{\mu\alpha\beta\gamma}$ is the Hodge dual with respect to the volume form $\epsilon = \mathrm{d}\mu_g$.

Remark A.2 (Components of the energy momentum tensor). The null decomposition of the energy momentum tensor is:

$$\begin{gathered} T_{uu} = \Bigl(\frac{\partial\phi}{\partial u}\Bigr)^2 \qquad T_{vv} = \Bigl(\frac{\partial\phi}{\partial v}\Bigr)^2 \\ T_{uv} = \bigl|\nabla\!\!\!/\phi\bigr|^2 \\ T_{AB} = E_A\phi\, E_B\phi - \frac{1}{2}g_{AB}\partial^\alpha\phi\,\partial_\alpha\phi \end{gathered} \tag{A.13}$$

Cauchy Problem. Let

$$^{R}\mathcal{P}_{\tau_1}^{\tau_2} = \Bigl\{(u,v):\ v-u\leq R\,, R+2\tau_1\leq v+u\leq R+2\tau_2\Bigr\} \tag{A.14}$$

$$^{R}\mathcal{D}_{\tau_1}^{\tau_2} = \Bigl\{(u,v):\ \tau_1\leq u\leq\tau_2\,, v-u\geq R\Bigr\} \tag{A.15}$$

and Σ_{τ_1} the past boundary of $\bigcup_{\tau_2\geq\tau_1}{}^{R}\mathcal{P}_{\tau_1}^{\tau_2}\cup{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$:

$$\Sigma_\tau = \Bigl\{(u,v):\ u+v=2\tau+R\,, v-u\leq R\Bigr\}\cup\Bigl\{(u,v):\ u=\tau\,, v-u\geq R\Bigr\} \tag{A.16}$$

Initial data for (A.7) is prescribed on Σ_{τ_1}, and our aim is here to prove estimates for the energy flux through Σ_τ $(\tau>\tau_1)$.

First, we have the classical conservation of energy, related to the timelike Killing vectorfield $T=\frac{\partial}{\partial t}$:

Proposition A.3 (Boundedness of energy). *Let ϕ be solution of* (A.7) *then*

$$\int_{\Sigma_{\tau_2}}\Bigl(J^T(\phi),n\Bigr)\leq\int_{\Sigma_{\tau_1}}\Bigl(J^T(\phi),n\Bigr) \qquad (\tau_2>\tau_1) \tag{A.17}$$

Proof. Choose $\mathcal{D}={}^{R}\mathcal{P}_{\tau_1}^{\tau_2}\cup{}^{R}\mathcal{D}_{\tau_1}^{\tau_2}$ and $V=T$ in (A.12), and observe that ${}^{(T)}\pi=0$. □

Second, we have spacetime integral estimate, originally due to [39].

Assumption A.4 (Integrated local energy decay). *Let ϕ be a solution to the wave equation* (A.7)*, then there exists a constant C(R) such that*

$$\int_{{}^{R}\mathcal{P}_{\tau_1}^{\tau_2}}\Bigl\{\bigl(\frac{\partial\phi}{\partial u}\bigr)^2+\bigl(\frac{\partial\phi}{\partial v}\bigr)^2+\bigl|\nabla\!\!\!/\phi\bigr|^2\Bigr\}\leq C(R)\int_{\Sigma_{\tau_1}}\Bigl(J^T(\phi),n\Bigr) \tag{A.18}$$

for all $\tau_2>\tau_1$.

Notation (Integration). The volume form is usually omitted. Moreover we write $\int_\Sigma(J^V,n)$ for the boundary terms arising in (A.12) if $\Sigma\subset\partial\mathcal{D}$ is spacelike or null; in fact for Σ_τ as defined in (A.16) we have

$$\int_{\Sigma_\tau}\Bigl(J^V(\phi),n\Bigr)\doteq\int_{\Sigma_\tau\cap\{r\leq R\}}\Bigl(J^V(\phi),\frac{\partial}{\partial t}\Bigr)+\int_{\tau+R}^\infty\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}r^2\Bigl(J^V(\phi),\frac{\partial}{\partial v}\Bigr) \tag{A.19}$$

Here it is stated as an assumption, because we are not concerned with the proof of (A.18), but rather with its implications for energy decay.

Theorem 8. *Let ϕ be a solution of the wave equation* (A.7) *and*

$$D \doteq \int_{\Sigma_{\tau_1}} \Big(J^T(\phi), n \Big) + \int_{\tau_1+R}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} r^2 \left(\frac{\partial(r\phi)}{\partial v}\right)^2 < \infty \tag{A.20}$$

for some fixed $\tau_1 > 0$. Then there exists a constant $C(R)$ such that

$$\int_{\Sigma_\tau} \Big(J^T(\phi), n \Big) \leq \frac{C(R)}{\tau^2} D \tag{A.21}$$

for all $\tau > \tau_1$.

The derivation of this energy decay result from the assumption A.4 on integrated local decay (and boundedness of energy) is subject of [22], and will also be apparent from the following.

Interior decay of the first order energy

We are going to show in this note the following result on interior decay of the first order energy:

Proposition A.5 (Interior first order energy decay)**.** *Let ϕ be a solution of the wave equation* (A.7)*, and*

$$\begin{aligned} D \doteq \int_{\tau_1+R}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \Bigg[& r^{4-\delta}\Big(\frac{\partial^2(r\phi)}{\partial v^2}\Big)^2 + r^2\Big(\frac{\partial(r\phi)}{\partial v}\Big)^2 + r^2\Big(\frac{\partial(rT\cdot\phi)}{\partial v}\Big)^2 \\ & + \sum_{j=1}^{3} r^2\Big(\frac{\partial(r\Omega_j\phi)}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^2\Big(\frac{\partial(r\Omega_j T\cdot\phi)}{\partial v}\Big)^2 + \sum_{j,k=1}^{3} r^2\Big(\frac{\partial(r\Omega_j\Omega_k T\cdot\phi)}{\partial v}\Big)^2 \Bigg]\Big|_{u=\tau_1} \\ + \int_{\Sigma_{\tau_1}} \Big(& J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j\phi) + \sum_{j=1}^{3} J^T(\Omega_j T\cdot\phi) + \sum_{j,k=1}^{3} J^T(\Omega_j\Omega_k T\cdot\phi), n \Big) \end{aligned} \tag{A.22}$$

be finite, for any chosen $R > 0$, $\tau_1 > 0$, and $0 < \delta < \frac{1}{2}$. Then there exists a constant $C(\delta, R)$ such that

$$\int_{\Sigma_\tau \cap \{r \leq R\}} \Big(J^T(T\cdot\phi), n \Big) \leq \frac{C(\delta, R)}{\tau^{4-2\delta}} D \tag{A.23}$$

for all $\tau > \tau_1$.

This holds under the same assumptions as Thm. 8, namely Prop. A.3 and Assumption A.4, with the exception that we also need a refinement of the latter: Let

$${}^R\mathcal{D}_{\tau_1}^{\tau_2} = \Big\{ (u, v) : \tau_1 \leq u \leq \tau_2\,, v - u \geq R\,, v \leq \tau_2 + R \Big\} \tag{A.24}$$

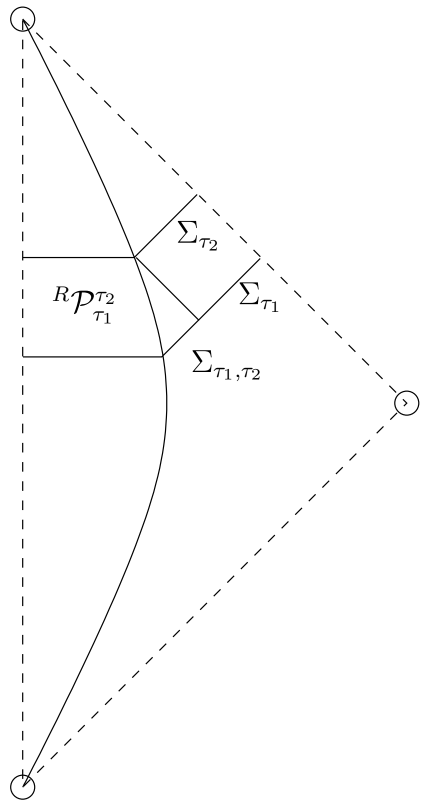

Figure A.1: The hypersurfaces (A.16) and (A.25) depicted in the Penrose diagram of Minkowski space.

and Σ_{τ_1,τ_2} the past boundary of $^R\mathcal{P}^{\tau_2}_{\tau_1} \cup {}^R\mathcal{D}^{\tau_2}_{\tau_1}$ (see also figure A.1):

$$\Sigma_{\tau_1,\tau_2} = \Big\{(u,v):\ u+v = 2\tau + R\,, v-u \leq R\Big\} \cup \Big\{(u,v):\ u=\tau\,, R+\tau_1 \leq v \leq R+\tau_2\Big\} \tag{A.25}$$

Assumption A.6 (Integrated local energy decay for finite regions)**.** *Let ϕ be a solution of* (A.7)*, then there is a constant C(R) such that*

$$\int_{^R\mathcal{P}^{\tau_2}_{\tau_1}} \Big\{ \Big(\frac{\partial\phi}{\partial u}\Big)^2 + \Big(\frac{\partial\phi}{\partial v}\Big)^2 + |\nabla\!\!\!/\,\phi|^2 \Big\} \leq$$
$$\leq C(R)\Big[\int_{\Sigma_{\tau_1,\tau_2}} \Big(J^T(\phi), n\Big) + \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}}\,(r\phi^2)(\tau_1, R+\tau_2)\Big] \tag{A.26}$$

for all $\tau_2 > \tau_1$.

Notation. In the following we use the short hand notation

$$\int_{\Sigma'_{\tau_1,\tau_2}} \Big(J^T(\phi), n\Big) \doteq \int_{\Sigma_{\tau_1,\tau_2}} \Big(J^T(\phi), n\Big) + \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}}\,(r\phi^2)\big|_{(u=\tau_1, v=R+\tau_2)}\,. \tag{A.27}$$

Sketch of Proof (of the refinement of Assumption A.4). The estimate (A.18) is readily proven using radial multiplier vectorfields. The *modified* currents associated to these multipliers give rise to zeroth order terms on the boundary, which are estimated by the energy using a Hardy inequality. For example, on the past boundary of ${}^R\mathcal{D}_{\tau_1}^{\tau_2}$, which is the null segment $u=\tau_1$, $r\geq R$, we would employ the estimate

$$\int_{\tau_1+R}^{\infty}\mathrm{d}v\, r^2\times\frac{1}{r^2}\phi^2\rvert_{u=\tau_1}\leq 4\int_{\tau_1+R}^{\infty}\Bigl(\frac{\partial\phi}{\partial v}\Bigr)^2 r^2\,\mathrm{d}v\rvert_{u=\tau_1}\,. \tag{A.28}$$

In order to prove (A.26), we would instead infer the following inequality, on the past boundary of ${}^R\mathfrak{D}_{\tau_1}^{\tau_2}$,

$$\frac{1}{2}\int_{R+\tau_1}^{R+\tau_2}\mathrm{d}v\, r^2\times\frac{1}{r^2}\phi^2\rvert_{u=\tau_1}\leq (r\phi^2)(\tau_1,R+\tau_2)+2\int_{R+\tau_1}^{R+\tau_2}\Bigl(\frac{\partial\phi}{\partial v}\Bigr)^2 r^2\,\mathrm{d}v\,, \tag{A.29}$$

which follows from a simple integration by parts:

$$\begin{aligned}\int_{R+\tau_1}^{R+\tau_2}\mathrm{d}v\, r^2\times\frac{1}{r^2}\phi^2 &= \int_{R+\tau_1}^{R+\tau_2}\mathrm{d}v\frac{\mathrm{d}}{\mathrm{d}v}\bigl(v-\tau_1\bigr)\,\phi^2\leq\\ &\leq (R+\tau_2-\tau_1)\phi^2\rvert_{v=R+\tau_2}+\int_{R+\tau_1}^{R+\tau_2}\mathrm{d}v\Bigl\{\frac{1}{2}\phi^2+2\bigl(\frac{\partial\phi}{\partial v}\bigr)^2r^2\Bigr\}\end{aligned} \tag{A.30}$$

Similarly for the future boundary of ${}^R\mathfrak{D}_{\tau_1}^{\tau_2}$. □

The refinement introduces the new boundary term in (A.26).

Lemma A.7 (Pointwise decay). *Let ϕ be a solution of the wave equation* (A.7)*, with initial data on Σ_{τ_0} ($\tau_0>0$) satisfying*

$$\begin{aligned}D\doteq\int_{\tau_0+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\Bigl[r^2\Bigl(\frac{\partial(r\phi)}{\partial v}\Bigr)^2+\sum_{j=1}^{3}r^2\Bigl(\frac{\partial(r\Omega_j\phi)}{\partial v}\Bigr)^2+\sum_{k,j=1}^{3}r^2\Bigl(\frac{\partial(r\Omega_k\Omega_j\phi)}{\partial v}\Bigr)^2\Bigr]\rvert_{u=\tau_0}\\ +\int_{\Sigma_{\tau_0}}\Bigl(J^T(\phi)+\sum_{j=1}^{3}J^T(\Omega_j\phi)+\sum_{k,j=1}^{3}J^T(\Omega_k\Omega_j\phi),n\Bigr)<\infty\,.\end{aligned} \tag{A.31}$$

Then there is a constant $C(R)$ such that

$$\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\phi^2\rvert_{(\tau_1,R+\tau_2)}\leq\frac{C(R)}{\tau_1}D \tag{A.32}$$

for any $\tau_2>\tau_1>\tau_0$.

Remark A.8. Note that by comparison to (A.26) we gain a power in r.

Proof. First, integrating from infinity,

$$\phi(\tau_1,R+\tau_1)=-\int_{\tau_1+R}^{\infty}\frac{\partial\phi}{\partial v}\,\mathrm{d}v \tag{A.33}$$

and then by Cauchy's inequality, and the Sobolev inequality on the sphere,

$$\begin{aligned}\phi^2(\tau_1,R+\tau_1)&\leq\int_{R+\tau_1}^{\infty}\frac{1}{r^2}\,\mathrm{d}v\times\int_{R+\tau_1}^{\infty}\Bigl(\frac{\partial\phi}{\partial v}\Bigr)^2r^2\,\mathrm{d}v\\&\leq\frac{1}{r}\Big|_{v=R+\tau_1}\int_{R+\tau_1}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\Bigl\{\Bigl(\frac{\partial\phi}{\partial v}\Bigr)^2+\sum_{j=1}^{3}\Bigl(\Omega_j\partial_v\phi\Bigr)^2+\sum_{j,k=1}^{3}\Bigl(\Omega_k\Omega_j\partial_v\phi\Bigr)^2\Bigr\}.\end{aligned}\tag{A.34}$$

Therefore, by Thm. 8,

$$(r\phi^2)(\tau_1,R+\tau_1)\leq\int_{\Sigma_{\tau_1}}\Bigl(J^T(\phi)+\sum_{j=1}^{3}J^T(\Omega_j\phi),n\Bigr)\leq\frac{C(R)}{\tau_1^2}\,D\,.\tag{A.35}$$

Now

$$\begin{aligned}r^2\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,\phi^2\,(\tau_1,R+\tau_2)&=\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}(r\phi)^2(\tau_1,R+\tau_1)+\int_{R+\tau_1}^{R+\tau_2}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\frac{\partial(r\phi)^2}{\partial v}\leq\\&\leq R^2\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\phi^2(\tau_1,R+\tau_1)+2\sqrt{\int_{R+\tau_1}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\phi^2}\sqrt{\int_{R+\tau_1}^{\infty}\mathrm{d}v\,r^2\Bigl(\frac{\partial(r\phi)}{\partial v}\Bigr)^2}\,,\end{aligned}\tag{A.36}$$

which proves (A.32) in view of the Hardy inequality (A.28), Thm. 8 and (A.35). □

Remark A.9. The decay rate in (A.32) is in fact not sufficient. However, we see from (A.36) that if a solution is already known to satisfy

$$\int_{\tau+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\Bigl(\frac{\partial(r\phi)}{\partial v}\Bigr)^2\Big|_{u=\tau}\leq\frac{C\,D}{\tau^2}\qquad(\tau>\tau_0)$$

then indeed also

$$\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\phi^2|_{(\tau,R+\tau')}\leq\frac{C\,D}{\tau^2}\qquad(\tau'>\tau)\,;$$

this is the case for the solution $T\cdot\phi$ (on "good" slices with a loss in the power of r) as shown below, and will be used in the argument.

Furthermore, it is important to point out that Assumption A.6 also allows us to control zeroth order terms on timelike boundaries.

Lemma A.10 (Control of zeroth order terms)**.** *Let ϕ be a solution of* (A.7)*, then there is a constant $C>0$ such that*

$$\begin{aligned}&\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}r^2\times\frac{1}{r^2}\phi^2|_{r=R}\leq\\&\qquad\leq C\Bigl\{\int_{\Sigma'_{\tau_1,\tau_2}}\Bigl(J^T(\phi),n\Bigr)+\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\Bigl[\Bigl(\frac{\partial\phi}{\partial u}\Bigr)^2+\Bigl(\frac{\partial\phi}{\partial v}\Bigr)^2+\bigl|\nabla\!\!\!\!/\,\phi\bigr|^2\Bigr]\Bigr\}\end{aligned}\tag{A.37}$$

for all $\tau_2>\tau_1$.

The proof of this fact is in the same vein as Assumption A.4, A.6 and not at the centre of our interest here, but we include the proof of a simplified version of Lemma A.10 without the refinement to finite regions; the difference to the proof given here amounts to keeping track of the boundary terms in the Hardy inequalities, (cf. Sketch of Proof of Assumption A.6).

Proposition A.11 (Example of an X-estimate). *Let ϕ be a solution of* (A.7), *then there is a constant $C > 0$ such that*

$$\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}r^2\times\frac{1}{r^2}\phi^2|_{r=R}\leq$$
$$\leq C\Big\{\int_{\Sigma_{\tau_1}}\Big(J^T(\phi),n\Big)+\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\Big[\big(\frac{\partial\phi}{\partial u}\big)^2+\big(\frac{\partial\phi}{\partial v}\big)^2+\big|\nabla\!\!\!/\phi\big|^2\Big]\Big\} \quad (A.38)$$

for all $\tau_2 > \tau_1$.

Proof. Consider the vectorfield

$$X=\frac{\partial}{\partial r}\,. \quad (A.39)$$

We find

$$K^X={}^{(X)}\pi^{\mu\nu}T_{\mu\nu}(\phi)=\frac{1}{r}(g^{-1})^{AB}T_{AB}(\phi)=\frac{1}{r}|\nabla\!\!\!/\phi|^2-\frac{1}{r}\partial^\alpha\phi\,\partial_\alpha\phi \quad (A.40)$$

and are thus led to the modified current

$$J^{X,1}_\mu=J^X_\mu+\frac{1}{2}\frac{1}{r}\partial_\mu\phi^2-\frac{1}{2}\partial_\mu\big(\frac{1}{r}\big)\phi^2\,, \quad (A.41)$$

with nonnegative divergence:

$$\nabla^\mu J^{X,1}_\mu=\frac{1}{r}\big|\nabla\!\!\!/\phi\big|^2 \quad (A.42)$$

Here we used that $\Box\phi^2=2\partial^\alpha\phi\,\partial_\alpha\phi$ for any solution of (A.7), and $\Box\big(\frac{1}{r}\big)=0$ away from the origin. Now apply (A.12) with (A.41) to the domain ${}^R\mathcal{D}_{\tau_1}^{\tau_2}$. Since

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}}{}^*J^{X,1}=$$
$$=-\int_{R+\tau_2}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\times\Big\{T\big(\frac{\partial}{\partial v},\frac{\partial}{\partial r}\big)+\frac{1}{r}\phi\frac{\partial\phi}{\partial v}+\frac{1}{2}\frac{1}{r^2}\phi^2\Big\}|_{u=\tau_2}$$
$$+\int_{R+\tau_1}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\times\Big\{T\big(\frac{\partial}{\partial v},\frac{\partial}{\partial r}\big)+\frac{1}{r}\phi\frac{\partial\phi}{\partial v}+\frac{1}{2}\frac{1}{r^2}\phi^2\Big\}|_{u=\tau_1}$$
$$-\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\times\Big\{T\big(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\big)+\frac{1}{r}\phi\frac{\partial\phi}{\partial r}+\frac{1}{2}\frac{1}{r^2}\phi^2\Big\}|_{r=R} \quad (A.43)$$

we have (using Cauchy's inequality for the mixed term $\phi\,\partial\phi/\partial r$) an estimate for

$$\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\,r^2\times\frac{1}{4}\frac{1}{r^2}\phi^2\leq$$

$$\leq \int_{R+\tau_1}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}}\, r^2 \times \Big\{ \Big(\frac{\partial\phi}{\partial v}\Big)^2 + \frac{1}{2}\big|\nabla\!\!\!/\phi\big|^2 + \frac{1}{r^2}\phi^2 \Big\}|_{u=\tau_1}$$
$$+ \int_{2\tau_1+R}^{2\tau_2+R} \mathrm{d}t \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}}\, r^2 \times \Big\{ \frac{1}{4}\Big(\frac{\partial\phi}{\partial v}\Big)^2 + \frac{1}{2}\big|\nabla\!\!\!/\phi\big|^2 + \frac{1}{4}\Big(\frac{\partial\phi}{\partial u}\Big)^2 + \Big(\frac{\partial\phi}{\partial r}\Big)^2 \Big\}|_{r=R} \quad \text{(A.44)}$$

which implies (A.38) in view of the Hardy inequality (A.28) and the fact that

$$\big(J^T(\phi), \frac{\partial}{\partial v}\big) = \frac{1}{2}\big(\frac{\partial\phi}{\partial v}\big)^2 + \frac{1}{2}\big|\nabla\!\!\!/\phi\big|^2 \qquad K^T(\phi) = 0\,.$$ ☐

Corollary A.12. *Let ϕ be a solution of (A.7), $R_0 > 0$ and $\tau_2 > \tau_1$. Then there is a constant $C(R_0)$ such that*

$$\int_{2\tau_1+R}^{2\tau_2+R} \mathrm{d}t \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \Big\{ \Big(\frac{\partial\phi}{\partial u}\Big)^2 + \Big(\frac{\partial\phi}{\partial v}\Big)^2 + \big|\nabla\!\!\!/\phi\big|^2 + \phi^2 \Big\}|_{r=R} \leq C(R_0) \int_{\Sigma'_{\tau_1,\tau_2}} \Big(J^T(\phi), n\Big) \quad \text{(A.45)}$$

for some $R_0 < R < R_0 + 1$.

Proof. Apply Assumption A.6 to the domain ${}^{R_0+1}\mathcal{P}_{\tau_1}^{\tau_2} \setminus {}^{R_0}\mathcal{P}_{\tau_1}^{\tau_2}$ and use the mean value theorem for the integration on $(R_0, R_0 + 1)$. ☐

Proof of Proposition A.5.

The method of proof is based on weighted energy identities. While [22] uses a weighted energy identity arising from the multiplier $r^p \frac{\partial}{\partial v}$ to prove Thm. 8, here we also use a commutation with $\frac{\partial}{\partial v}$ to obtain energy decay of the solution $\frac{\partial\phi}{\partial t}$ of (A.7).

Weighted energy identity. We shall first consider the current that yields the weighted energy identity of [22]:

$$\overset{r}{J}_\mu(\phi) \doteq T_{\mu\nu}(\psi) V^\nu \quad \text{(A.46)}$$
$$\psi = r\phi \qquad V = r^q \frac{\partial}{\partial v} \qquad q = p - 2 \qquad 0 < p \leq 2$$

Since

$$ {}^{(V)}\pi_{uu} = 2q\, r^{q-1} \qquad {}^{(V)}\pi_{uv} = -q\, r^{q-1} \qquad {}^{(V)}\pi_{vv} = 0$$
$$ {}^{(V)}\pi_{AB} = r^{q-1} g_{AB} \quad \text{(A.47)}$$

we find

$$K^V(\phi) = {}^{(V)}\pi^{\alpha\beta} T_{\alpha\beta}(\phi) = \frac{1}{2} q\, r^{q-1} \Big(\frac{\partial\phi}{\partial v}\Big)^2 - \frac{1}{2} q\, r^{q-1} \big|\nabla\!\!\!/\phi\big|^2 + r^{q-1} (\partial_u\phi)(\partial_v\phi)\,. \quad \text{(A.48)}$$

Notation. To make the dependence on p explicit, we denote by

$$\overset{r}{K}_p(\phi) \doteq \nabla^\mu \overset{r}{J}_\mu(\phi)\,. \quad \text{(A.49)}$$

We calculate

$$\overset{r}{K}_p(\phi) = \Box(\psi)V\cdot\psi + K^V(\psi) \\ = \frac{2}{r}\frac{\partial\psi}{\partial r}r^q\frac{\partial\psi}{\partial v} + \frac{1}{2}q\,r^{q-1}\Big(\frac{\partial\psi}{\partial v}\Big)^2 - \frac{1}{2}q\,r^{q-1}\big|\nabla\!\!\!/\psi\big|^2 + r^{q-1}\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v} \\ = \frac{1}{2}p\,r^{q-1}\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \frac{1}{2}(2-p)r^{q-1}\big|\nabla\!\!\!/\psi\big|^2 \tag{A.50}$$

which is nonnegative for $p\leq 2$; by (A.12) this immediately implies

$$\int_{\tau_2+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\,\frac{1}{2}r^p\Big(\frac{\partial\psi}{\partial v}\Big)^2\Big|_{u=\tau_2} + \int_{\tau_1}^{\tau_2}\mathrm{d}u\int_{u+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\frac{p}{2}\,r^{p-1}\Big(\frac{\partial\psi}{\partial v}\Big)^2\leq \\ \leq\int_{\tau_1+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\,\frac{1}{2}r^p\Big(\frac{\partial\psi}{\partial v}\Big)^2\Big|_{u=\tau_1} + \int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \big|\nabla\!\!\!/\psi\big|^2\Big\}\Big|_{r=R} \tag{A.51}$$

for $p=1,2$ in particular, (and all $\tau_2>\tau_1$).

Weighted energy and commutation. Now consider the current:

$$\overset{v}{J}_\mu(\phi) \doteq T_{\mu\nu}(\chi)V^\nu \tag{A.52}$$

$$\chi = \partial_v\psi = \frac{\partial(r\phi)}{\partial v} \qquad V = r^q\frac{\partial}{\partial v} \tag{A.53}$$

$$q = p-2 \qquad 0<p<4 \qquad \delta = 4-p$$

Notation. We denote similarly by

$$\overset{v}{K}_p(\phi) \doteq \nabla^\mu\,\overset{v}{J}_\mu(\phi)\,. \tag{A.54}$$

The error terms for $\overset{v}{K}$, and $\overset{r}{K}$, – in comparison to (A.11) – arise from the fact that χ, and ψ respectively, are not solutions of (A.7); here we find:

$$\Box\chi = \frac{2}{r}\frac{\partial\chi}{\partial r} + \frac{2}{r}\Delta\!\!\!/\psi \tag{A.55}$$

Hence

$$\overset{v}{K}_p(\phi) = \Box(\chi)\,V\cdot\chi + K^V(\chi) \\ = \frac{1}{2}p\,r^{q-1}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + 2\,r^{q-1}\,(\Delta\!\!\!/\psi)\,\frac{\partial\chi}{\partial v} + \frac{1}{2}\,(2-p)\,r^{q-1}\,\big|\nabla\!\!\!/\chi\big|^2, \tag{A.56}$$

which is not positive definite. However, we have

$$\frac{1}{4}p\,r^{p-1}\Big(\frac{\partial^2(r\phi)}{\partial v^2}\Big)^2\leq \\ \leq\overset{v}{K}_p(\phi)\,r^2 + \frac{4}{p}r^{(p-2)-1}\Big(\Delta\!\!\!/(r\phi)\Big)^2r^2 + \frac{1}{2}(p-2)r^{p-1}|\nabla\!\!\!/\partial_v(r\phi)|^2\,, \tag{A.57}$$

and are able to bound the second term using a commutation with Ω_i, and the third term with an integration by parts argument.

Lemma A.13. *For any function* $\phi \in \mathrm{H}^2(S_r)$ *we have* $\not\Delta\phi \in \mathrm{L}^2(S_r)$*, and there exists a constant* $C > 0$ *such that it holds*

$$\int_{\mathbb{S}^2} \Big(\not\Delta(r\phi)\Big)^2 r^2 \,\mathrm{d}\mu_{\overset{\circ}{\gamma}} \leq C \int_{\mathbb{S}^2} \Big\{ \sum_{i=1}^{3} \Big|\not\nabla(r\Omega_i\phi)\Big|^2 + \Big|\not\nabla(r\phi)\Big|^2 \Big\} \,\mathrm{d}\mu_{\overset{\circ}{\gamma}} \,. \tag{A.58}$$

If $p \leq 2$ (A.57) holds without the last term; and if $p > 2$ we can "interchange" the derivatives of $|\not\nabla\partial_v\psi|^2$ by integrating by parts twice, (such that we can absorb the resulting $\partial_v\chi$ term in the left hand side):

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{u+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{p-1}|\not\nabla\partial_v(r\phi)|^2 = \\
&\quad = -\int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{u+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^{p-1}\partial_v\big(\not\Delta(r\phi)\big)\,\partial_v(r\phi) + r^{p-1}\frac{2}{r}\not\Delta(r\phi)\,\partial_v(r\phi) \Big\} \\
&\quad = -\int_{\tau_1}^{\tau_2} \mathrm{d}u \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{p-1}\not\Delta(r\phi)\,\partial_v(r\phi)\Big|_{u+R}^{\infty} \\
&\quad + \int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{u+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ (p-1)r^{p-2}\not\Delta(r\phi)\,\partial_v(r\phi) \\
&\qquad + r^{p-1}\not\Delta(r\phi)\,\partial_v^2(r\phi) + 2r^{p-2}\not\Delta(r\phi)\,\partial_v(r\phi) \Big\} \\
&\quad \leq \int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{p-1}\not\Delta(r\phi)\,\partial_v(r\phi)\big|_{v=u+R} \\
&\quad + \int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{u+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ \big[2 + (p-1) + 2(p-2)\frac{8}{p}\big] r^{(p-2)-1}\Big(\not\Delta(r\phi)\Big)^2 r^2 \\
&\qquad + \big[2 + (p-1)\big] r^{(p-2)-1}\Big(\frac{\partial(r\phi)}{\partial v}\Big)^2 + \frac{2}{p-2}\frac{p}{8} r^{p-1}\Big(\frac{\partial^2(r\phi)}{\partial v^2}\Big)^2 \Big\}
\end{aligned} \tag{A.59}$$

Therefore

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{u+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{p-1}\Big(\frac{\partial^2(r\phi)}{\partial v^2}\Big)^2 \leq \\
&\quad \leq C(p,\delta) \int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} \mathrm{d}\mu_g \Big\{ \overset{v}{K}_p(\phi) + \overset{r}{K}_{p-2}(\phi) + \sum_{j=1}^{3} \overset{r}{K}_{p-2}(\Omega_j\phi) \Big\} \\
&\qquad + C\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}}\, r^{p-2} \Big\{ \sum_{j=1}^{3} |\not\nabla r\Omega_j\phi|^2 + \Big(\frac{\partial(r\phi)}{\partial v}\Big)^2 \Big\}\Big|_{r=R} .
\end{aligned} \tag{A.60}$$

Note that we need $p \leq 4$ in order to control the $(\partial_v\psi)^2$ term by (A.50) with $p \leq 2$, and strictly $p < 4$ to control the $(\not\Delta\psi)^2$ term by (A.50) using Lemma A.13.

Thus after turning the divergences into boundary terms using (A.12), we arrive at the following *weighted energy inequality* for χ:

$$\begin{aligned}
&\int_{\tau_2+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^p\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r^{p-2}\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^{p-2}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 \Big\}\Big|_{u=\tau_2} \\
&\qquad + \int_{\tau_1}^{\tau_2}\mathrm{d}u \int_{u+R}^{\infty}\mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{p-1}\Big(\frac{\partial\chi}{\partial v}\Big)^2
\end{aligned}$$

$$\leq C(p,\delta)\int_{\tau_1+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^p\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^{p-2}\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^{p-2}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_1}$$
$$+C(p,\delta)\int_{2\tau_1+R}^{2\tau_2+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^p\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^{p-2}\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^{p-2}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2$$
$$+r^p\big|\nabla\!\!\!/\chi\big|^2+r^{p-2}\big|\nabla\!\!\!/\psi\big|^2+\sum_{j=1}^{3}r^{p-2}\big|\nabla\!\!\!/\Omega_j\psi\big|^2\Big\}\Big|_{r=R}\qquad(\tau_2>\tau_1)\quad\text{(A.61)}$$

The fact that this inequality holds for up to $p<4$ – which appears as the weight r^p in the boundary terms – is directly related to the decay rate in (A.23). Correspondingly we will proceed in a hierarchy of four steps:

$\boxed{p=4-\delta:}$ Let $\tau_1>0$, and $\tau_{j+1}=2\tau_j\quad(j\in\mathbb{N})$. In a first step we use (A.61) with $p=4-\delta$ and (A.51) with $p=2$ as an estimate for the spacetime integral of $\partial_v\chi$, $\partial_v\psi$, and $\partial_v(\Omega_j\psi)$ on ${}^R\mathcal{D}_{\tau_j}^{\tau_{j+1}}$, and in a second step as an estimate for the corresponding integral on the future boundary of ${}^R\mathcal{D}_{\tau_1}^{\tau_j}$:

$$\int_{\tau_j}^{\tau_{j+1}}\mathrm{d}u\int_{u+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^{3-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j+1}^{3}r\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\leq$$
$$\leq C(\delta)\int_{\tau_j+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_j}$$
$$+C(\delta)\int_{2\tau_j+R}^{2\tau_{j+1}+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2$$
$$+r^{4-\delta}\big|\nabla\!\!\!/\chi\big|^2+r^2\big|\nabla\!\!\!/\psi\big|^2+\sum_{j=1}^{3}r^2\Big(\nabla\!\!\!/\Omega_j\psi\Big)^2\Big\}\Big|_{r=R}\leq$$
$$\leq C(\delta)\int_{\tau_1+R}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_1}$$
$$+C(\delta)\int_{2\tau_1+R}^{2\tau_{j+1}+R}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2$$
$$+r^{4-\delta}\big|\nabla\!\!\!/\chi\big|^2+r^2\big|\nabla\!\!\!/\psi\big|^2+\sum_{j=1}^{3}r^2\Big(\nabla\!\!\!/\Omega_j\psi\Big)^2\Big\}\Big|_{r=R}\quad\text{(A.62)}$$

Observe that,

$$\Big(\frac{\partial\psi}{\partial v}\Big)^2=\Big(\frac{\partial(r\phi)}{\partial v}\Big)^2\leq 2\phi^2+2r^2\Big(\frac{\partial\phi}{\partial v}\Big)^2\qquad\text{(A.63)}$$
$$\Big(\frac{\partial\chi}{\partial v}\Big)^2=\Big(\frac{\partial^2(r\phi)}{\partial v^2}\Big)^2\leq 4\Big(\frac{\partial\phi}{\partial v}\Big)^2+2r^2\Big(\frac{\partial^2\phi}{\partial v^2}\Big)^2,\qquad\text{(A.64)}$$

and using first the wave equation (A.7),

$$\frac{\partial^2\phi}{\partial v^2}=2\frac{\partial T\cdot\phi}{\partial v}-\frac{1}{r}\frac{\partial\phi}{\partial v}+\frac{1}{r}\frac{\partial\phi}{\partial u}-\frac{1}{r^2}\overset{\circ}{\Delta\!\!\!\!/}\phi\qquad\text{(A.65)}$$

and then Lemma A.13,

$$\int_{\mathbb{S}^2} r^2\Big(\frac{\partial^2\phi}{\partial v^2}\Big)^2 \mathrm{d}\mu_{\overset{\circ}{\gamma}} \leq$$
$$\leq C\int_{\mathbb{S}^2}\Big[r^2\Big(\frac{\partial(T\cdot\phi)}{\partial v}\Big)^2 + \Big(\frac{\partial\phi}{\partial v}\Big)^2 + \Big(\frac{\partial\phi}{\partial u}\Big)^2 + \sum_{j=1}^{3}\big|\nabla\!\!\!/\,\Omega_j\phi\big|^2 + \big|\nabla\!\!\!/\,\phi\big|^2\Big]; \quad (A.66)$$

moreover,

$$\big|\nabla\!\!\!/\,\chi\big|^2 = \big|\nabla\!\!\!/\,\partial_v(r\phi)\big|^2 \leq 2\big|\nabla\!\!\!/\,\phi\big|^2 + 2\sum_{j=1}^{3}\Big(\frac{\partial\Omega_j\phi}{\partial v}\Big)^2. \quad (A.67)$$

Therefore, by virtue of Assumption A.4 and Proposition A.11, we can choose a $R_0 \in (R, R+1)$ such that

$$\int_{2\tau_1+R_0}^{2\tau_{j+1}+R_0}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{\Big(\frac{\partial\chi}{\partial v}\Big)^2 + \Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2$$
$$+ \big|\nabla\!\!\!/\,\chi\big|^2 + \big|\nabla\!\!\!/\,\psi\big|^2 + \sum_{j=1}^{3}\Big(\nabla\!\!\!/\,\Omega_j\psi\Big)^2\Big\}\Big|_{r=R_0} \leq$$
$$\leq C(R)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j\phi), n\Big) \quad (A.68)$$

and henceforth there exists a sequence $\tau_j' \in (\tau_j, \tau_{j+1})$ such that

$$\int_{\tau_j'+R_0}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{r^{3-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j+1}^{3} r\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_j'} \leq$$
$$\leq \frac{C(\delta)}{\tau_j}\int_{\tau_1+R_0}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_1}$$
$$+ \frac{C(\delta,R)}{\tau_j}\int_{\Sigma_{\tau_1}}\Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j\phi), n\Big). \quad (A.69)$$

$\boxed{p = 3-\delta:}$ Here we apply (A.61) with $p = 3-\delta$ and (A.51) with $p = 1$ to the domain ${}^{R_j}\mathcal{D}_{\tau_{2j-1}'}^{\tau_{2j+1}'}$,

$$\int_{\tau_{2j-1}'}^{\tau_{2j+1}'}\mathrm{d}u\int_{u+R_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{r^{2-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + \Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\} \leq$$
$$\leq C\int_{\tau_{2j-1}'+R_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{r^{3-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_{2j-1}'}$$
$$+ C\int_{2\tau_{2j-1}'+R_j}^{2\tau_{2j+1}'+R_j}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{r^{3-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2$$
$$+ r^{3-\delta}\big|\nabla\!\!\!/\,\chi\big|^2 + r\big|\nabla\!\!\!/\,\psi\big|^2\sum_{j=1}^{3} r\big|\nabla\!\!\!/\,\Omega_j\psi\big|^2\Big\}\Big|_{r=R_j} \quad (A.70)$$

where $R_j \in (R_0, R_0+1)$ is chosen by Assumption A.4 and Prop. A.11 such that

$$\int_{2\tau'_{2j-1}+R_j}^{2\tau'_{2j+1}+R_j} \mathrm{d}t \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ \Big(\frac{\partial\chi}{\partial v}\Big)^2 + \Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 + \big|\nabla\!\!\!/\chi\big|^2 + \big|\nabla\!\!\!/\psi\big|^2 + \sum_{j=1}^{3}\big|\nabla\!\!\!/\Omega_j\psi\big|^2 \Big\}\Big|_{r=R_j} \leq$$

$$\leq C(R) \int_{\Sigma_{\tau'_{2j-1}}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j\phi), n\Big) . \quad \text{(A.71)}$$

Therefore, there exists a sequence $\tau''_j \in (\tau'_{2j-1}, \tau'_{2j+1})$ such that

$$\int_{\tau''_j+R_j}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^{2-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + \Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 \Big\}\Big|_{u=\tau''_j} \leq$$

$$\leq \frac{C(\delta)}{(\tau''_j)^2} \int_{\tau_1+R_0}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 \Big\}\Big|_{u=\tau_1}$$

$$+ \frac{C(\delta,R)}{(\tau''_j)^2} \int_{\Sigma_{\tau_1}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j\phi), n\Big)$$

$$+ C(R) \int_{\Sigma_{\tau_{2j-1}}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j), n\Big) \quad \text{(A.72)}$$

where we have used (A.69) for the first term in (A.70), However, by virtue of Thm. 8 the last term decays with the same rate, and we obtain

$$\int_{\tau''_j+R_j}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^{2-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + \Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3}\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 \Big\}\Big|_{u=\tau''_j} \leq$$

$$\leq \frac{C(\delta,R)}{(\tau''_j)^2} \int_{\tau_1+R_0}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial T\cdot\psi}{\partial v}\Big)^2 \Big\}\Big|_{u=\tau_1}$$

$$+ \frac{C(\delta,R)}{(\tau''_j)^2} \int_{\Sigma_{\tau_1}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} J^T(\Omega_j\phi), n\Big) . \quad \text{(A.73)}$$

In fact,

$$r^{2-\delta}\Big(\frac{\partial(rT\cdot\phi)}{\partial v}\Big)^2 = r^{2-\delta}\Big(\frac{\partial(T\cdot\psi)}{\partial v}\Big)^2 = r^{2-\delta}\Big(\frac{1}{2}\frac{\partial^2(r\phi)}{\partial v^2} + \frac{1}{2}\frac{\partial^2(r\phi)}{\partial u\partial v}\Big)^2 \leq$$

$$\leq \frac{1}{2} r^{2-\delta}\Big(\frac{\partial^2(r\phi)}{\partial v^2}\Big)^2 + \frac{1}{2} r^{2-\delta}\Big(\Delta\!\!\!/(r\phi)\Big)^2 \quad \text{(A.74)}$$

so by Lemma A.13

$$\int_{\tau'_{2j-1}}^{\tau'_{2j+1}} \mathrm{d}u \int_{u+R_j}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{2-\delta}\Big(\frac{\partial rT\cdot\phi}{\partial v}\Big)^2 \leq$$

$$\leq C \int_{\tau'_{2j-1}}^{\tau'_{2j+1}} \mathrm{d}u \int_{u+R_j}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big\{ r^{2-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + \overset{r}{K}_{1-\delta}(\phi)\, r^2 + \sum_{j=1}^{3} \overset{r}{K}_{1-\delta}(\Omega_j\phi)\, r^2 \Big\} \quad \text{(A.75)}$$

generates boundary terms that are already present in (A.70), (cf. (A.50); consequently we have

$$\int_{\tau_j''+R_j}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigl\{ r^{2-\delta}\Bigl(\frac{\partial r T\cdot\phi}{\partial v}\Bigr)^2\Bigr\}\Bigr|_{u=\tau_j''} \leq$$

$$\leq \frac{C(\delta,R)}{(\tau_j'')^2}\int_{\tau_1+R_0}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigl\{ r^{4-\delta}\Bigl(\frac{\partial\chi}{\partial v}\Bigr)^2 + r^2\Bigl(\frac{\partial\psi}{\partial v}\Bigr)^2 + \sum_{j=1}^3 r^2\Bigl(\frac{\partial\Omega_j\psi}{\partial v}\Bigr)^2 + r^2\Bigl(\frac{\partial T\cdot\psi}{\partial v}\Bigr)^2\Bigr\}\Bigr|_{u=\tau_1}$$

$$+\frac{C(\delta,R)}{(\tau_j'')^2}\int_{\Sigma_{\tau_1}}\Bigl(J^T(\phi)+J^T(T\cdot\phi)+\sum_{j=1}^3 J^T(\Omega_j\phi),n\Bigr)\,. \quad \text{(A.76)}$$

Remark A.14. This statement should be compared to Thm. 8, where all that one can assert is

$$\int_{\tau+R}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigl\{ r^2\Bigl(\frac{\partial(r\phi)}{\partial v}\Bigr)^2\Bigr\}\Bigr|_{u=\tau} < \infty \qquad (\tau > \tau_1)\,. \quad \text{(A.77)}$$

We will now proceed along the lines of the proof of Thm. 8, just that we have (A.76) as a starting point for the solution $T\cdot\phi$ of (A.7), (and (A.21)); however, as opposed to Thm. 8 the hierarchy does not descend from $p=2$ but $p<2$, which introduces a degeneracy in the last step, and requires the refinement of Assumption A.4 to Assumption A.6 (and Prop. A.11 to Prop. A.10), summarized in Corollary A.12.

$\boxed{p=2-\delta:}$ By (A.50),

$$\int_{\tau_{2j-1}''}^{\tau_{2j+1}''}\mathrm{d}u\int_{u+R_j'}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times r^{1-\delta}\Bigl(\frac{\partial r\,T\cdot\phi}{\partial v}\Bigr)^2 \leq$$

$$\leq \int_{\tau_{2j-1}''}^{\tau_{2j+1}''}\mathrm{d}u\int_{u+R_j'}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times 2\,\overset{r}{K}_{2-\delta}\,(T\cdot\phi)\,r^2 \leq$$

$$\leq \int_{\tau_{2j-1}''+R_j'}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigl\{ r^{2-\delta}\Bigl(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Bigr)^2\Bigr\}\Bigr|_{u=\tau_{2j-1}''}$$

$$+\int_{2\tau_{2j-1}''+R_j'}^{2\tau_{2j+1}''+R_j'}\mathrm{d}t\int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \frac{1}{2}r^{2-\delta}\Bigl\{\Bigl(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Bigr)^2 + \bigl|\nabla\!\!\!/(r\,T\cdot\phi)\bigr|^2\Bigr\}\Bigr|_{r=R_j'} \leq$$

$$\leq \frac{C(\delta,R)}{(\tau_{2j-1}'')^2}\int_{\tau_1+R_0}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigl\{ r^{4-\delta}\Bigl(\frac{\partial\chi}{\partial v}\Bigr)^2$$

$$+ r^2\Bigl(\frac{\partial\psi}{\partial v}\Bigr)^2 + \sum_{j=1}^3 r^2\Bigl(\frac{\partial\Omega_j\psi}{\partial v}\Bigr)^2 + r^2\Bigl(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Bigr)^2\Bigr\}\Bigr|_{u=\tau_1}$$

$$+\frac{C(\delta,R)}{(\tau_{2j-1}'')^2}\int_{\Sigma_{\tau_1}}\Bigl(J^T(\phi)+J^T(T\cdot\phi)+\sum_{j=1}^3 J^T(\Omega_j\phi),n\Bigr) \quad \text{(A.78)}$$

where in the last step we have chosen $R_j'\in(R_0+1,R_0+2)$ suitably using Assumption A.4 and Prop. A.10, and subsequently applied Thm. 8 to the flux through the past boundary $\Sigma_{\tau_{2j-1}''}$. Therefore there exists a sequence $\tau_j'''\in(\tau_{2j-1}'',\tau_{2j+1}'')$ $(j\in\mathbb{N})$ such that

$$\int_{\tau_j'''+R_j''}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigl\{ r^{1-\delta}\Bigl(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Bigr)^2\Bigr\}\Bigr|_{u=\tau_j'''} \leq$$

$$\leq \frac{C(\delta,R)}{(\tau'''_j)^3}\int_{\tau_1+R_0}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Big)^2+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_1}+\frac{C(\delta,R)}{(\tau'''_j)^3}\int_{\Sigma_{\tau_1}}\Big(J^T(\phi)+J^T(T\cdot\phi)+\sum_{j=1}^{3}J^T(\Omega_j\phi),n\Big)\,. \quad (A.79)$$

$\boxed{p=1-\delta:}$ Since

$$\int_{u+R}^{\infty}\mathrm{d}v\frac{1}{r^\delta}\Big(\frac{\partial\psi}{\partial v}\Big)^2=\int_{u+R}^{\infty}\mathrm{d}v\frac{1}{r^\delta}\Big\{\phi^2+2r\phi\frac{\partial\phi}{\partial v}+r^2\Big(\frac{\partial\phi}{\partial v}\Big)^2\Big\}=\int_{u+R}^{\infty}\mathrm{d}v\frac{1}{r^\delta}\Big\{\frac{\partial(r\phi^2)}{\partial v}+r^2\Big(\frac{\partial\phi}{\partial v}\Big)^2\Big\}=r^{1-\delta}\phi^2\Big|_{u+R}^{\infty}+\int_{u+R}^{\infty}\mathrm{d}v\Big\{\frac{\delta}{r^{1+\delta}}r\phi^2+r^{2-\delta}\Big(\frac{\partial\phi}{\partial v}\Big)^2\Big\} \quad (A.80)$$

we can estimate the degenerate energy density of $T\cdot\phi$ by:

$$\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}u\int_{u+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\frac{1}{r^\delta}\Big\{\Big(\frac{\partial(T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(T\cdot\phi)\big|^2\Big\}r^2\leq \leq\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}u\int_{u+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\frac{1}{r^\delta}\Big\{\Big(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(r\,T\cdot\phi)\big|^2\Big\}+\int_{2\tau'''_{2j-1}+R''_j}^{2\tau'''_{2j+1}+R''_j}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{r^{1-\delta}\Big(T\cdot\phi\Big)^2\Big\}\Big|_{r=R''_j} \quad (A.81)$$

On one hand we have,

$$\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}u\int_{u+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\frac{1}{r^\delta}\Big\{\Big(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(r\,T\cdot\phi)\big|^2\Big\}\leq \leq C(\delta)\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}u\int_{u+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\overset{r}{K}_{1-\delta}\,(T\cdot\phi)\,r^2 \quad (A.82)$$

and on the other hand, *now by Cor. A.12* (recall also the notation introduced on pg. 183), we can choose R''_j such that

$$\int_{2\tau'''_{2j-1}+R''_j}^{2\tau'''_{2j+1}+R''_j}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{r^{1-\delta}\Big(T\cdot\phi\Big)^2\Big\}\Big|_{r=R''_j}\leq C(R)\int_{\Sigma'_{\tau'''_{2j-1},\tau'''_{2j+1}}}\Big(J^T(T\cdot\phi),n\Big)\,. \quad (A.83)$$

Therefore, by virtue of (A.79),

$$\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}u\int_{u+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\frac{1}{r^\delta}\Big\{\Big(\frac{\partial(T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(T\cdot\phi)\big|^2\Big\}r^2\leq$$

$$\leq C(\delta)\int_{\tau'''_{2j-1}+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{r^{1-\delta}\Big(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Big)^2\Big\}\Big|_{u=\tau'''_{2j-1}}$$
$$+C(R)\int_{2\tau'''_{2j-1}+R''_j}^{2\tau'''_{2j+1}+R''_j}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{\Big(\frac{\partial(T\cdot\phi)}{\partial u}\Big)^2+\Big(\frac{\partial(T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(T\cdot\phi)\big|^2\Big\}\Big|_{r=R''_j}$$
$$+C(R)\int_{\Sigma'_{\tau'''_{2j-1},\tau'''_{2j+1}}}\Big(J^T(T\cdot\phi),n\Big)$$
$$\leq\frac{C(\delta,R)}{(\tau'''_{2j-1})^3}\int_{\tau_1+R_0}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big\{r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Big)^2$$
$$+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big\}\Big|_{u=\tau_1}$$
$$+\frac{C(\delta,R)}{(\tau'''_{2j-1})^3}\int_{\Sigma_{\tau_1}}\Big(J^T(\phi)+J^T(T\cdot\phi)+\sum_{j=1}^{3}J^T(\Omega_j\phi),n\Big)$$
$$+C(R)\int_{\Sigma'_{\tau'''_{2j-1},\tau'''_{2j+1}}}\Big(J^T(T\cdot\phi),n\Big)\,. \quad (A.84)$$

Remark A.15. Moreover, by Assumption A.6, we can extend the integral on the left hand side to ${}^{R''_j}\mathcal{P}_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}$ while leaving the right hand side unchanged; i.e. (A.84) holds with the left hand side replaced by

$$\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}\tau\int_{\Sigma_\tau}\zeta_\delta(r)\Big(J^T(T\cdot\phi),n\Big)\qquad\text{, where }\zeta_\delta(r)=\begin{cases}1 & r<R''_j\\ (\frac{R''_j}{r})^\delta & r>R''_j\end{cases}.$$

This inequality can in itself be improved so as to show the same decay rate for all terms on the right hand side; for certainly by Thm. 8 applied to the last term[1],

$$\int_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}\mathrm{d}u\int_{u+R''_j}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\frac{1}{r^\delta}\Big\{\Big(\frac{\partial(T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(T\cdot\phi)\big|^2\Big\}r^2\leq$$
$$\leq\frac{C(\delta,R)}{(\tau'''_{2j-1})^2}\Big\{\int_{\tau_1+R_0}^{\infty}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}\times\Big[r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2+r^2\Big(\frac{\partial(r\,T\cdot\phi)}{\partial v}\Big)^2$$
$$+r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2+\sum_{j=1}^{3}r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2\Big]\Big|_{u=\tau_1}$$
$$+\int_{\Sigma_{\tau_1}}\Big(J^T(\phi)+J^T(T\cdot\phi)+\sum_{j=1}^{3}J^T(\Omega_j\phi),n\Big)\Big\}\,. \quad (A.85)$$

But then, for a sequence $\tau''''_j\in(\tau'''_{2j-1},\tau'''_{2j+1})$, we obtain

$$\int_{\tau''''_j+R''_j}^{\tau'''_{2(j+1)+1}+R''_{j+1}}\mathrm{d}v\int_{\mathbb{S}^2}\mathrm{d}\mu_{\mathring{\gamma}}r^2\times\Big\{\Big(\frac{\partial(T\cdot\phi)}{\partial v}\Big)^2+\big|\nabla\!\!\!/(T\cdot\phi)\big|^2\Big\}\Big|_{u=\tau''''_j}\leq$$

[1]more precisely by applying Theorem 8 to the last term in the analogue of (A.84) that results from not using the refinement in (A.83), but Assumption A.4 and Proposition A.11 as above.

$$\leq \left(\tau'''_{2(j+1)+1} + R''_{j+1} - \tau''''_j\right)^\delta \int_{\tau''''_j + R''_j}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} r^2 \times \frac{1}{r^\delta} \Big\{ \Big(\frac{\partial (T\cdot\phi)}{\partial v}\Big)^2 + \big|\nabla\!\!\!/ (T\cdot\phi)\big|^2 \Big\}\Big|_{u=\tau''''_j}$$

$$\leq \frac{C(\delta,R)}{(\tau''''_j)^{3-\delta}} \Big\{ \int_{\tau_1+R_0}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big[r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial (r\,T\cdot\phi)}{\partial v}\Big)^2 + r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^3 r^2 \Big(\frac{\partial \Omega_j\psi}{\partial v}\Big)^2\Big]\Big|_{u=\tau_1} + \int_{\Sigma_{\tau_1}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^3 J^T(\Omega_j\phi), n\Big)\Big\}; \quad \text{(A.86)}$$

since by Lemma A.7 (compare Remark A.9)

$$\int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}}\, r (T\cdot\phi)^2\Big|_{(u=\tau'''_{2j-1}, v=R''_j+\tau'''_{2j+1})} \leq$$

$$\leq \frac{R}{(\tau'''_{2j+1} + R''_j - \tau'''_{2j-1})} \frac{C(R)}{(\tau'''_{2j-1})^2} \times$$

$$\times \Big\{ \int_{\tau_1+R_0}^\infty \mathrm{d}v \int_s \mathrm{d}\mu_{\overset{\circ}{\gamma}} \Big[r^2 \Big(\frac{\partial (r\,T\cdot\phi)}{\partial v}\Big)^2 + \sum_{j=1}^3 r^2 \Big(\frac{\partial (r\Omega_j T\cdot\phi)}{\partial v}\Big)^2 + \sum_{j,k=1}^3 r^2 \Big(\frac{\partial (r\Omega_j\Omega_k T\cdot\phi)}{\partial v}\Big)^2 \Big] + \int_{\Sigma_{\tau_1}} \Big(J^T(T\cdot\phi) + \sum_{j=1}^3 J^T(\Omega_j T\cdot\phi) + \sum_{j,k=1}^3 J^T(\Omega_j\Omega_k T\cdot\phi), n\Big)\Big\}$$

$$+ \frac{2}{(\tau'''_{2j+1} + R''_j - \tau'''_{2j-1})^{1-\delta}} \sqrt{\int_{R''_j+\tau'''_{2j+1}}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} (T\cdot\phi)^2} \sqrt{\int_{R''_j+\tau'''_{2j+1}}^\infty \mathrm{d}v\, r^{2-\delta} \Big(\frac{\partial (rT\phi)}{\partial v}\Big)^2} \quad \text{(A.87)}$$

this shows in particular (recall here our previous Remark on pg. 195) that

$$\int_{\Sigma'_{\tau'''_{2j-1},\tau'''_{2j+1}}} \Big(J^T(T\cdot\phi), n\Big) \leq$$

$$\leq \int_{\Sigma_{\tau''''_{j-1},\tau'''_{2j+1}}} \Big(J^T(T\cdot\phi), n\Big) + \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \Big(r(T\cdot\phi)^2\Big)\Big|_{(\tau'''_{2j-1}, R''_j+\tau'''_{2j+1})}$$

$$\leq \frac{C(\delta,R)}{(\tau'''_{2j-1})^{3-\delta}} \Big\{ \int_{\tau_1+R_0}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Big[r^{4-\delta}\Big(\frac{\partial\chi}{\partial v}\Big)^2 + r^2 \Big(\frac{\partial (T\cdot\psi)}{\partial v}\Big)^2 + r^2\Big(\frac{\partial\psi}{\partial v}\Big)^2 + \sum_{j=1}^3 r^2\Big(\frac{\partial\Omega_j\psi}{\partial v}\Big)^2 + \sum_{j=1}^3 r^2 \Big(\frac{\partial (\Omega_j T\cdot\psi)}{\partial v}\Big)^2 + \sum_{j,k=1}^3 r^2\Big(\frac{\partial(\Omega_j\Omega_k T\cdot\psi)}{\partial v}\Big)^2\Big]\Big|_{u=\tau_1}$$

$$+ \int_{\Sigma_{\tau_1}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^3 J^T(\Omega_j\phi) + \sum_{j=1}^3 J^T(\Omega_j T\phi) + \sum_{j,k=1}^3 J^T(\Omega_j\Omega_k T\phi)\Big\}. \quad \text{(A.88)}$$

Returning to (A.84) we can now conclude that there is a sequence $\tau''''_j \in (\tau'''_{2j-1}, \tau'''_{2j+1})$ such that

$$\int_{\tau''''_j+R''_j}^\infty \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} r^2 \times \frac{1}{r^\delta}\Big\{ \Big(\frac{\partial (T\cdot\phi)}{\partial v}\Big)^2 + \big|\nabla\!\!\!/ (T\cdot\phi)\big|^2\Big\} \leq \frac{C(\delta,R)}{(\tau''''_j)^{4-\delta}} \times D \qquad \text{(A.89)}$$

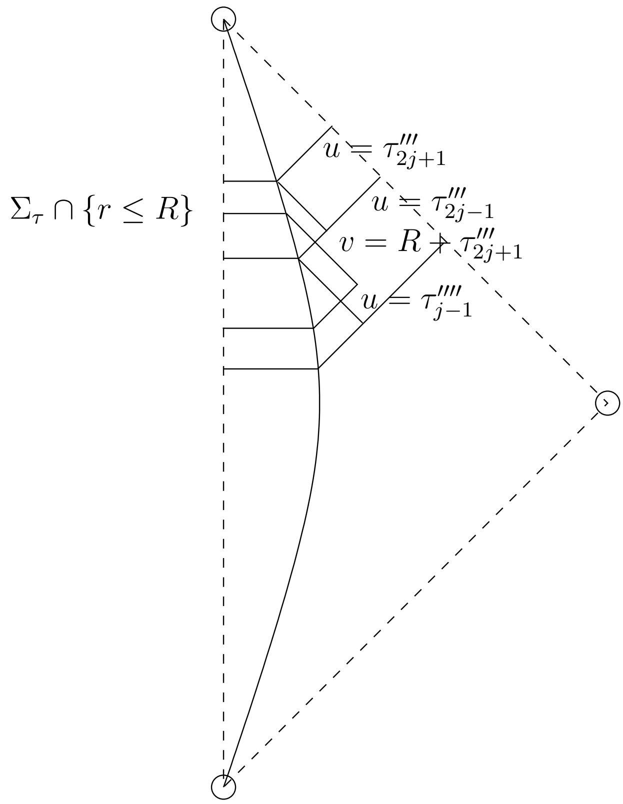

Figure A.2: Removal of the restriction to the dyadic sequence (τ''''_j).

where

$$D = \int_{\tau_1+R_0}^{\infty} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} \times \Bigg\{ r^{4-\delta}\Big(\frac{\partial \chi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial \psi}{\partial v}\Big)^2 + r^2\Big(\frac{\partial T\cdot\psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^2\Big(\frac{\partial \Omega_j \psi}{\partial v}\Big)^2 + \sum_{j=1}^{3} r^2\Big(\frac{\partial \Omega_j T\cdot\psi}{\partial v}\Big)^2 + \sum_{j,k=1}^{3} r^2\Big(\frac{\partial \Omega_j\Omega_k T\cdot\psi}{\partial v}\Big)^2 \Bigg\}\Big|_{u=\tau_1} + \int_{\Sigma_{\tau_1}} \Big(J^T(\phi) + J^T(T\cdot\phi) + \sum_{j=1}^{3} (\Omega_j\phi) + \sum_{j=1}^{3} J^T(\Omega_j T\cdot\phi) + \sum_{j,k=1}^{3} J^T(\Omega_j\Omega_k T\cdot\phi), n \Big)\,; \tag{A.90}$$

or alternatively,

$$\int_{\tau''''_{j-1}+R''_{j-1}}^{\tau'''_{2j-1}+R''_j} \mathrm{d}v \int_{\mathbb{S}^2} \mathrm{d}\mu_{\overset{\circ}{\gamma}} r^2 \times \Bigg\{ \Big(\frac{\partial (T\cdot\phi)}{\partial v}\Big)^2 + \big|\nabla\!\!\!/ (T\cdot\phi)\big|^2 \Bigg\} \leq \frac{C(\delta,R)}{(\tau''''_j)^{4-2\delta}} \times D\,. \tag{A.91}$$

Remark A.16. As previously noted, (as a consequence of Assumption A.6 used for the extension to ${}^{R''_j}\mathcal{P}_{\tau'''_{2j-1}}^{\tau'''_{2j+1}}$ in (A.84)) the left hand side of (A.89) may be replaced by

$$\int_{\Sigma_{\tau''''_j}} \zeta_\delta\Big(J^T(T\cdot\phi), n\Big)$$

while leaving the right hand side unchanged.

Therefore

$$\int_{\Sigma_\tau \cap \{r \leq R\}} \Big(J^T(T \cdot \phi), n \Big) \leq \frac{C(\delta, R)}{\tau^{4-2\delta}} \times D \,, \tag{A.92}$$

because for any given $\tau > \tau_1$ we can choose $j \in \mathbb{N}$ such that $\tau \in (\tau'''_{2j-1}, \tau'''_{2j+1})$ (compare also figure A.2) and

$$\int_{\Sigma_\tau \cap \{r \leq R\}} \Big(J^T(T \cdot \phi), n \Big) \leq \int_{\Sigma_{\tau''''_{j-1}, \tau'''_{2j+1}}} \Big(J^T(T \cdot \phi), n \Big) \,. \tag{A.93}$$

Appendix B

Reference for Chapter 1

B.1 Notation

Contraction

We sum over repeated indices. Also we use interchangeably

$$\begin{aligned} g(V,N) &\doteq (V,N) \doteq V_\mu N^\mu \\ J\cdot N &\doteq (J,N) \doteq J_\mu N^\mu\,, \end{aligned} \tag{B.1}$$

where V, N are vectorfields, and J is a 1-form.

Integration

Let $\mathcal{D}$ in $\mathcal{M}$ be a domain bounded by two homologous hypersurfaces, Σ_1 and Σ_2 being its past and future boundary respectively. We then write $\int_{\Sigma_1}(J,n)$ for the boundary terms on Σ_1 arising from a general current J in the expression $\int_{\partial\mathcal{D}} {}^*J$. If $\mathcal{S}\subset\Sigma_1$ is spacelike, then $(J,n)=g(J,n)$ is in fact the inner product of J with the timelike normal n to Σ_1; e.g. on constant t-slices $\overline{\Sigma}_t$ (see Section 1.2) we have $n=(1-\frac{2m}{r^{n-2}})^{-\frac{1}{2}}\frac{\partial}{\partial t}$. If $\mathcal{U}\subset\Sigma_1$ is an outgoing null segment then $\int_{\mathcal{U}}(J,n)$ denotes an integral of the form $\int \mathrm{d}v\int_{\mathbb{S}}\mathrm{d}\mu_\gamma g(J,\frac{\partial}{\partial v})$; e.g. on the outgoing null segments of the hypersurfaces Σ_τ (see Section 1.4) we have

$$\int_{\Sigma_\tau\cap\{r\geq R\}}(J,n) \doteq \int_{\tau+R^*}^{\infty}\mathrm{d}v^*\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,r^{n-1}\Big(J,\frac{\partial}{\partial v^*}\Big)\,. \tag{B.2}$$

The volume form is usually omitted,

$$\int_{\mathcal{D}} f \doteq \int_{\mathcal{D}} f\;\mathrm{d}\mu_g \qquad (\mathcal{D}\subset\mathcal{M})\,.$$

B.2 Formulas

In this appendix we summarize a few formulas for reference.

The wave equation

The d'Alembert operator can we written out in any coordinate system according to

$$\Box_g \phi = (g^{-1})^{\mu\nu} \nabla_\mu \partial_\nu \phi \tag{B.1}$$

where ∇ denotes the covariant derivative of the Levi-Civita connection of g.

Components of the energy momentum tensor

The components of the energy momentum tensor

$$T_{\mu\nu}(\phi) = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \, \partial^\alpha \phi \, \partial_\alpha \phi$$

tangential to $\mathcal{Q}$ are given in (u^*, v^*)-coordinates by

$$T_{u^*u^*} = \Big(\frac{\partial \phi}{\partial u^*}\Big)^2 \tag{B.2a}$$

$$T_{v^*v^*} = \Big(\frac{\partial \phi}{\partial v^*}\Big)^2 \tag{B.2b}$$

$$T_{u^*v^*} = \Big(1 - \frac{2m}{r^{n-2}}\Big) \big|\nabla\!\!\!/ \phi\big|^2_{r^2 \overset{\circ}{\gamma}_{n-1}} . \tag{B.2c}$$

We also refer to (B.2) as the *null decomposition* of the energy momentum tensor. Note here that

$$\partial^\alpha \phi \, \partial_\alpha \phi = -\frac{1}{1 - \frac{2m}{r^{n-2}}} \Big(\frac{\partial \phi}{\partial u^*}\Big)\Big(\frac{\partial \phi}{\partial v^*}\Big) + \big|\nabla\!\!\!/ \phi\big|^2_{r^2 \overset{\circ}{\gamma}_{n-1}}$$

and

$$\frac{1}{r^2} \overset{\circ}{\gamma}_{n-1}^{\ AB} \, T_{AB} = \big|\nabla\!\!\!/ \phi\big|^2_{r^2 \overset{\circ}{\gamma}_{n-1}} - \frac{1}{2}(n-1) \partial^\alpha \phi \, \partial_\alpha \phi \, .$$

Integration

A typical domain of integration that we use is

$${}^R\mathcal{D}_{\tau_1}^{\tau_2} = \Big\{ (u^*, v^*) : \tau_1 \leq u^* \leq \tau_2 \, , v^* - u^* \geq R^* \Big\} . \tag{B.3}$$

In local coordinates we have, by calculating the volume form from (1.2.25), that

$$\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} \mathrm{d}\mu_g = \int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{u^*+R^*}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \, 2\Big(1 - \frac{2m}{r^{n-2}}\Big) r^{n-1} . \tag{B.4}$$

For a general current J the energy identity on this domain reads

$$\int_{{}^R\mathcal{D}_{\tau_1}^{\tau_2}} K^X \, \mathrm{d}\mu_g = \int_{\partial {}^R\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J \,, \tag{B.5}$$

where the right hand side is given more explicitly by

$$\begin{aligned}\int_{\partial {}^R\mathcal{D}_{\tau_1}^{\tau_2}} {}^*J = &-\int_{R^*+\tau_2}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \; r^{n-1} \, g(J, \frac{\partial}{\partial v^*})|_{u^*=\tau_2} \\ &-\int_{\tau_1}^{\tau_2} \mathrm{d}u^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \; r^{n-1} g(J, \frac{\partial}{\partial u^*})|_{v^*\to\infty} \\ &+\int_{R^*+\tau_1}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \; r^{n-1} \, g(J, \frac{\partial}{\partial v^*})|_{u^*=\tau_1} \\ &-\int_{R^*+2\tau_1}^{R^*+2\tau_2} \mathrm{d}t \int_{\mathbb{S}^{n-1}} r^{n-1} g(J, \frac{\partial}{\partial r^*})|_{r=R} \,.\end{aligned} \tag{B.6}$$

Dyadic sequences

In our argument, Section 1.5.3 in particular, we construct a hierarchy of *dyadic sequences*, beginning with a sequence of real numbers $(\tau_j)_{j\in\mathbb{N}}$ where $\tau_1 > 0$ and $\tau_{j+1} = 2\tau_j$ $(j \in \mathbb{N})$. We then obtain (by the mean value theorem of integration) a sequence $(\tau_j')_{j\in\mathbb{N}}$ with τ_j' in the interval (τ_j, τ_{j+1}) of length τ_j for all $j \in \mathbb{N}$. We then built up on these values another sequence $(\tau_j'')_{j\in\mathbb{N}}$ which takes values (as selected by the mean value theorem) in the intervals $(\tau_{2j-1}', \tau_{2j+1}') \ni \tau_j''$; note that their length is at least $\tau_{2j+1}' - \tau_{2j-1}' \geq \tau_{2j+1} - \tau_{2j} = \tau_{2j}$. In the same fashion the sequence $(\tau_j''')_{j\in\mathbb{N}}$ is built upon $(\tau_j'')_{j\in\mathbb{N}}$, etc.

B.2.1 Rational functions

For completeness, we include in this appendix the elementary integration of the rational function $(x^n - 1)^{-1}$ needed in Section 1.2. Consider the real polynomial of degree $n \in \mathbb{N}$,

$$p(z) = z^n - 1 \quad (z \in \mathbb{C}) \,.$$

The zeros $p(z_j) = 0$ are

$$z_j = e^{2\pi i \frac{j}{n}} \quad (j \in \mathbb{Z})$$

where

$$\begin{aligned}|j| &\leq \frac{n-1}{2} \quad \text{if n is odd} \\ |j| &\leq \frac{n-2}{2}, \; j = \frac{n}{2} \quad \text{if n is even.}\end{aligned}$$

Therefore in factor representation

$$p(z) = \prod_{|j|\leq[\frac{n-1}{2}]} (z - z_j) \begin{cases} 1 & \text{, n odd} \\ (z+1) & \text{, n even} \end{cases}$$

By division into partial fractions there are numbers $c_j \in \mathbb{C}$ such that

$$\frac{1}{p(z)} = \sum_{|j| \le [\frac{n-1}{2}]} \frac{c_j}{z - z_j} + \begin{cases} 0 & \text{, n odd} \\ \frac{c_{\frac{n}{2}}}{z+1} & \text{, n even} \end{cases} .$$

In fact

$$\begin{aligned} c_j &= \lim_{z \to z_j} \frac{z - z_j}{p(z)} = \left(n z_j^{n-1}\right)^{-1} \\ &= \frac{1}{n} e^{-2\pi i j \frac{n-1}{n}} , \end{aligned}$$

so in particular

$$c_0 = \frac{1}{n} \qquad c_{\frac{n}{2}} = -\frac{1}{n} \quad \text{(n even)}.$$

Note that $c_{-j} = \overline{c_j}$. Now, for $x \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{x^n - 1} &= \sum_{j=1}^{[\frac{n-1}{2}]} \left(\frac{c_j}{x - z_j} + \frac{\overline{c_j}}{x - \overline{z_j}}\right) + \frac{c_0}{x-1} + \begin{cases} 0 & \text{, n odd} \\ \frac{c_{\frac{n}{2}}}{x+1} & \text{, n even} \end{cases} \\ &= \sum_{j=1}^{[\frac{n-1}{2}]} \frac{(c_j + \overline{c_j})x - (c_j \overline{z_j} + \overline{c_j} z_j)}{x^2 - (z_j + \overline{z_j})x + |z_j|^2} + \frac{c_0}{x-1} + \begin{cases} 0 & \text{, n odd} \\ \frac{c_{\frac{n}{2}}}{x+1} & \text{, n even} \end{cases} \\ &= \sum_{j=1}^{[\frac{n-1}{2}]} \frac{2}{n} \frac{\cos(2\pi j \frac{n-1}{n})x - 1}{x^2 - 2\cos(\frac{2\pi j}{n})x + 1} + \frac{1}{n}\frac{1}{x-1} + \begin{cases} 0 & \text{, n odd} \\ -\frac{1}{n}\frac{1}{x+1} & \text{, n even} \end{cases} \end{aligned}$$

since

$$\Re[c_j] = \frac{1}{n}\cos(2\pi j \frac{n-1}{n}) \qquad \Re[c_j \overline{z_j}] = \frac{1}{n} .$$

We may now integrate

$$\begin{aligned} &\int \frac{1}{x^n - 1}\,\mathrm{d}x = \\ &= \frac{1}{n} \sum_{j=1}^{[\frac{n-1}{2}]} \cos\big(2\pi j \frac{n-1}{n}\big) \int \frac{2x - 2\cos(\frac{2\pi j}{n})}{x^2 - 2\cos(\frac{2\pi j}{n})x + 1}\,\mathrm{d}x \\ &\quad + \frac{2}{n} \sum_{j=1}^{[\frac{n-1}{2}]} \Big(\cos\big(2\pi j \frac{n-1}{n}\big)\cos\big(\frac{2\pi j}{n}\big) - 1\Big) \int \frac{1}{x^2 - 2\cos(\frac{2\pi j}{n})x + 1}\,\mathrm{d}x \\ &\quad + \frac{1}{n}\int \frac{1}{x-1}\,\mathrm{d}x + \begin{cases} 0 & \text{, n odd} \\ -\frac{1}{n}\int \frac{1}{x+1}\,\mathrm{d}x & \text{, n even} \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{j=1}^{[\frac{n-1}{2}]} \cos\bigl(2\pi j\frac{n-1}{n}\bigr)\log\bigl|x^2 - 2\cos(\frac{2\pi j}{n})x + 1\bigr| \\
&\quad + \frac{2}{n}\sum_{j=1}^{[\frac{n-1}{2}]} \sin\bigl(2\pi j\frac{n-1}{n}\bigr)\arctan\Bigl(\frac{x-\cos\bigl(\frac{2\pi j}{n}\bigr)}{\sin\bigl(\frac{2\pi j}{n}\bigr)}\Bigr) \\
&\quad + \frac{1}{n}\log|x-1| + \begin{cases} 0 & \text{, n odd} \\ -\frac{1}{n}\log|x+1| & \text{, n even} \end{cases}
\end{aligned}
$$

because

$$
\begin{aligned}
&\int \frac{1}{x^2 - 2\cos(\frac{2\pi j}{n}) + 1}\,\mathrm{d}x = \\
&\qquad = \int \frac{1}{t^2 + \sin^2(\frac{2\pi j}{n})}\,\mathrm{d}t\Big|_{t=x-\cos(\frac{2\pi j}{n})} = \frac{1}{\sin(\frac{2\pi j}{n})}\arctan\Bigl(\frac{x-\cos\bigl(\frac{2\pi j}{n}\bigr)}{\sin\bigl(\frac{2\pi j}{n}\bigr)}\Bigr)\ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\exp\Bigl[\int \frac{n}{x^n - 1}\,\mathrm{d}x\Bigr] = \\
&= |x-1| \begin{cases} 1 & \text{, n odd} \\ |x+1|^{-1} & \text{, n even} \end{cases} \\
&\quad \times \prod_{j=1}^{[\frac{n-1}{2}]} \Bigl|x^2 - 2\cos(\frac{2\pi j}{n})x + 1\Bigr|^{\cos(2\pi j\frac{n-1}{n})} \\
&\quad \times \prod_{j=1}^{[\frac{n-1}{2}]} \exp\Bigl[2\sin\bigl(2\pi j\frac{n-1}{n}\bigr)\arctan\bigl(\frac{x-\cos(\frac{2\pi j}{n})}{\sin(\frac{2\pi j}{n})}\bigr)\Bigr]\ .
\end{aligned}
$$

B.2.2 Radial functions

In this appendix we prove some statements on the relation between

$$
r^* = \int_{(nm)^{\frac{1}{n-2}}}^{r} \frac{1}{1 - \frac{2m}{r^{n-2}}}\,\mathrm{d}r \tag{B.7}
$$

and r.

The rational function that essentially appears is integrated already in Appendix B.2.1, and we obtain

$$\begin{aligned}
r^* &= r - (nm)^{\frac{1}{n-2}} + (2m)^{\frac{1}{n-2}} \int_{(\frac{n}{2})^{\frac{1}{n-2}}}^{\frac{r}{\sqrt[n-2]{2m}}} \frac{1}{x^{n-2}-1}\,\mathrm{d}x \\
&= r - (nm)^{\frac{1}{n-2}} + (2m)^{\frac{1}{n-2}} \times \\
&\quad \times \Bigg[\frac{1}{n-2}\log|x-1| + \begin{cases} 0 & \text{n odd} \\ -\frac{1}{n-2}\log|x+1| & \text{n even} \end{cases} \\
&\qquad + \frac{1}{n-2}\sum_{j=1}^{[\frac{n-3}{2}]} \cos\big(2\pi j\frac{n-3}{n-2}\big)\log\Big|x^2 - 2\cos(\frac{2\pi j}{n-2}) + 1\Big| \\
&\qquad + \frac{2}{n-2}\sum_{j=1}^{[\frac{n-3}{2}]} \sin\big(2\pi j\frac{n-3}{n-2}\big)\arctan\Big(\frac{x-\cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big)\Bigg]_{(\frac{n}{2})^{\frac{1}{n-2}}}^{x=\frac{r}{\sqrt[n-2]{2m}}}
\end{aligned}$$

It is useful to rewrite the last two terms as follows:

$$\begin{aligned}
&\sum_{j=1}^{[\frac{n-3}{2}]} \cos\big(2\pi j\frac{n-3}{n-2}\big)\log\Big|x^2 - 2\cos(\frac{2\pi j}{n-2})x + 1\Big| \\
&= \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \cos\Big(2\pi\frac{(n-3)-(j-1)}{n-2}\Big)\log\Big|x^2 - 2\cos(\frac{2\pi j}{n-2})x + 1\Big| \\
&\quad + \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \cos\Big(2\pi\frac{[\frac{n-2}{2}]+j}{n-2}\Big)\log\Big|x^2 + 2\cos(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even} \end{cases}))x + 1\Big| \\
&\quad + \begin{cases} 0 & [\frac{n-3}{2}] \text{ even} \\ \cos\Big(\frac{3}{2}\pi \begin{cases} 1-\frac{1}{3(n-2)} & \text{n odd} \\ 1 & \text{n even} \end{cases}\Big)\log\Big|x^2 - 2\cos(\frac{\pi}{2}\begin{cases} \frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even} \end{cases})x + 1\Big| & [\frac{n-3}{2}] \text{ odd} \end{cases} \\
&= \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \cos(\frac{2\pi j}{n-2})\log\Big|x^2 - 2\cos(\frac{2\pi j}{n-2})x + 1\Big| \\
&\quad - \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \cos\Big(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even} \end{cases})\Big)\log\Big|x^2 + 2\cos(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even} \end{cases}))x + 1\Big| \\
&\quad + \begin{cases} 0 & [\frac{n-3}{2}] \text{ even} \\ -\cos\Big(\frac{\pi}{2}\begin{cases} \frac{n-3}{n-2} & \text{n odd} \\ 1 & \text{n even} \end{cases}\Big)\log\Big|x^2 - 2\cos(\frac{\pi}{2}\begin{cases} \frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even} \end{cases})x + 1\Big| & [\frac{n-3}{2}] \text{ odd} \end{cases}
\end{aligned}$$

$$\sum_{j=1}^{[\frac{n-3}{2}]} \sin\big(2\pi j \frac{n-3}{n-2}\big) \arctan\Big(\frac{x-\cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big)$$

$$= - \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \sin(\frac{2\pi j}{n-2}) \arctan\Big(\frac{x-\cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big)$$

$$- \sum_{j=1}^{[\frac{1}{2}\frac{n-3}{2}]]} \sin\Big(\frac{2\pi}{n-2}\big(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases}\big)\Big) \arctan\Bigg(\frac{x+\cos\big(\frac{2\pi}{n-2}\big(j-\begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases}\big)\big)}{\sin\big(\frac{2\pi}{n-2}\big(j-\begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases}\big)\big)}\Bigg)$$

$$+ \begin{cases} 0 & [\frac{n-3}{2}] \text{ even} \\ -\sin\Big(\frac{\pi}{2}\begin{cases}\frac{n-3}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\Big) \arctan\Bigg(\frac{x-\cos\Big(\frac{\pi}{2}\begin{cases}\frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\Big)}{\sin\Big(\frac{\pi}{2}\begin{cases}\frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\Big)}\Bigg) & [\frac{n-3}{2}] \text{ odd} \end{cases}$$

It is now easy to see that *for $x \geq 1$ these sums are in fact negative or at most zero*, and are bounded below by

$$\sum_{j=1}^{[\frac{n-3}{2}]} \cos\big(2\pi j \frac{n-3}{n-2}\big) \log\Big|x^2 - 2\cos(\frac{2\pi j}{n-2})x + 1\Big|$$

$$\geq \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \log \frac{\big|\sin^2(\frac{2\pi j}{n-2})\big|^{\cos(\frac{2\pi j}{n-2})}}{\big|(x+1)^2\big|} - \cos(\frac{\pi}{2}\frac{n-3}{n-2}) \log\big|(x+1)^2\big|$$

$$\geq \frac{n-3}{2} \log\sin\frac{2\pi}{n-2} - \frac{n+1}{2}\log(1+x)$$

$$\sum_{j=1}^{[\frac{n-3}{2}]} \sin\big(2\pi j \frac{n-3}{n-2}\big) \arctan\Big(\frac{x-\cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\Big) \geq -\frac{n-1}{4}\pi$$

for $x \geq 1$, and of course $n \geq 5$ ($= 0$ for $n = 3, 4$).

Proposition B.1. *For all $n \geq 3$,*

$$\lim_{\frac{r}{\sqrt[n-2]{2m}} \to \infty} \frac{r^*}{r} = 1\,.$$

Proof.

Step 1. (Upper bound on $\frac{r^*}{r}$) Let $r^* \geq 0$, $r \geq \sqrt[n-2]{nm}$. Then using the insights from above

$$\begin{aligned}
r^* \leq\ & r - (nm)^{\frac{1}{n-2}} + (2m)^{\frac{1}{n-2}} \times \\
& \times \Bigg[\frac{1}{n-2} \log \frac{\left| \frac{r}{\sqrt[n-2]{2m}} - 1 \right|}{\left| (\frac{n}{2})^{\frac{1}{n-2}} - 1 \right|} + \begin{cases} 0 & \text{n odd} \\ \frac{1}{n-2} \log \left| (\frac{n}{2})^{\frac{1}{n-2}} + 1 \right| & \text{n even} \end{cases} \\
& \quad + \begin{cases} 0 & n = 3, 4 \\ \frac{1}{n-2} \Big(-\frac{n-3}{2} \log \sin \frac{2\pi}{n-2} + \frac{n+1}{2} \log \big(1 + (\frac{n}{2})^{\frac{1}{n-2}} \big) \Big) & n \geq 5 \\ + \frac{2}{n-2} \frac{n-1}{4} \pi & \end{cases} \Bigg] \\
\leq\ & r - (nm)^{\frac{1}{n-2}} + (2m)^{\frac{1}{n-2}} \times \\
& \times \Bigg[\frac{1}{n-2} \log \frac{\left| \frac{r}{\sqrt[n-2]{2m}} - 1 \right|}{\left| (\frac{n}{2})^{\frac{1}{n-2}} - 1 \right|} \\
& \quad + \begin{cases} \log \frac{5}{2} & n = 3, 4 \\ 3 \log \frac{5}{2} + \pi - \frac{1}{2} \log \sin \frac{2\pi}{n-2} & n \geq 5 \end{cases} \Bigg]
\end{aligned}$$

Since

$$\lim_{x \to \infty} \frac{\log x}{x} = 0$$

we have

$$\lim_{\frac{r}{\sqrt[n-2]{2m}} \to \infty} \frac{r^*}{r} \leq 1 \,.$$

Step 2. (Lower bound on $\frac{r^*}{r}$)

Since $\left(1 - \frac{2m}{r^{n-2}}\right) \leq 1$,

$$r^* = \int_{(nm)^{\frac{1}{n-2}}}^{r} \frac{1}{1 - \frac{2m}{r^{n-2}}} \, \mathrm{d}r \geq r - \sqrt[n-2]{nm} \,.$$

Hence $r \leq \sqrt[n-2]{nm} + r^*$ and

$$\lim_{r^* \to \infty} \frac{r}{r^*} \leq 1 \,.$$

□

While this fact concerns the region $r^* \geq 0$ and is essentially due to $\lim_{x \to \infty} \frac{\log x}{x} = 0$, the next concerns $r^* \leq 0$ and is similarly due to $\lim_{x \to 0} x \log x = 0$.

Proposition B.2. *For all $n \geq 3$,*

$$\lim_{\frac{r}{\sqrt[n-2]{2m}} \to 1} \Big(1 - \frac{2m}{r^{n-2}}\Big)(-r^*) = 0 \,.$$

Proof. Using the formulas from above for $r^* \leq 0$,

$$\begin{aligned}
\big(1-\frac{2m}{r^{n-2}}\big)(-r^*) &= \big(1-\frac{2m}{r^{n-2}}\big)\big(\sqrt[n-2]{nm}-r\big) \\
&\quad + \big(1-\frac{2m}{r^{n-2}}\big)(2m)^{\frac{1}{n-2}}\int_{\frac{r}{\sqrt[n-2]{2m}}}^{(\frac{n}{2})^{\frac{1}{n-2}}}\frac{1}{x^{n-2}-1}\,\mathrm{d}x \\
&\leq \big(1-\frac{2m}{r^{n-2}}\big)\big((\frac{n}{2})^{\frac{1}{n-2}}-1\big)(2m)^{\frac{1}{n-2}} + \big(1-\frac{2m}{r^{n-2}}\big)(2m)^{\frac{1}{n-2}}\times \\
&\quad \times\Bigg[\frac{1}{n-2}\log\frac{\big|(\frac{n}{2})^{\frac{1}{n-2}}-1\big|}{\big|\frac{r}{\sqrt[n-2]{2m}}-1\big|} \\
&\quad + \begin{cases} 0 & n=3,4 \\ -\frac{1}{2}\log\sin\frac{2\pi}{n-2}+\log(\frac{r}{\sqrt[n-2]{2m}}+1)+\pi & n\geq 5\end{cases}\Bigg]
\end{aligned}$$

Now recall

$$x^{n-2}-1=(x-1)\times\prod_{j=1}^{[\frac{n-3}{2}]}\big(x^2-2\cos(\frac{2\pi j}{n-2})x+1\big)\times\begin{cases}1 & \text{n odd}\\ (x+1) & \text{n even}\end{cases}$$

so for $x \geq 1$

$$\begin{aligned}
x^{n-2}-1 &\leq (x-1)(x+1)\times\prod_{j=1}^{[\frac{n-3}{2}]}(x^2+2x+1) \\
&\leq (x-1)(x+1)^{n-2}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\big(1-\frac{2m}{r^{n-2}}\big)\log\frac{\big|(\frac{n}{2})^{\frac{1}{n-2}}-1\big|}{\big|\frac{r}{\sqrt[n-2]{2m}}-1\big|} &\leq \Big(1+\frac{r}{\sqrt[n-2]{2m}}\Big)^{n-2}\times \\
&\times\Big[-\big(\frac{r}{\sqrt[n-2]{2m}}-1\big)\log\big|\frac{r}{\sqrt[n-2]{2m}}-1\big|+\big(\frac{r}{\sqrt[n-2]{2m}}-1\big)\log\big|(\frac{n}{2})^{\frac{1}{n-2}}-1\big|\Big]
\end{aligned}$$

and

$$\begin{aligned}
\big(1-\frac{2m}{r^{n-2}}\big)\Big(\frac{-r^*}{(2m)^{\frac{1}{n-2}}}\Big) &\leq \big(1-\frac{2m}{r^{n-2}}\big)\big((\frac{n}{2})^{\frac{1}{n-2}}-1\big) \\
&\quad -\frac{1}{n-2}\Big(1+(\frac{n}{2})^{\frac{1}{n-2}}\Big)^{n-2}\big(\frac{r}{\sqrt[n-2]{2m}}-1\big)\log\big|\frac{r}{\sqrt[n-2]{2m}}-1\big| \\
&\quad + \big(1-\frac{2m}{r^{n-2}}\big)\begin{cases}0 & n=3,4\\ \frac{1}{2}\log\frac{n-2}{4}+\pi+\log(1+(\frac{n}{2})^{\frac{1}{n-2}}) & n\geq 5\end{cases}
\end{aligned}$$

We see clearly

$$\lim_{\frac{r}{\sqrt[n-2]{2m}}\to 1}\big(1-\frac{2m}{r^{n-2}}\big)(-r^*)\leq 0$$

because

$$\lim_{x\to 0} x\log x = 0\,. \qquad \square$$

While the previous propositions could be proven with fairly rough bounds on r^*, the following propositions concerning the region $r^* \leq 0$ require the (error) terms to be arranged more carefully. With

$$q_0(x) = \frac{x-1}{x+1} \tag{B.8}$$

$$q_{\alpha,\beta}(x) = \frac{(x^2 - 2\alpha x + 1)^\alpha}{(x^2 + 2\beta x + 1)^\beta} \tag{B.9}$$

we see in view of the above that

$$\begin{aligned}
-r^* =& (nm)^{\frac{1}{n-2}} - r + \frac{(2m)^{\frac{1}{n-2}}}{n-2} \log \frac{\left|(\frac{n}{2})^{\frac{1}{n-2}} - 1\right|}{\left|\frac{r}{\sqrt[n-2]{2m}} - 1\right|} \\
&+ \begin{cases} 0 & \text{n odd} \\ -\frac{(2m)^{\frac{1}{n-2}}}{n-2} \log \frac{\left|(\frac{n}{2})^{\frac{1}{n-2}}+1\right|}{\left|\frac{r}{\sqrt[n-2]{2m}}+1\right|} & \text{n even} \end{cases} \\
&+ \frac{(2m)^{\frac{1}{n-2}}}{n-2} \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \log \frac{q_{\cos(\frac{2\pi j}{n-2}),\cos\left(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases})\right)}\left((\frac{n}{2})^{\frac{1}{n-2}}\right)}{q_{\cos(\frac{2\pi j}{n-2}),\cos\left(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases})\right)}\left(\frac{r}{\sqrt[n-2]{2m}}\right)} \\
&+ \begin{cases} 0 & [\frac{n-3}{2}] \text{ even} \\ -\frac{(2m)^{\frac{1}{n-2}}}{n-2} \cos\left(\frac{\pi}{2} \begin{cases} \frac{n-3}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\right) \times & \\ \quad \times \log \frac{\left|(\frac{n}{2})^{\frac{2}{n-2}} - 2\cos\left(\frac{\pi}{2}\begin{cases} \frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\right)(\frac{n}{2})^{\frac{1}{n-2}} + 1\right|}{\left|\frac{r^2}{(2m)^{\frac{2}{n-2}}} - 2\cos\left(\frac{\pi}{2}\begin{cases} \frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\right)\frac{r}{(2m)^{\frac{1}{n-2}}} + 1\right|} & [\frac{n-3}{2}] \text{ odd} \end{cases} \\
&- \frac{2(2m)^{\frac{1}{n-2}}}{n-2} \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \left\{ \int_{\frac{r}{\sqrt[n-2]{2m}}}^{(\frac{n}{2})^{\frac{1}{n-2}}} \frac{1}{1 + \left(\frac{y - \cos(\frac{2\pi j}{n-2})}{\sin(\frac{2\pi j}{n-2})}\right)^2} \mathrm{d}y \right. \\
&\left. \qquad + \int_{\frac{r}{\sqrt[n-2]{2m}}}^{(\frac{n}{2})^{\frac{1}{n-2}}} \frac{1}{1 + \left(\frac{y + \cos\left(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases})\right)}{\sin\left(\frac{2\pi}{n-2}(j - \begin{cases} 1/2 & \text{n odd} \\ 0 & \text{n even}\end{cases})\right)}\right)^2} \mathrm{d}y \right\} \\
&+ \begin{cases} 0 & [\frac{n-3}{2}] \text{ odd} \\ -\frac{2(2m)^{\frac{1}{n-2}}}{n-2} \int_{\frac{r}{\sqrt[n-2]{2m}}}^{(\frac{n}{2})^{\frac{1}{n-2}}} \frac{1}{1 + \left(\frac{y - \cos\left(\frac{\pi}{2}\begin{cases} \frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\right)}{\sin\left(\frac{\pi}{2}\begin{cases} \frac{n-1}{n-2} & \text{n odd} \\ 1 & \text{n even}\end{cases}\right)}\right)^2} \mathrm{d}y & [\frac{n-3}{2}] \text{ even} \end{cases}
\end{aligned}$$

In addition to Prop. B.2 we in fact have:

Proposition B.3. *For $r^* < 0$,*

$$\Big(1 - \frac{2m}{r^{n-2}}\Big) \le \frac{(2m)^{\frac{1}{n-2}}}{(-r^*)}\,.$$

This being an upper bound on $(-r^*)$ we will also need a lower bound:

Proposition B.4. *For $r^* \le 0$,*

$$(-r^*) \ge \frac{(2m)^{\frac{1}{n-2}}}{n-2} \log \frac{q_0\Big(\big(\frac{n}{2}\big)^{\frac{1}{n-2}}\Big)}{q_0\Big(\frac{r}{\sqrt[n-2]{2m}}\Big)}\,.$$

We only give a proof for the case where n *is even* because

$$q_{\alpha,\alpha}(x) = \Big(\frac{x^2 - 2\alpha x + 1}{x^2 + 2\alpha x + 1}\Big)^{\alpha}$$

is monotone increasing which simplifies the proof; indeed

$$q'_{\alpha,\alpha}(x) = 4\alpha^2 \frac{\big(x^2 - 2\alpha x + 1\big)^{\alpha-1}}{\big(x^2 + 2\alpha x + 1\big)^{\alpha+1}}\big(x^2 - 1\big) \ge 0$$

for $0 < \alpha < 1$, $x \ge 1$, and

$$q_{\alpha,\alpha}(1) = \Big(\frac{1-\alpha}{1+\alpha}\Big)^{\alpha}$$

$$\lim_{x\to\infty} q_{\alpha,\alpha}(x) = 1\,.$$

Proof (of Prop.B.3, n even). From the above we get the upper bound

$$\begin{aligned}\frac{(-r^*)}{(2m)^{\frac{1}{n-2}}} \le \big(\frac{n}{2}\big)^{\frac{1}{n-2}} - \frac{r}{(2m)^{\frac{1}{n-2}}} + \frac{1}{n-2} \log \frac{\big|\big(\frac{n}{2}\big)^{\frac{1}{n-2}} - 1\big|}{\big|\frac{r}{\sqrt[n-2]{2m}} - 1\big|} \\ + \frac{1}{n-2} \sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]} \log\Big[\Big(\frac{1+\cos\big(\frac{2\pi j}{n-2}\big)}{1-\cos\big(\frac{2\pi j}{n-2}\big)}\Big)^{\cos(\frac{2\pi j}{n-2})} q_{\cos(\frac{2\pi j}{n-2}),\cos(\frac{2\pi j}{n-2})}\Big(\big(\frac{n}{2}\big)^{\frac{1}{n-2}}\Big)\Big]\end{aligned}$$

Since $(\frac{n}{2})^{\frac{1}{n-2}} \to 1$ the last term tends to zero; in fact it is bounded by $\frac{1}{100}$. Along the lines of the proof of Prop. B.2 we now recall that for n even

$$\begin{aligned}x^{n-2} - 1 &= (x-1)(x+1) \times \prod_{j=1}^{[\frac{n-4}{4}]} \big(x^2 - 2\cos(\frac{2\pi j}{n-2})x + 1\big) \times \begin{cases} 1 & \frac{n-4}{2} \text{ even} \\ x^2+1 & \frac{n-4}{2} \text{ odd}\end{cases} \\ &= (x-1) \times q_{n-2}(x)\end{aligned}$$

thus

$$1 - \frac{1}{x^{n-2}} = \frac{1}{x^{n-2}}(x-1)q_{n-2}(x)$$

and

$$\lim_{x\to 1}\frac{1}{x^{n-2}}q_{n-2}(x)=\lim_{x\to 1}\frac{1}{x^{n-2}}\frac{x^{n-2}-1}{x-1}=n-2\,;$$

in fact

$$\frac{1}{n-2}\big(1-\frac{1}{x^{n-2}}\big)\leq x-1\,.$$

Therefore,

$$\begin{aligned}\big(1-\frac{2m}{r^{n-2}}\big)\frac{(-r^*)}{(2m)^{\frac{1}{n-2}}}\leq&\big((\frac{n}{2})^{\frac{1}{n-2}}-1\big)(1-\frac{2}{n})\\&-\big(\frac{r}{\sqrt[n-2]{2m}}-1\big)\log\frac{|\frac{r}{\sqrt[n-2]{2m}}-1|}{|(\frac{n}{2})^{\frac{1}{n-2}}-1|}+\frac{1}{100}\big(1-\frac{2}{n}\big)\\\leq&\frac{1}{2}+\frac{1}{4}+\frac{1}{100}\leq 1\end{aligned}$$

□

Proof (of Prop. B.4, n even). Substracting the last terms from the first we get,

$$\begin{aligned}(-r^*)\geq&\frac{1}{n-2}\big((nm)^{\frac{1}{n-2}}-r\big)+\frac{(2m)^{\frac{1}{n-2}}}{n-2}\log\frac{q_0\Big((\frac{n}{2})^{\frac{1}{n-2}}\Big)}{q_0\Big(\frac{r}{\sqrt[n-2]{2m}}\Big)}\\&+\frac{(2m)^{\frac{1}{n-2}}}{n-2}\sum_{j=1}^{[\frac{1}{2}[\frac{n-3}{2}]]}\log\frac{q_{\cos(\frac{2\pi j}{n-2}),\cos(\frac{2\pi j}{n-2})}\Big((\frac{n}{2})^{\frac{1}{n-2}}\Big)}{q_{\cos(\frac{2\pi j}{n-2}),\cos(\frac{2\pi j}{n-2})}\Big(\frac{r}{\sqrt[n-2]{2m}}\Big)}\\\geq&\frac{(2m)^{\frac{1}{n-2}}}{n-2}\log\frac{q_0\Big((\frac{n}{2})^{\frac{1}{n-2}}\Big)}{q_0\Big(\frac{r}{\sqrt[n-2]{2m}}\Big)}\end{aligned}$$

because $q_{\alpha,\alpha}$ is monotone increasing. □

B.3 Boundary Integrals and Hardy Inequalities

In this appendix we prove appropriate Hardy inequalities that are needed in our argument to estimate boundary terms that typically arise in the energy identities.

X-type currents. Let $X=f(r^*)\frac{\partial}{\partial r^*}$ and recall the modification (1.4.15).

Proposition B.5 (Boundary terms near null infinity)**.** *Let $f=\mathcal{O}(1)$, $f'=\mathcal{O}(\frac{1}{r})$, and $f''=\mathcal{O}(\frac{1}{r^2})$, then there exists a constant $C(n,m)$ such that*

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}\setminus\{r=R\}}{}^*J^{X,1}\leq C(n,m)\int_{\Sigma_{\tau_1}}\Big(J^T(\phi),n\Big)\,. \tag{B.1}$$

Proof. For the boundary integrals on the null segments $u^* = \tau_1, \tau_2$ we find

$$\Big|\int_{R^*+\tau_i}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} g(J^{X,1}, \frac{\partial}{\partial v^*})\, r^{n-1}\Big| \leq \\ \leq C(n) \int_{R^*+\tau_i}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\, r^{n-1} \Big\{ \Big(\frac{\partial\phi}{\partial v^*}\Big)^2 + \big|\nabla\!\!\!/\phi\big|^2 \\ + \Big[\frac{|f|}{r^2} + \frac{|f'|}{r} + |f'|^2 + |f''|\Big]\phi^2 \Big\} \quad \text{(B.2)}$$

and in view of the Hardy inequality Lemma B.6

$$\int_{R^*+\tau_i}^{\infty} \mathrm{d}v^* \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} \frac{1}{r^2}\phi^2\, r^{n-1}|_{u^*=\tau_i} \leq C(n,m) \int_{\Sigma_{\tau_i}} \Big(J^T(\phi), n\Big)\,; \quad \text{(B.3)}$$

note that the corresponding zero order terms vanish at future null infinity, cf. Remark B.7. Then (B.1) follows from the energy identity for T on ${}^R\mathcal{D}_{\tau_1}^{\tau_2}$. □

Lemma B.6 (Hardy inequality). *Let $\phi \in \mathrm{C}^1([a,\infty))$, $a > 0$, with $|\phi(a)| < \infty$ and*

$$\lim_{x\to\infty} x^{\frac{n-2}{2}}\phi(x) = 0\,, \quad \text{(B.4)}$$

then a constant $C(n) > 0$ exists such that

$$\int_a^{\infty} \frac{1}{x^2}\phi^2(x)\, x^{n-1}\,\mathrm{d}x \leq C(n) \int_a^{\infty} (\frac{\mathrm{d}\phi}{\mathrm{d}x})^2\, x^{n-1}\,\mathrm{d}x\,. \quad \text{(B.5)}$$

Proof. This is a consequence of the Cauchy-Schwarz inequality, after integration by parts

$$\int_a^{\infty} \frac{1}{x^2}\phi^2(x)\, x^{n-1}\,\mathrm{d}x = \int_a^{\infty} g'(x)\phi^2(x)\,\mathrm{d}x$$

with

$$g(x) = \int_a^x y^{n-3}\,\mathrm{d}y\,.$$ □

Remark B.7. The conditions of the Lemma on ϕ are in fact satisfied for any solution of the wave equation (1.1.1). By a density argument we may assume without loss of generality that the initial data is compactly supported. Then for a fixed τ, and v^* large enough $\phi(\tau, v^*) = 0$ and for $u^* \geq \tau$

$$\phi(u^*, v^*) = \int_{\tau}^{u^*} \frac{\partial\phi}{\partial u^*}\,\mathrm{d}u^*\,.$$

Thus

$$\phi(u^*, v^*) \leq \Big(\int_{\tau}^{u^*} (\frac{\partial\phi}{\partial u^*})^2\, r^{n-1}\,\mathrm{d}u^*\Big)^{\frac{1}{2}} \Big(\int_{\tau}^{u^*} \frac{1}{r^{n-1}}\,\mathrm{d}u^*\Big)^{\frac{1}{2}}.$$

On one hand

$$\int_{\tau}^{u^*} \int_{\mathbb{S}^{n-1}} (\frac{\partial\phi}{\partial u^*})^2\, r^{n-1}\,\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}\,\mathrm{d}u^* \leq \int_{\Sigma_\tau} \big(J^T(\phi), n\big) < \infty\,,$$

whereas on the other hand

$$\int_\tau^{u^*} \frac{1}{r^{n-1}}\,\mathrm{d}u^* = \frac{1}{n-2}\int_\tau^{u^*}(1-\frac{2m}{r^{n-2}})^{-1}\frac{\partial}{\partial u^*}(\frac{1}{r^{n-2}})\,\mathrm{d}u^*$$
$$\leq \frac{1}{n-2}\Big(1-\frac{2m}{R^{n-2}}\Big)^{-1}\Big(1-\Big(\frac{r(u^*,v^*)}{r(\tau,v^*)}\Big)^{n-2}\Big)\frac{1}{r^{n-2}}$$

if we restrict $u^*\geq\tau$ to $r(u^*,v^*)\geq R$. Hence

$$\lim_{v^*\to\infty} r^{\frac{n-2}{2}}\phi = 0\,.$$

Instead of (B.5) which requires (B.4) one can prove the corresponding Hardy inequality for finite intervals:

Lemma B.8 (Hardy inequality for finite intervals)**.** *Let* $0<a<b$, *and* $\phi\in\mathrm{C}^1((a,b))$ *then*

$$\frac{1}{2}\int_a^b\frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x \leq \frac{1}{n-2}b^{n-2}\phi^2(b)+2\Big(\frac{2}{n-2}\Big)^2\int_a^b\Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2x^{n-1}\,\mathrm{d}x\,. \tag{B.6}$$

Proof. Let

$$g(x)=\int_a^x y^{n-3}\,\mathrm{d}y=\frac{1}{n-2}y^{n-2}|_a^x$$

then by integration by parts and using Cauchy's inequality

$$\int_a^b\frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x = g\,\phi^2|_a^b-\int_a^b g(x)2\phi(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x\leq$$
$$\leq g(b)\phi^2(b)+2\epsilon\int_a^b\frac{1}{x^2}\phi^2(x)x^{n-1}\,\mathrm{d}x+\frac{1}{2\epsilon}\int_a^b\frac{g(x)^2}{x^{n-3}}\Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2\mathrm{d}x$$

where $\epsilon>0$; (B.6) follows for $\epsilon=\frac{1}{4}$ because

$$g(b)\leq\frac{1}{n-2}b^{n-2}$$
$$\frac{g(x)^2}{x^{n-3}}\leq\frac{2}{n-2}\Big(1+(\frac{a}{x})^{2(n-2)}\Big)x^{n-1}\,.$$

□

Recall the domain (1.4.105); by using Lemma B.8 instead of Lemma B.6 we can prove the following refinement of Prop. B.5 to bounded domains:

Proposition B.9 (Boundary terms on bounded domains)**.** *Let* $f=\mathcal{O}(1)$, $f'=\mathcal{O}(\frac{1}{r})$, *and* $f''=\mathcal{O}(\frac{1}{r^2})$, *then there exists a constant* $C(n,m)$ *such that*

$$\int_{\partial^R\mathcal{D}_{\tau_1}^{\tau_2}\setminus\{r=R\}}{}^*J^{X,1}\leq$$
$$\leq C(n,m)\Big\{\int_{\Sigma_{\tau_1}^{\tau_2}}\Big(J^T(\phi),n\Big)+\int_{\mathbb{S}^{n-1}}\mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}}r^{n-2}\phi^2|_{(u^*=\tau_1,v^*=R^*+\tau_2)}\Big\}\,. \tag{B.7}$$

Recall the domain (1.4.2).

Proposition B.10 (Boundary terms near the event horizon)**.** *Let* $f = \mathcal{O}(1)$*,* $f' = \mathcal{O}(\frac{1}{|r^*|^4})$*, and* $f'' = \mathcal{O}(\frac{1}{|r^*|^5})$*, and*

$$\pi_l \phi = 0 \qquad (0 \leq l < L)\,,$$

for some $L \in \mathbb{N}$*, then there exists a constant* $C(n, m, L)$ *such that*

$$\int_{\partial \mathcal{R}^{\infty}_{\tau_0, r_1}(t_0)} {}^* J^{X,1} \leq C(n, m, L) \int_{\Sigma_{\tau_0}} \Big(J^T(\phi), n \Big)\,. \tag{B.8}$$

where $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$*.*

The proof is given in Section 1.4.4 in the special case $f = f_{\gamma,\alpha}$ using the following Lemma.

Lemma B.11 (Hardy inequality)**.** *Let* $a > 0$*,* $\phi \in \mathrm{C}^1([a, \infty))$ *with*

$$\lim_{x \to \infty} |\phi(x)| < \infty\,.$$

Then

$$\int_a^\infty \frac{1}{1 + x^2} \phi^2(x)\,\mathrm{d}x \leq$$

$$\leq 8 \frac{1 + a^2}{a^2} \int_a^\infty \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x + 2\pi \int_a^{a+1} \Big\{ \phi^2 + \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2 \Big\}\,\mathrm{d}x\,. \tag{B.9}$$

Proof. Let us first assume that $\phi(a) = 0$. Define

$$g(x) = -\int_x^\infty \frac{1}{1 + y^2}\,\mathrm{d}y$$

then

$$\begin{aligned}\int_a^\infty \frac{1}{1 + x^2} \phi^2(x)\,\mathrm{d}x &= \int_a^\infty g'(x) \phi^2(x)\,\mathrm{d}x \\ &= g(x)\phi^2(x)\big|_a^\infty - 2 \int_a^\infty g(x)\phi(x) \frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x \\ &\leq 2 \Big(\int_a^\infty \frac{g(x)^2}{g'(x)} \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_a^\infty g'(x)\phi^2(x)\,\mathrm{d}x \Big)^{\frac{1}{2}}\,.\end{aligned}$$

Since $|g(x)| \leq \frac{1}{x}$ we have

$$\frac{g(x)^2}{g'(x)} \leq \frac{1 + x^2}{x^2} \leq \frac{1 + a^2}{a^2}$$

and therefore

$$\int_a^\infty \frac{1}{1 + x^2} \phi^2(x)\,\mathrm{d}x \leq 4 \int_a^\infty \frac{g(x)^2}{g'(x)} \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x \leq 4 \frac{1 + a^2}{a^2} \int_a^\infty \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x\,.$$

Without the assumption $\phi(a) = 0$ this applied to the function $\phi(x) - \phi(a)$ yields

$$\int_a^\infty \frac{1}{1+x^2}\phi^2(x)\,\mathrm{d}x \leq$$
$$\leq 2\int_a^\infty \frac{1}{1+x^2}\big(\phi(x)-\phi(a)\big)^2\,\mathrm{d}x + 2\int_a^\infty \frac{1}{1+x^2}\phi(a)^2\,\mathrm{d}x$$
$$\leq 8\frac{1+a^2}{a^2}\int_a^\infty \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\,\mathrm{d}x + \pi\phi(a)^2\,.$$

We conclude the proof with the following pointwise bound: On one hand for some $a' \in (a, a+1)$

$$\int_a^{a+1} \phi(x)^2\,\mathrm{d}x = \phi(a')^2$$

and on the other hand

$$\phi(a')^2 - \phi(a)^2 = \int_a^{a'} \frac{\mathrm{d}}{\mathrm{d}x}\phi(x)^2\,\mathrm{d}x \leq \int_a^{a'} \Big\{\phi(x)^2 + \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\Big\}\,\mathrm{d}x\,.$$

Hence

$$\phi(a)^2 \leq \int_a^{a'} \Big\{\phi(x)^2 + \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\Big\}\,\mathrm{d}x + \int_a^{a+1} \phi(x)^2\,\mathrm{d}x$$
$$\leq 2\int_a^{a+1} \Big\{\phi(x)^2 + \big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\big)^2\Big\}\,\mathrm{d}x\,. \quad \square$$

Auxiliary currents. For auxiliary currents of the form

$$J^{\mathrm{aux}}_\mu = \frac{1}{2}h(r)\partial_\mu(\phi^2) \tag{B.10}$$

we have the same results.

Proposition B.12. *Let $h = \mathcal{O}(\frac{1}{r})$, then there exists a constants $C(n,m)$ such that*

$$\int_{\partial^R\mathcal{D}^{\tau_2}_{\tau_1}\setminus\{r=R\}} {}^*J^{aux} \leq C(n,m)\int_{\Sigma_{\tau_1}} \Big(J^T(\phi), n\Big)\,, \tag{B.11}$$

and moreover for a constant $C(n,m)$ we have the refinement

$$\int_{\partial^R\mathcal{D}^{\tau_2}_{\tau_1}\setminus\{r=R\}} {}^*J^{aux} \leq$$
$$\leq C(n,m)\Big\{\int_{\Sigma^{\tau_2}_{\tau_1}} \Big(J^T(\phi), n\Big) + \int_{\mathbb{S}^{n-1}} \mathrm{d}\mu_{\overset{\circ}{\gamma}_{n-1}} r^{n-2}\phi^2|_{(\tau_1, R^*+\tau_2)}\Big\}\,. \tag{B.12}$$

Proof. Note that here, in comparison to the proof of Prop. B.5,

$$\Big|g(J^{\mathrm{aux}}, \frac{\partial}{\partial v^*})\Big| \leq h^2\phi^2 + \Big(\frac{\partial\phi}{\partial v^*}\Big)^2\,. \quad \square$$

Proposition B.13. *Let $h = \mathcal{O}(\frac{1}{|r^*|})$, then there exists a constant $C(n,m)$ such that*

$$\int_{\partial \mathcal{R}^{\infty}_{r_0,r_1}(t_0)} {}^*J^{aux} \leq C(n,m) \int_{\Sigma_{\tau_0}} \Big(J^T(\phi), n \Big) . \tag{B.13}$$

where $\tau_0 = \frac{1}{2}(t_0 - r_1^*)$.

Remark B.14. Note that in view of Prop. B.3 the function $h = \frac{1}{r}\Big(1 - \frac{2m}{r^{n-2}}\Big)$ satisfies the assumption of the Proposition.

Appendix C

Reference for Chapter 2

C.1 Decomposition formulas

In this appendix we shall carry out several decompositions relative to the foliation of $\mathcal{R}$ by level sets of the area radius. Computations are significantly simplified by the use of the coordinates (r,σ) introduced in Section 2.2.1.3 which defines a "convenient frame" in the sense that

$$[\frac{\partial}{\partial r},\frac{\partial}{\partial \sigma}]=0\,. \tag{C.1}$$

Recall the first fundamental form

$$\overline{g}_{\sigma\sigma}=\frac{1}{4}\frac{3}{\Lambda}\Bigl(\frac{\Lambda}{3}r^2-1\Bigr)\frac{1}{\sigma^2} \tag{C.2a}$$

$$\overline{g}_{\sigma A}=0 \qquad \overline{g}_{AB}=r^2\,\overset{\circ}{\gamma}_{AB}\,, \tag{C.2b}$$

and by the first variational formula we have

$$k_{ij}=\frac{1}{2\phi}\frac{\partial\overline{g}_{ij}}{\partial r} \tag{C.3}$$

for the second fundamental form k of the foliation (Σ_r). Thus it is easily checked that

$$\mathrm{tr}\,k=\overline{g}_r^{ij}k_{ij}=\frac{\Lambda}{3}\phi\,r+\frac{2}{r}\frac{1}{\phi} \tag{C.4a}$$

$$|k|^2=\overline{g}_r^{im}\overline{g}_r^{jn}k_{mn}k_{ij}=\Bigl(\frac{\Lambda}{3}\phi\,r\Bigr)^2+\frac{2}{(r\phi)^2}\,, \tag{C.4b}$$

and

$$\frac{\partial\,\mathrm{tr}\,k}{\partial r}=\Lambda\phi-\phi|k|^2 \tag{C.5}$$

in agreement with the general second variation formula.

Wave equation. Let us derive the explicit form of the wave equation

$$\Box_g \psi = 0 \tag{C.6}$$

for a metric of the form (2.2.39), i.e.

$$g = -\phi^2 \, \mathrm{d}r^2 + \overline{g}_r \, . \tag{C.7}$$

Since

$$\Gamma^r_{rr} = \frac{1}{\phi}\frac{\partial \phi}{\partial r} \tag{C.8a}$$

$$\Gamma^i_{rr} = \phi \, \overline{g}_r^{\;ij} \, \partial_j \phi \tag{C.8b}$$

$$\Gamma^r_{ij} = \frac{1}{2}\frac{1}{\phi^2}\, \partial_r \overline{g}_{r\,ij} = \frac{1}{\phi}\, k_{ij} \tag{C.8c}$$

we obtain

$$\Box_g \psi = (g^{-1})^{rr} \nabla_r \partial_r \psi + (g^{-1})^{ij} \nabla_i \partial_j \psi = \\ = -\frac{1}{\phi^2}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{\phi}\Big[\frac{1}{\phi^2}\frac{\partial \phi}{\partial r} - \mathrm{tr}\, k\Big]\frac{\partial \psi}{\partial r} + \frac{1}{\phi} \overline{\nabla} \phi \cdot \overline{\nabla} \psi + \overline{\triangle} \psi \, . \tag{C.9}$$

For the lapse (2.2.38) we have

$$\frac{1}{\phi^2}\frac{\partial \phi}{\partial r} = -\frac{\Lambda}{3} \phi \, r \tag{C.10}$$

$$\overline{\nabla} \phi = 0 \, , \tag{C.11}$$

so that here

$$\Box_g \psi = -\frac{1}{\phi^2}\frac{\partial^2 \psi}{\partial r^2} - 2\Big[\frac{\Lambda}{3}\big(r\phi\big) + \frac{1}{(r\phi)}\Big]\frac{1}{\phi}\frac{\partial \psi}{\partial r} + \overline{\triangle}\psi \, . \tag{C.12}$$

C.2 Coercivity inequality on the sphere

In this appendix we shall prove the coercivity formula for the standard sphere, and recall the classic Sobolev inequality on the sphere.

Let us denote by

$$S_r = \Big\{ x \in \mathbb{R}^3 : \quad |x| = r \Big\} \tag{C.1}$$

the sphere of radius r in Euclidean space, a submanifold of

$$\Big(\mathbb{R}^3 , e = (\,\mathrm{d}x^1)^2 + (\,\mathrm{d}x^2)^2 + (\,\mathrm{d}x^3)^2 \Big). \tag{C.2}$$

We denote the metric of the round sphere S_r by

$$\gamma_r = e \big|_{\mathrm{T}S_r} = r^2 \, \mathring{\gamma} \, , \tag{C.3}$$

where $\overset{\circ}{\gamma}$ is the standard metric on the unit sphere $\mathbb{S}^2$.

Let

$$\Omega_{(i)} = \epsilon_{ijk}\, x^j \,\frac{\partial}{\partial x^k} \qquad i = 1,2,3\,, \tag{C.4}$$

where ϵ is the volume form of e. We have

$$\sum_{i=1}^{3} \Omega^m_{(i)} \Omega^n_{(i)} = |x|^2 \delta_{mn} - x^n x^m\,, \tag{C.5}$$

and thus for all $x \in \mathbb{R}^3 \backslash \{0\}$ and $u : \mathbb{R}^3 \to \mathbb{R}$ differentiable:

$$\sum_{i=1}^{3} \bigl(\Omega_{(i)} u\bigr)^2(x) = |x|^2 \Bigl[|\nabla u|^2 - \bigl\langle \frac{x}{|x|}, \nabla u \bigr\rangle^2\Bigr] = |x|^2 |\nabla\!\!\!/ u|^2\,. \tag{C.6}$$

This is the coercivity formula on the sphere. Here $\nabla\!\!\!/ = \Pi\nabla$, and

$$\Pi^b_a(\xi) = \delta^b_a - \xi_a\, \xi^b\,, \qquad \xi \in \mathbb{S}^2\,, \tag{C.7}$$

is the projection to the sphere; by uniqueness $\nabla\!\!\!/$ is the connection of γ_r.

Lemma C.1 (Coercivity inequalities on the sphere)**.** *Let u be a smooth function on S_r, then*

$$r^2 |\nabla\!\!\!/ u|^2_{\gamma_r} \leq \sum_{i=1}^{3} \bigl(\Omega_{(i)} u\bigr)^2\,, \tag{C.8}$$

and

$$r^4 |\nabla\!\!\!/^2 u|^2_{\gamma_r} \leq \sum_{i,j=1}^{3} \bigl(\Omega_{(i)} \Omega_{(j)} u\bigr)^2\,. \tag{C.9}$$

We have already shown the first inequality. For the second inequality we can use that more generally for any S_r-1-form (a 1-form on $\mathbb{R}^3$ such that $\theta \cdot X = \theta \cdot \Pi X$) it holds

$$\sum_{i=1}^{3} \bigl|\mathcal{L}\!\!\!/_{\Omega_{(i)}} \theta\bigr|^2_{\gamma_r} = r^2 \bigl|\nabla\!\!\!/ \theta\bigr|^2_{\gamma_r} + \bigl|\theta\bigr|^2_{\gamma_r}\,, \tag{C.10}$$

where $\mathcal{L}\!\!\!/$ denotes the Lie derivative on S_r. (To prove (C.10) one can proceed analogeously to Lemma 11.2 in [14].) By substituting

$$\theta = d\!\!\!/ u \doteq \mathrm{d}u|_{\mathrm{T}S_r}\,, \tag{C.11}$$

we then obtain

$$r^2 \bigl|\nabla\!\!\!/^2 u\bigr|^2_{\gamma_r} = r^2 \bigl|\nabla\!\!\!/\, d\!\!\!/ u\bigr|^2_{\gamma_r} \leq$$

$$\leq \sum_{i=1}^{3} \bigl|\mathcal{L}\!\!\!/_{\,\Omega_{(i)}}\, d\!\!\!/ u\bigr|^2_{\gamma_r} = \sum_{i=1}^{3} \bigl|\, d\!\!\!/ \mathcal{L}\!\!\!/_{\,\Omega_{(i)}} u\bigr|^2_{\gamma_r} = \sum_{i=1}^{3} \bigl|\nabla\!\!\!/ (\Omega_{(i)})\bigr|^2_{\gamma_r}\,, \tag{C.12}$$

because Lie derivatives commute with exterior derivatives. Inequality (C.9) then follows from (C.12) using (C.8).

We recall the classical Sobolev inequality on the sphere.

Lemma C.2 (Sobolev embedding on $\mathbb{S}^2$)**.** *Let $u \in \mathrm{H}^2(\mathbb{S}^2)$, then $u \in \mathrm{L}^\infty(\mathrm{S}^2)$ and*

$$\|u\|_{\mathrm{L}^\infty(\mathbb{S}^2)} \leq C\,\|u\|_{\mathrm{H}^2(\mathbb{S}^2)}\,. \tag{C.13}$$

This can be shown similarly to the Sobolev embedding on $\mathbb{R}^2$ using the fact that $\overset{\circ}{\gamma}$ is conformal to the Euclidean metric on the plane. Indeed, given any $x \in \mathbb{S}^2 \subset \mathbb{R}^3$, we can introduce stereograhic coordinates $y \in \mathbb{R}^2$, such that $y(x) = 0$ and $\overset{\circ}{\gamma}$ takes the form

$$\overset{\circ}{\gamma} = \frac{1}{(1 + \frac{1}{4}|y|^2)^2}|\,\mathrm{d}y|^2\,. \tag{C.14}$$

It is then easy to verify Morrey's inequality, namely that $\mathrm{W}^{1,p}(\mathbb{S}^2) \subset \mathrm{L}^\infty(\mathbb{S}^2)$ for all $p > 2$. On the other hand, using two stereographic charts to cover the sphere, it is easy to check that also the classic embedding $\mathrm{W}^{1,1}(\mathbb{S}^2) \subset \mathrm{L}^2(\mathbb{S}^2)$ is valid, and therefore also $\mathrm{W}^{1,q}(\mathbb{S}^2) \subset \mathrm{L}^p(\mathbb{S}^2)$ for $1/p = 1/q - 1/2$. Thus in particular $\mathrm{W}^{1,\frac{3}{2}}(\mathbb{S}^2) \subset \mathrm{L}^6(\mathbb{S}^2)$, while $\mathrm{W}^{1,6}(\mathbb{S}^2) \subset \mathrm{L}^\infty(\mathbb{S}^2)$ and

$$\begin{aligned}\|u\|_\infty &\leq C\Big(\|u\|_{\mathrm{L}^6(\mathbb{S}^2)} + \|\nabla u\|_{\mathrm{L}^6(\mathbb{S}^2)}\Big) \leq \\ &\leq C\Big(\|u\|_{\mathrm{L}^{\frac{3}{2}}(\mathbb{S}^2)} + \|\nabla u\|_{\mathrm{L}^{\frac{3}{2}}(\mathbb{S}^2)} + \|\nabla^2 u\|_{\mathrm{L}^{\frac{3}{2}}(\mathbb{S}^2)}\Big) \leq C\|u\|_{\mathrm{H}^2(\mathbb{S}^2)}\,.\end{aligned} \tag{C.15}$$

Given a function u on S_r we can apply Lemma C.2 to $u \circ h$, where

$$h : \mathbb{S}^2 \to S_r, \quad \xi \mapsto r\xi\,. \tag{C.16}$$

Since, in the coordinates

$$\gamma_r = r^2 \overset{\circ}{\gamma} = r^2 \overset{\circ}{\gamma}_{AB}\ \mathrm{d}y^A\,\mathrm{d}y^B\,, \tag{C.17}$$

h is the identity mapping, we get

$$|\overset{\circ}{\nabla}(u \circ h)|^2_{\overset{\circ}{\gamma}} = r^2|(\nabla u) \circ h|^2_{\gamma_r}\,, \quad |\overset{\circ}{\nabla}{}^2(u \circ h)|^2_{\overset{\circ}{\gamma}} = r^4|(\nabla^2 u) \circ h|^2_{\gamma_r}\,, \tag{C.18}$$

and thus

$$\begin{aligned} r|u|\Big|_{S_r} &\leq C\Big(\int_{S_r}|u|^2\,\mathrm{d}\mu_{\gamma_r}\Big)^{\frac{1}{2}} + C\Big(\int_{S_r} r^2|\nabla u|^2\,\mathrm{d}\mu_{\gamma_r}\Big)^{\frac{1}{2}} \\ &\quad + C\Big(\int_{S_r} r^4|\nabla^2 u|^2\,\mathrm{d}\mu_{\gamma_r}\Big)^{\frac{1}{2}}\,.\end{aligned} \tag{C.19}$$

Corollary C.3. *Let $u \in \mathrm{H}^2(S_r)$, then*

$$\begin{aligned} r|u|\Big|_{S_r} &\leq C\Big(\int_{S_r}|u|^2\,\mathrm{d}\mu_{\gamma_r}\Big)^{\frac{1}{2}} + C\Big(\int_{S_r}\sum_{i=1}^{3}\big(\Omega_{(i)}u\big)^2\,\mathrm{d}\mu_{\gamma_r}\Big)^{\frac{1}{2}} \\ &\quad + C\Big(\int_{S_r}\sum_{i,j=1}^{3}\big(\Omega_{(i)}\Omega_{(j)}u\big)^2\,\mathrm{d}\mu_{\gamma_r}\Big)^{\frac{1}{2}}\,.\end{aligned} \tag{C.20}$$

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