ON THE LIFESPAN OF SOLUTIONS OF NONLINEAR WAVE EQUATIONS WITH SMALL INITIAL DATA.[1]

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Introduction. We will consider the Cauchy problem in $\mathbf{R}_+ \times \mathbf{R}^3$

$$\square\, u = \partial_t^2 u - \sum_{i=1}^{3} \partial_{x_i}^2 u = G(u, u', u''), \quad u = \epsilon u_0, \partial_t u = \epsilon u_1 \quad \text{when } t = 0, \tag{0.1}$$

where $u_0, u_1 \in C_0^\infty$ and G is a smooth function of u, $\{u_j'\}_{j=0}^3$ and $\{u_{jk}''\}_{j,k=0}^3$ vanishing to second order at the origin. In case $G(u, u', u'') = G(u', u'')$ it was proved in John-Klainerman [7] that the equation (0.1) will have a C^∞ solution u for $0 \le t < T_\epsilon$, where T_ϵ satisfies

$$\log T_\epsilon \ge c/\epsilon, \tag{0.2}$$

when ϵ is sufficiently small. "Without loss of generality" they assumed in addition that $G(u', u'')$ was linear in u''. It was mentioned without proof that (0.2) should also hold if $G(u, u', u'') = \sum_0^3 \partial_j H_j(u, u')$. Here we shall show that (0.2) holds in the case when $G(u, u', u'')$ also depends on u and satisfies $G_{uu}''(0,0,0) = 0$. We shall also show that in general, when $G_{uu}''(0,0,0) \ne 0$, then

$$T_\epsilon \ge c/\epsilon^2. \tag{0.3}$$

Actually, in case $G(u, u', u'') = u^2 + H(u')$, where H is a positive definite quadratic form, nothing better holds. (See John [5] and Lindblad [11].)

In section 1 we state some well known results of Hörmander and Klainerman. The new results here that will enable us to get control of the L^2 and L^∞ norms of u are provided by Proposition 1.8 and Proposition 1.9. Also Lemma 1.10 will be useful. In Section 2 we give the theorems on the lifespan. L. Hörmander's $L^1 - L^\infty$ estimate is so important in this paper that we give a new and simplified proof, due to L. Hörmander, in an appendix.

1. L^2 and $L^1 - L^\infty$ estimates for the wave operator. For $(t, x) \in \mathbf{R}^{1+3}$ denote ∂_t by ∂_0 and ∂_{x_j} by ∂_j for $j = 1, 2, 3$. Let

$$Z_{jk} = \lambda_j x_j \partial_k - \lambda_k x_k \partial_j, \quad 0 \le j < k \le 3, \tag{1.1}$$

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where $\lambda = (1, -1, -1, -1)$ and $x_0 = t$, which all commute with $\Box$. Let

$$Z_{00} = \sum_0^3 x_j \partial_j \tag{1.2}$$

which satisfies $[\Box, Z_{00}] = 2\Box$. Set $Z_{jk} = 0$, if $j \geq k > 0$. We will write Z^I for a product of $|I|$ of the vector fields (1.1), (1.2) and ∂_j, $j = 0, 1, 2, 3$. Let $\tilde{Z}^I$ be defined by $\Box Z^I = \tilde{Z}^I \Box$, which means that $\tilde{Z}^I$ differs from Z^I only in that Z_{00} should be replaced by $Z_{00} + 2$. We note that $[\partial_i, Z_{jk}]$ is either 0 or else equal to $\pm\partial_l$ for some l. We will write $|u'|^2 = |\partial u|^2 = \sum_0^3 |\partial_j u|^2$. It follows that

$$|Z^I \partial_j u| \leq C \sum_{|J| \leq |I|} |\partial Z^J u| \tag{1.3}$$

Introduce the usual polar coordinates $r^2 = |x|^2 = \sum_1^3 |x_j|^2$, $\omega = x/|x|$ and $\partial_r = \sum_1^3 \omega_j \partial_j$. Let E be the fundamental solution of $\Box$. For the solution of $\Box v = f$, with initial data 0, where $f \in C([0, T) \times \mathbf{R}^3)$, we extend f to be 0, when $t < 0$ and write $v = E * f$.

The operators $\{Z_{jk}\}_{j,k=0}^3$ span the tangent space at every point where $t \neq |x|$. But when $t = |x|$ they only span the tangent space of the cone $t = |x|$, which explains the Lemma 1.1 below. (The 1 in the left-hand side of (1.4) is due to the fact that ∂_j, $j = 0, ..., 3$, are included in the right5-hand side.) Lemma 1.2 then gives an inequality for L^2 norms which will be used together with (1.4) to estimate L^2 norms of products; see Lemma 1.10. Inequaliy (1.5) shows that (1.4) can be improved when $t + |x|$ if we also have an estimate for a derivative which is not tangential to the cone $t = |x|$. Such an estimate can be obtained from a bound of $\Box u$; see Lemma 1.7.

Lemma 1.1. *Let $u \in C^1([0, t] \times \mathbf{R}^3)$. Then*

$$(|t - |x|| + 1)^2 |\partial u|^2 \leq 4 \sum_{|I|=1} |Z^I u|^2, \tag{1.4}$$

$$(t + |x|)|\partial u| \leq 2\big(2|x||\partial_t u| + \sum_{j,k} |Z_{jk} u|\big), \tag{1.5}$$

$$\sum_{0<j<k} |Z_{jk} u| \leq \frac{\sqrt{6}|x|}{t + |x|} \sum_{j,k} |Z_{jk} u|. \tag{1.6}$$

Proof. (1.4)–(1.6) are implied by the rotationally invariant estimates

$$(t - |x|)^2 \sum_{j=0}^3 |\partial_j u|^2 \leq \sum_{j,k} |Z_{jk} u|^2, \tag{1.7}$$

$$(t^2 + |x|^2) \sum_{j=0}^3 |\partial_j u|^2 \leq 2\big(4|x|^2 |\partial_t u|^2 + \sum_{j,k} |Z_{jk} u|^2\big), \tag{1.8}$$

$$\sum_{0<j<k} |Z_{jk} u|^2 \leq \frac{|x|^2}{t^2 + |x|^2} \sum_{j<k} |Z_{jk} u|^2. \tag{1.9}$$

In fact when proving (1.4) we may assume that $|t-|x|| \geq 1$. We note that $t^2+|x|^2 \geq (t+|x|)^2/2$ so (1.5) and (1.6) follow from (1.8) and (1.9). When proving these estimates we may assume that $x_2 = x_3 = 0$, $t > 0$ and $x_1 > 0$. Then

$$t^2|x|^2 \sum_{j=2}^{3} |\partial_j u|^2 = t^2 \sum_{0<j<k} |Z_{jk}u|^2 = |x|^2 \sum_{k=2}^{3} |Z_{0k}u|^2,$$

which proves (1.9) and shows that

$$(t^2+|x|^2) \sum_{j=2}^{3} |\partial_j u|^2 = \sum_{0<j<k} |Z_{jk}u|^2 + \sum_{k=2}^{3} |Z_{0k}u|^2, \tag{1.10}$$

With $Z_{00} = t\partial_t + x_1\partial_1$ and $Z_{01} = t\partial_1 + x_1\partial_t$ we have

$$(t^2-|x|^2)\partial_t u = tZ_{00}u - x_1 Z_{01}u, \quad (t^2-|x|^2)\partial_1 u = -x_1 Z_{00}u + tZ_{01}u.$$

Hence

$$(t-|x|)^2(|\partial_t u|^2 + |\partial_1 u|^2) \leq (|Z_{00}u|^2 + |Z_{01}u|^2),$$

which in view of (1.10) proves (1.7). We also have

$$\begin{aligned}(t^2+|x|^2)\partial_t u &= tZ_{00}u - x_1 Z_{01}u + 2|x|^2\partial_t u,\\ (t^2+|x|^2)\partial_1 u &= x_1 Z_{00}u + tZ_{01}u - 2tx_1\partial_t u.\end{aligned}$$

Hence

$$\begin{aligned}&(t^2+|x|^2)^2(|\partial_t u|^2 + |\partial_1 u|^2)\\ &\quad \leq 2|tZ_{00}u - x_1 Z_{01}u|^2 + 8|x|^4|\partial_t u|^2 + 2|x_1 Z_{00}u + tZ_{01}u|^2 + 8t^2|x|^2|\partial_t u|^2.\end{aligned}$$

It follows that

$$(t^2+|x|^2)(|\partial_t u|^2 + |\partial_1 u|^2) \leq 2\big(4|x|^2|\partial_t u|^2 + |Z_{00}u|^2 + |Z_{01}u|^2\big),$$

which in view of (1.10) proves (1.8). $\square$

Lemma 1.2. *If $u \in C^1([0,t]\times \mathbf{R}^3)$ and $t-|x| \geq -\rho$, in $\operatorname{supp} u$ then*

$$||u(t,\cdot)/(|t-r(\cdot)|+1)||_2 \leq C||\partial_r u(t,\cdot)||_2, \quad \textit{where } r(x) = |x|. \tag{1.11}$$

Proof. We claim that if $|x| < R$ in supp u, then

$$\int \frac{|u(x)|^2}{(R-|x|)^2}\,dx \leq 4\int |\partial_r u|^2\,dx.$$

In fact if we introduce polar coordinates this is implied by

$$\int_0^R \frac{|u|^2 r^2}{(R-r)^2}\,dr \leq \int_0^R |\partial_r u|^2 r^2\,dr,$$

for every ω. If we set $v = ru$ and note that

$$\int (v'^2 - r^2 u'^2)\, dr = \int u(u + 2ru')\, dr = \int d(ru^2) = 0,$$

this inequality reduces to Hardy's inequality

$$\int_0^R \frac{v^2}{(R-r)^2}\, dr \le 4 \int_0^R v'^2\, dr,$$

proved immediately by a partial integration. $\square$

Below are some further standard results. Proposition 1.3 states the most classical energy estimate, Lemma 1.4 shows the decay of solutions of the wave equations, and Proposition 1.5 is a Sobolev type of lemma; it gives a bound for the L^∞ norm in terms of L^2 norms. Proposition 1.6, on the other hand, gives the L^∞ norm of u in terms ofL^1 norms of $\square\, u$. In Lemma 1.7 we also use a bound of $\square\, u$ to get control of the L^∞ norm of $|\partial u|$. Bounds of $\square\, u$ can the be otained using the equation for u.

Proposition 1.3. *Let $u \in C^2$ satisfy*

$$\square\, u = f, \quad 0 \le t < T$$

and assume that $u = 0$ for large x. Then it follows for $0 \le t < T$ that

$$||u'(t,\cdot)||_2 \le (||u'(0,\cdot)||_2 + \int_0^t ||f(s,\cdot)||_2\, ds).$$

Lemma 1.4. *Let w be the solution of*

$$\square\, w = 0$$

with initial data $w(0,x) = w_0(x)$, $w_t'(0,x) = w_1(x) \in C_0^\infty$ such that $|x| \le R$ in $\operatorname{supp} w_j$, $j = 0, 1$. *Then*

$$||w(t,\cdot)||_2 \le CR||\partial w(0,\cdot)||_2, \tag{1.12}$$

$$(R+t)||w(t,\cdot)||_\infty \le CR^2||\partial w(0,\cdot)||_\infty. \tag{1.13}$$

Proof. Since $|t - |x|| \le R$ in $\operatorname{supp} w$ it follows from Lemma 1.2 and Proposition 1.3 that

$$||w(t,\cdot)||_2 \le CR||\partial w(t,\cdot)||_2 \le CR||\partial w(0,\cdot)||_2,$$

which proves (1.12). The proof of (1.13) is an immediate consequence of Kirchoff's formula

$$w(t,x) = t\int_{|\omega|=1} \big(w_1(x + t\omega) + \langle w_0'(x + t\omega), \omega\rangle\big)\, dS(\omega) + \int_{|\omega|=1} w_0(x + t\omega)\, dS(\omega),$$

where $dS(\omega)$ is the normalized surface measure on S^2. In the support of the integrand we have $|x + t\omega| < R$, hence $|\omega + x/t| < R/t$, which means that the measure is $\le CR^2/(t+R)^2$. Since $|w_0| \le R \sup |w_0'|$, we get the bound

$$CR^2(t+R)^{-2}(t+R) \sup |w'(0,.)|.$$

Proposition 1.5. *There is a constant C such that*

$$(1+t+|x|)(1+||t|-|x||)^{1/2}|w(t,x)| \le C \sum_{|I|\le 2} ||Z^I w(t,\cdot)||_2,$$

for $w \in C_0^2$ in $[0,t]\times\mathbf{R}^n$, say.

Proof. See Klainerman [8]. $\square$

Proposition 1.6. *Let $g \in C^2([0,t]\times\mathbf{R}^3)$ and assume that $g(t,x)=0$ for large $|x|$. Let v be the solution of*

$$\square\, v = g, \quad t \ge 0,$$

with initial data 0. Then

$$|v(t,x)|(1+t+|x|) \le C \sum_{|I|\le 2} \int_0^t ||(Z^I g)(s,\cdot)/(1+s+r(\cdot))||_1\, ds, \tag{1.15}$$

where $r(x)=|x|$, and

$$||v(t,\cdot)||_1 \le (1+t)\int_0^t ||g(s,\cdot)||_1\, ds. \tag{1.16}$$

Proof. (1.15) follows from Hörmander [2]. (See also the appendix, Klainerman [8] and Hörmander [4].) Since $|v| = |E*g| \le E*|g|$ we may assume that $g\ge 0$ and hence $v\ge 0$. Then

$$\partial_t^2 \int v(t,x)\,dx = \int \square\, v(t,x)\,dx = \int g(t,x)\,dx,$$

which proves that

$$\int v(t,x)\,dx = \int_0^t (t-s)\,ds \int g(s,x)\,dx. \quad \square$$

Lemma 1.7. *If $u\in C_0^2((0,t]\times\mathbf{R}^3)$ then*

$$\begin{aligned}(t+|x|)|\partial u| &\le 2\sum_{j,k}|Z_{jk}u| \\ &+ 4\int_0^t \Big(||r(\cdot)\,\square\, u(s,\cdot)||_\infty + \sqrt{6}\sum_{j,k,l,m}||(Z_{jk}Z_{lm}u)(s,\cdot)/(s+r(\cdot))||_\infty\Big)\,ds,\end{aligned} \tag{1.17}$$

where $r(x)=|x|$.

Proof. We have

$$(\partial_t^2-\partial_r^2)(ru) = r\,\square\, u + r^{-1}\sum_{0<j<k} Z_{jk}^2 u.$$

Regarding this as a two dimensional Cauchy problem and using (1.6) we obtain

$$|\partial_t(ru)| \le \int_0^t \Big(||r(\cdot)\,\square\, u(s,\cdot)||_\infty + \sqrt{6}\sum_{j,k,l,m}||(Z_{jk}Z_{lm}u)(s,\cdot)/(s+r(\cdot))||_\infty\Big)\,ds.$$

Hence the lemma follows from (1.5).

Proposition 1.3 together with Proposition 1.5 will give us the L^2 and L^∞ norm of u'. To get hold of the L^2 and L^∞ norms of u we have Proposition 1.8 and Proposition 1.9, which are the main technical inovations of this paper. It turns out to be important that Proposition 1.8 does not involve any estimates of $|\partial f|$ and that, in Proposition 1.9 we can put all derivatives on one factor. To motivate these propositions let us give some idea of how they are going to be used. (This will be explained in more detail in the beginning of section 2.)

$$\Box\, u = \sum a_j \partial_j u^2 + \sum b_{jk} (\partial_j u)(\partial_k u),$$

with initial data ϵu_0, ϵu_1 into three parts w_1, w_2 and w_3:

$$\Box\, w_1 = \sum a_j \partial_j u^2, \quad \Box\, w_2 = \sum b_{jk} (\partial_j u)(\partial_k u), \quad \Box\, w_3 = 0,$$

where w_1 and w_2 have initial data 0 and w_3 has the same initial data as u. Proposition 1.8 will the give us estimates for w_1 and Proposition 1.9 gives estimates for w_2. Finally, w_3 can be estimated by Lemma 1.4.

Proposition 1.8. *Suppose that $f \in C^4([0,t] \times \mathbf{R}^3)$ and $t - |x| \geq -\rho$ in $\operatorname{supp} f$. Let v be the solution of*

$$\Box\, v = \sum_0^3 a_j \partial_j f, \quad a_j \in \mathbf{R},$$

with initial data 0. Then there is a constant C depending on ρ and a_j such that

$$||v(t,\cdot)||_2 \leq C\Big(\int_0^t ||f(s,\cdot)||_2 \, ds + ||f(0,\cdot)||_2\Big), \tag{1.18}$$

(1.19)
$$(1+t)||v(t,\cdot)||_\infty \leq C \int_0^t ||f(s,\cdot)||_\infty (1+s)\, ds + C \sum_{|I|\leq 4} \int_0^t ||Z^I f(s,\cdot)||_1 \frac{ds}{(1+s)^2}.$$

Proof. Let u be the solution of

$$\Box\, u = f,$$

with initial data 0 and let g be the solution of

$$\Box\, g = 0,$$

with initial data $g(0,x) = 0$, $\partial_t g(0,x) = f(0,x)$. Then

$$v = \sum_0^3 a_j \partial_j u - a_0 g.$$

Hence

$$||v(t,\cdot)||_2 \leq |a|(||\partial u(t,\cdot)||_2 + ||g(t,\cdot)||_2).$$

Since $|t-|x||$ is bounded in the support of g it follows from Lemma 1.2 and Proposition 1.3 that

$$||g(t,\cdot)||_2 \le C||\partial g(t,\cdot)||_2 \le C||f(0,\cdot)||_2.$$

Again by Proposition 1.3 we have

$$||\partial u(t,\cdot)||_2 \le \int_0^t ||f(s,\cdot)||_2\, ds,$$

which proves (1.18). By Proposition 1.6 we have

$$(1+t)||v(t,\cdot)||_\infty \le C \sum_{|I|\le 2} \sum_{j=0}^{3} \int_0^t ||Z^I \partial_j f(s,\cdot)||_1 \frac{ds}{1+s},$$

which proves (1.19) if $f \in C_0^\infty([0,2)\times \mathbf{R}^3)$, say. On the other hand if $f \in C_0^\infty((1,t]\times \mathbf{R}^3)$ then $v = \sum_0^3 a_j \partial_j u$. Note that $Z^I u = Z^I E * f = E * (\tilde{Z}^I f)$, where $\tilde{Z}^I$ differs from Z^I only in that Z_{00} should be replaced by $Z_{00}+2$. Then if we use Proposition 1.6 to estimate $||Z^I u(t,\cdot)||_\infty$ it follows from Lemma 1.7 that

$$\begin{aligned}(t+|x|)|\partial u| \le{}& \frac{C}{t+1} \sum_{|I|\le 3} \int_0^t ||Z^I f(s,\cdot)||_1 \frac{ds}{1+s} \\ &+ C\int_0^t ||r(\cdot) f(s,\cdot)||_\infty\, ds + C\int_0^t \Big(\sum_{|I|\le 4} \int_0^s ||Z^I f(v,\cdot)||_1 \frac{dv}{1+v}\Big) \frac{ds}{(1+s)^2}.\end{aligned}$$

If we note that $r(x) \le C(1+t)$ in supp f and change the order of integration in the last integral (1.19) follows. In general we obtain (1.19) by writing

$$f(t,x) = \chi(t) f(t,x) + (1-\chi(t)) f(t,x),$$

where $\chi \in C_0^\infty$ is equal to 1 when $0 \le t < 1$ and $t < 2$ in supp χ. $\square$

Proposition 1.9. *Assume that $u_j \in C^2([0,t]\times \mathbf{R}^3)$ and let v be the solution of*

$$\square\, v = |u_1 u_2|,$$

with initial data 0. *Then*
(1.20)

$$(1+t)^2||v(t,\cdot)||_\infty^2 \le C\int_0^t \big(\sum_{|I|\le 2} ||Z^I u_1(s,\cdot)||_2\big)^2 \frac{ds}{1+s} \int_0^t \big(\sum_{|I|\le 2} ||Z^I u_2(s,\cdot)||_2\big)^2 \frac{ds}{1+s}$$

and

$$||v(t,\cdot)||_2^2 \le C\int_0^t \big(\sum_{|I|\le 2} ||Z^I u_1(s,\cdot)||_2\big)^2 \frac{ds}{\sqrt{1+s}} \int_0^t ||u_2(s,\cdot)||_2^2 \frac{ds}{\sqrt{1+s}}. \tag{1.21}$$

proof. By Cauchy-Schwarz' inequality and the positivity of the fundamental solution ,E, we have $v^2 = (E * |u_1 u_2|) \le (E * u_1^2)(E * u_2^2)$ so (1.20) follows from (1.15) in Proposition 1.6. In the same way with

$$v_1(t,x) = u_1(t,x)(1+t^2+|x|^2)^{1/8} \quad \text{and} \quad v_2(t,x) = u_2(t,x)/(1+t^2+|x|^2)^{1/8},$$

we obtain

$$v^2 = \big(E * |v_1 v_2|\big)^2 \le (E * v_1^2)(E * v_2^2).$$

Hence

$$||v(t,\cdot)||_2^2 \le ||E * v_1^2(t,\cdot)||_\infty ||E * v_2^2(t,\cdot)||_1,$$

and thus by (1.15) and (1.16) of Proposition 1.6 we have

$$||v(t,\cdot)||_2^2 \le C \int_0^t \big(\sum_{|I|\le 2} ||Z^I v_1^2(s,\cdot)/(1+s+r(\cdot))||_1\big)\, ds \int_0^t ||v_2(s,\cdot)||_2^2\, ds.$$

Since

$$|Z^I(1+t^2+|x|^2)^{1/8}| \le C_I (1+t^2+|x|^2)^{1/8},$$

this proves (1.21). $\square$

Lemma 1.10. *Suppose that $v_j \in C^\infty$, $j = 1, 2$, with $t - |x| \ge -\rho$ in the supports. Then*

$$||(\partial_j v_1) v_2(t,\cdot)||_2 \le C_\rho \sum_{|I|=1} ||Z^I v_1(t,\cdot)||_\infty ||\partial v_2(t,\cdot)||_2.$$

Proof. By Lemma 1.1 and Lemma 1.2 we have with $r(x) = |x|$

$$||(\partial_j v_1) v_2(t,\cdot)||_2 \le C \sum_{|I|\le 1} |||Z^I v_1(t,\cdot)| \frac{v_2(t,\cdot)}{|t - r(\cdot)| + 1}||_2 \le C \sum_{|I|\le 1} ||Z^I v_1(t,\cdot)||_\infty ||\partial v_2(t,\cdot)||_2. \ \square$$

Proposition 1.11. *Let $u \in C^2$ satisfy*

$$\Box\, u + \sum_{j,k=0}^{3} \gamma^{jk}(t,x)\partial_j \partial_k u = f, \quad 0 \le t \le T,$$

and assume that $u = 0$ for large x. If

$$|\gamma| = \sum |\gamma^{jk}| \le \frac{1}{2}, \quad 0 \le t \le T,$$

it follows for $0 \le t \le T$ that

$$||u'(t,\cdot)||_2 \le 2(||u'(0,\cdot)||_2 + \int_0^t ||f(s,\cdot)||_2\, ds) \exp\Big(\int_0^t 2|\gamma'(s)|\, ds\Big),$$

where

$$|\gamma'(t)| = \sup |\partial_i \gamma^{jk}(t,\cdot)|.$$

Proof. See Klainerman [10]. $\square$

2. The lifespan estimates. We will start by proving the estimate (0.2) for the lifespan of the solution of

$$\Box\, u = G(u,u') = \sum a_j \partial_j u^2 + \sum b_{jk}(\partial_j u)(\partial_k u), \quad u = \epsilon u_0, \partial_t u = \epsilon u_1 \quad \text{when } t = 0,$$

in Theorem 2.1. This will be done with a continuity argument. We will assume that we have bounds

$$||\partial Z^I u(t,\cdot)||_2 \le M_1 \epsilon, \quad ||Z^I u(t,\cdot)||_2 \le M_2 \epsilon \sqrt{t+1}, \quad \text{for } |I| \le k, \tag{2.1}$$

and

$$(1+t)||\partial Z^I u(t,\cdot)||_\infty \le N_1 \epsilon, \quad (1+t)||Z^I u(t,\cdot)||_\infty \le N_2 \epsilon, \quad \text{for } |I| \le k-4, \tag{2.2}$$

where k is an integer such that $2(k-5) \ge k$. Then we use the differentiated equations $\Box\, Z^I u = \tilde{Z}^I G(u,u')$ to obtain bounds for the the quantities in (2.1) and (2.2) in terms of integrals of these quantities for smaller values of t, which will be used to show that the estimates (2.1) and (2.2) for smaller values of t implies the same estimates diveded by 2 if $\epsilon \log{(t+1)}$ is sufficiently small. It follows from the beginning of section 1 that we can write

$$\Box\, Z^I u = \tilde{Z}^I G = \sum_{2|I_1|,\,|I_2| \le |I|} c_{jI_1I_2} \partial_j (Z^{I_1} u Z^{I_2} u) + \sum_{2|I_1|,\,|I_2| \le |I|} b_{ijI_1I_2} (\partial_i Z^{I_1} u)(\partial_j Z^{I_2} u). \tag{2.3}$$

To get hold of $||\partial Z^I u(t,\cdot)||_2$, for $|I| \le k$, we shall use the energy integral method, Proposition 1.3. This involves estimates of $||\tilde{Z}^I G(u,u')(t,\cdot)||_2$. which by Lemma 1.10 can be estimated by $||\partial Z^{I_1} u(t,\cdot)||_\infty$ and $||Z^{I_1} u(t,\cdot)||_\infty$for $|I_1| \le [k/2]+1$ multiplied by $||\partial Z^{I_2} u(t,\cdot)||_2$ for $|I_2| \le k$, (for ∂_j acts on at least one factor in every term in $\tilde{Z}^I G$). To get hold of $||Z^I u(t,\cdot)||_2$, for $|I| \le k$, we first note that $E * (\tilde{Z}^I G)$ only differs from $Z^I u$ by the solution of $\Box\, v = 0$ with the same initial data as $Z^I u$, and for this Lemma 1.4 gives an estimate. To estimate $E * (\tilde{Z}^I G)$ we apply Proposition 1.8 to the first sum in (2.3) and Proposition 1.9 to the second. By the same propositions we get an estimate for $||Z^I u(t,\cdot)||_\infty$, for $|I| \le k-4$ and the estimate for $||\partial Z^I u(t,\cdot)||_\infty$, for $|I| \le k-4$, follows from Proposition 1.5.

In Theorem 2.2 we shall show the shorter lifespan estimate, (0.3), when u^2 is present in G. In Theorem 2.3 we shall generalize these results to the case when $G(u,u',u'')$ is any smooth function vanishing to second order at the origin. The principle will be the same (see the discussion before Theorem 2.3).

Theorem 2.1. *Let $u_0, u_1 \in C_0^\infty$. Then there exist constants δ and ϵ_0 such that for $\epsilon < \epsilon_0$*

$$\Box\, u = \sum a_j \partial_j u^2 + \sum b_{jk} (\partial_j u)(\partial_k u) \tag{2.4}$$

has a C^∞ solution with initial data $\epsilon u_0, \epsilon u_1$ for $0 \le t < T_\epsilon = \exp(\delta/\epsilon)$.

Proof. Let k be an integer such that $2(k-5) \ge k$. From the local existence theory (see e.g. John [6] or Klainerman [10]) we know that it suffices to prove that if a solution exists for $0 \le t < T$ there are bounds

$$M_1(t) = \sum_{|I|\le k} ||\partial Z^I u(t,\cdot)||_2 \le M_1\epsilon, \tag{2.5}$$

$$M_2(t) = \sum_{|I|\le k} ||Z^I u(t,\cdot)||_2 \le M_2\epsilon\sqrt{t+1}, \tag{2.6}$$

$$N_1(t) = \sum_{|I|\le k-2} ||\partial Z^I u(t,\cdot)||_\infty \le N_1\epsilon/(t+1), \tag{2.7}$$

$$N_2(t) = \sum_{|I|\le k-4} ||Z^I u(t,\cdot)||_\infty \le N_2\epsilon/(t+1), \tag{2.8}$$

for $0 \le t < T$, which are independent of T if $\epsilon \log T \le \delta$. (In fact then the solution can be continued for some further time.) Again by the local existence theory this is clear when T is small if the constants are large enough. We will use (2.4) to prove that these bounds will imply the same bounds divided by 2 for $t \le T \le \exp(\delta/\epsilon)$ if ϵ and δ are sufficiently small and the constants M_1, M_2, N_1, N_2 are sufficiently large. Hence by continuity we conclude that (2.5)–(2.8) hold.

From the discussion in the beginning of section 1 we know that $Z^I\partial_j$ may be written as a linear combination of $\partial_k Z^J$ for $|J| \le |I|$. Hence if we apply Z^I to both sides of (2.4) we obtain

$$\Box\, Z^I u = \sum_{2|I_1|,\,|I_2|\le|I|} c_{jI_1I_2}\partial_j(Z^{I_1}uZ^{I_2}u) + \sum_{2|I_1|,\,|I_2|\le|I|} b_{ijI_1I_2}(\partial_i Z^{I_1}u)(\partial_j Z^{I_2}u), \tag{2.9}$$

where the sums are also over all $i, j = 0,1,2,3$. If $|I| \le k$ then $|I_2| \le k$ and $|I_1| \le k/2 < k-4$ by assumption. Hence

$$||(\partial_i Z^{I_1}u)(\partial_j Z^{I_2}u)||_2 \le N_1(t)M_1(t)$$

It follows from Lemma 1.10 that

$$\begin{aligned} ||\partial_j\big((Z^{I_1}u)(Z^{I_2}u)\big)(t,\cdot)||_2 &\le ||(Z^{I_1}u\partial_j Z^{I_2}u)(t,\cdot)||_2 + ||\big((\partial_j Z^{I_1}u)(Z^{I_2}u)\big)(t,\cdot)||_2 \\ &\le ||Z^{I_1}u(t,\cdot)||_\infty ||\partial_j Z^{I_2}u(t,\cdot)||_2 + C\sum_{|J|\le 1} ||Z^J Z^{I_1}u(t,\cdot)||_\infty ||\partial Z^{I_2}u(t,\cdot)||_2 \\ &\le C' N_2(t)M_1(t), \end{aligned}$$

since $1 + |I_1| \le 1 + k/2 \le k-4$, by assumption. Hence by Proposition 1.3

$$||\partial Z^I u(t,\cdot)||_2 \le ||\partial Z^I u(0,\cdot)||_2 + C\int_0^t (N_1(s)+N_2(s))M_1(s)\,ds.$$

It follows that

$$M_1(t) \le C\int_0^t (N_1(s)+N_2(s))M_1(s)\,ds + M_1(0). \tag{2.10}$$

Since by (1.3) $|Z^I\partial_j v| \le C\sum_{|J|\le|I|}|\partial Z^J v|$ it follows from Proposition 1.5 that

$$(1+t)|\partial_j Z^I u| \le C\sum_{|J|\le 2+|I|} ||\partial Z^J u(t,\cdot)||_2$$

Hence

$$(1+t)N_1(t) \le CM_1(t). \tag{2.11}$$

It we convolute (2.9) with the fundamental solution E, we obtain

$$Z^I u = \sum_{2|I_1|,\,|I_2|\le|I|} c_{jI_1I_2} E * \big(\partial_j(Z^{I_1}uZ^{I_2}u)\big) + \sum_{2|I_1|,\,|I_2|\le|I|} b_{ijI_1I_2} E * \big((\partial_i Z^{I_1}u)(\partial_j Z^{I_2}u)\big) + w^I,$$

where w^I is the solution of $\Box\, w^I = 0$ with the same initial data as $Z^I u$. Hence by Proposition 1.8, Proposition 1.9 and Lemma 1.4 we have

$$\begin{aligned}
||Z^I u(t,\cdot)||_2 \le C' &\sum_{2|I_1|,\,|I_2|\le|I|} \Big(\int_0^t ||Z^{I_1}uZ^{I_2}u(s,\cdot)||_2\, ds + ||Z^{I_1}uZ^{I_2}u(0,\cdot)||_2\Big) \\
+C' \sum_{2|I_1|,\,|I_2|\le|I|} &\Big(\int_0^t \big(\sum_{|J|\le 2} ||Z^J\partial_j Z^{I_1}u(s,\cdot)||_2\big)^2 \frac{ds}{\sqrt{1+s}} \int_0^t ||\partial_j Z^{I_2}u(s,\cdot)||_2^2 \frac{ds}{\sqrt{1+s}}\Big)^{1/2} \\
&+ C'||\partial Z^I u(0,\cdot)||_2.
\end{aligned}$$

It follows that

$$M_2(t) \le C\Big(\int_0^t N_2(s)M_2(s)\, ds + N_2(0)M_2(0) + \int_0^t M_1(s)^2 \frac{ds}{\sqrt{1+s}} + M_1(0)\Big). \tag{2.12}$$

Finally by the same propositions

$$\begin{aligned}
||Z^I u(t,\cdot)||_\infty(1+t) \le C' &\sum_{2|I_1|,\,|I_2|\le|I|} \Big(\int_0^t ||Z^{I_1}uZ^{I_2}u(s,\cdot)||_\infty (1+s)\, ds \\
&+ \sum_{|J|\le 4} \int_0^t ||Z^J(Z^{I_1}uZ^{I_2}u)(s,\cdot)||_1 \,\frac{ds}{(1+s)^2} \\
+\Big(\int_0^t \big(\sum_{|J|\le 2} ||Z^J\partial Z^{I_1}u(s,\cdot)||_2\big)^2 &\frac{ds}{1+s} \int_0^t \big(\sum_{|J|\le 2} ||Z^J\partial Z^{I_2}u(s,\cdot)||_2\big)^2 \frac{ds}{1+s}\Big)^{1/2}\Big) \\
&+ C'||\partial Z^I u(0,\cdot)||_\infty.
\end{aligned}$$

If we take $|I| \le k-4$ it follows that

$$N_2(t)(1+t) \le C\Big(\int_0^t N_2(s)^2(1+s)\, ds + \int_0^t M_2(s)^2 \frac{ds}{(1+s)^2} + \int_0^t M_1(s)^2 \,\frac{ds}{1+s} + N_1(0)\Big). \tag{2.13}$$

If we use (2.5)–(2.8) it follows from (2.10), (2.12) and (2.13) that

$$\begin{aligned}
M_1(t) &\le C(N_1+N_2)M_1\epsilon^2 \log(t+1) + M_1(0), \\
M_2(t) &\le C\big(N_2M_2\epsilon^2 2\sqrt{1+t} + N_2M_2\epsilon^2 + M_1^2\epsilon^2 2\sqrt{1+t} + M_1(0)\big), \\
(1+t)N_2(t) &\le C\big((N_2^2+M_2^2+M_1^2)\epsilon^2 \log(t+1) + N_1(0)\big),
\end{aligned} \tag{2.14}$$

and for $N_1(t)$ we have the estimate (2.11). Choose M_1, M_2 and N_2 such that

$$M_1\epsilon \geq 4M_1(0), \quad M_2\epsilon = CM_1\epsilon \geq 4CM_1(0), \quad N_2\epsilon \geq 4CN_1(0), \tag{2.15}$$

and set $N_1 = CM_1$. Then for $\epsilon \log(t+1) \leq \delta$ we have

$$\begin{aligned} M_1(t) &\leq \big(C\delta(N_1+N_2)+1/4\big)M_1\epsilon, \\ M_2(t) &\leq \big(C\epsilon 3N_2 + 2C\epsilon M_1^2/M_2 + 1/4\big)M_2\epsilon\sqrt{1+t}, \\ (1+t)N_2(t) &\leq \big(C(N_2+M_2^2/N_2+M_1^2/N_2)\delta + 1/4\big)N_2\epsilon. \end{aligned} \tag{2.16}$$

Hence the assertion follows if δ and ϵ are chosen sufficiently small. $\square$

Theorem 2.2. *Let $u_0, u_1 \in C_0^\infty$. Then there exist constants μ and ϵ_0 such that for $\epsilon < \epsilon_0$*

$$\Box\, u = \sum_{|\alpha|,|\beta|\leq 1} c_{\alpha\beta}(\partial^\alpha u)(\partial^\beta u) \tag{2.17}$$

with initial data $\epsilon u_0, \epsilon u_1$ has a C^∞ solution for $0 \leq t < T_\epsilon = \mu^2/\epsilon^2$.

Proof. Let $2(k-3) \geq k$ and let $M_1(t), M_2(t)\, N_1(t)$ be defined as in the proof of Theorem 2.1 except that we sum now over $|I| \leq k-2$ also in $N_2(t)$. We are going to show that there exist ϵ_0 and μ such that if $\epsilon < \epsilon_0$ and $\epsilon\sqrt{t+1} \leq \mu$ then

$$M_i(t) \leq M_i\epsilon, \quad (1+t)N_i(t) \leq N_i\epsilon. \quad i = 1,2. \tag{2.18}$$

We have

$$\Box\, Z^I u = \sum_{|\alpha|,|\beta|\leq 1,\ 2|I_1|,\,|I_2|\leq |I|} c_{\alpha\beta I_1 I_2}(\partial^\alpha Z^{I_1}u)(\partial^\beta Z^{I_2}u). \tag{2.19}$$

Hence it follows from Proposition 1.3 that

$$M_1(t) \leq C\int_0^t (N_1(s)+N_2(s))(M_1(s)+M_2(s))\, ds + M_1(0) \tag{2.20}$$

It follows from (2.19) that

$$Z^I u = \sum_{|\alpha|,|\beta|\leq 1,\ 2|I_1|,\,|I_2|\leq |I|} c_{\alpha\beta I_1 I_2} E * \big((\partial^\alpha Z^{I_1}u)(\partial^\beta Z^{I_2}u)\big) + w^I, \tag{2.21}$$

where w^I is the solution of $\Box\, w^I$ with the same initial data as $Z^I u$. It follows from Proposition 1.9 and Lemma 1.4 that

$$M_2(t) \leq C\Big(\int_0^t (M_1(s)+M_2(s))^2 \frac{ds}{\sqrt{1+s}} + M_1(0)\Big). \tag{2.22}$$

We have

$$(1+t)N_i(t) \leq CM_i(t), \quad i = 1,2.$$

In fact this follows in the same way as for $i = 1$ in the proof of Theorem 2.1. If we use (2.18) in (2.20) and (2.22) we obtain

$$\begin{aligned} &M_1(t) \le C(N_1+N_2)(M_1+M_2)\epsilon^2 \log(1+t) + M_1(0), \\ &M_2(t) \le C(M_1+M_2)^2\epsilon^2 2\sqrt{1+t} + CM_1(0). \end{aligned} \tag{2.23}$$

Let $\epsilon \log(t+1) \le \delta$ and $\epsilon\sqrt{1+t} \le \mu$ and choose M_i, $i = 1,2$ such that

$$M_1\epsilon \ge 4M_1(0), \quad M_2\epsilon = CM_1\epsilon \ge 4CM_1(0), \quad N_i = CM_i, \quad i = 1,2.$$

Then

$$\begin{aligned} &M_1(t) \le (C(N_1+N_2)(1+M_2/M_1)\delta + 1/4)M_1\epsilon, \\ &M_2(t) \le (C(M_1+M_2)(M_1/M_2+1)2\mu + 1/4)M_2\epsilon. \end{aligned} \tag{2.24}$$

Hence the theorem follows if we choose δ and μ sufficiently small. $\square$

In Theorem 2.3 we shall generalize the results in Theorem 2.1 and Theorem 2.2 to the case when $G(u,u',u'')$ is any smooth function vanishing to second order at the origin. Below we shall briefly discuss the generalization of the case when $G(u,u')$ is a quadratic form without the u^2 term to the case when $G''_{uu}(0,0,0) = 0$. The principle will be the same but we must take extra care of the terms in $\tilde{Z}^I G$, when $|I| = k$, that contain $\partial^\alpha Z^J u$, with $|\alpha| + |J| = 2+k$, as a factor because these can not be estimated by the quantities in (2.1) and (2.2). We must also estimate third order terms, but that will be easy because by (2.2) we then have an extra factor $\epsilon/(1+t)$ which means that in places where we used to have $\partial Z^J u$ we at worst instead have $\sqrt{\epsilon} Z^J u/\sqrt{1+t}$ and by (2.1) the L^2 norms of these will be smaller than the L^2 norm of $\partial Z^J u$. The terms in $\tilde{Z}^I G$, for $|I| = k$ that contain $\partial^\alpha Z^J u$, with $|\alpha| + |J| = 2+k$, as a factor are $(\partial G/\partial u''_{ij})\partial_i\partial_j Z^I u$ so we can use Proposition 1.11, with $\gamma^{ij} = -(\partial G/\partial u''_{ij})$, instead of Proposition 1.3 to get an estimate for $||\partial Z^I u(t,\cdot)||_2$ for $|I| = k$. Since $G''_{uu}(0,0,0) = 0$ we can write

$$G(u,u',u'') = \sum_{i=0}^{3} \partial_i G_i(u,u') + G_4(u',u'') + G_5(u,u',u''),$$

where G_i for $i = 0, ..., 4$ are quadratic forms and G_5 is a smooth function vanishing to third order at the origin. If $|I| < k$ the estimate for $||Z^I u(t,\cdot)||_2$ follows as before but when $|I| = k$ we must take care of the terms in $\tilde{Z}^I G_m$, for $m = 4,5$, that contain $\partial^\alpha Z^J u$, with $|\alpha| + |J| = 2+k$, as a factor. These are $(\partial G_m/\partial u''_{ij})\partial_i\partial_j Z^I u$ which is the same as

$$\partial_i\big((\partial G_m/\partial u''_{ij})\partial_j Z^I u\big) - (\partial_i(\partial G_m/\partial u''_{ij}))\partial_j Z^I u.$$

If we convolute with the fundamental solution E, and use Proposition 1.8 we see that the first term can be estimated by means of the quantities in (2.1) and (2.2). To the second term we can apply Proposition 1.9.

Theorem 2.3. *Let $u_0, u_1 \in C_0^\infty$ and let $G(u,u',u'')$ be a smooth function of u, $\{u_j'\}_{j=0}^3$ and $\{u_{jk}''\}_{j,k=0}^3$ vanishing to second order at the origin. Then there exist constants δ and ϵ_0 such that for $\epsilon<\epsilon_0$*

$$\Box\, u = G(u,u',u''), \tag{2.25}$$

with initial data $\epsilon u_0, \epsilon u_1$ has a C^∞ solution for $0\le t<T_\epsilon$, where

$$T_\epsilon = \delta/\epsilon^2, \qquad \textit{if } G_{uu}''(0,0,0)\neq 0, \tag{2.26}$$
$$T_\epsilon = \exp(\delta/\epsilon), \quad \textit{if } G_{uu}''(0,0,0)= 0. \tag{2.27}$$

Proof. First we shall prove (2.27). Let k and l be positive integers such that $k-4\ge l\ge [k/2]+1$. Set

$$\begin{aligned} M_1(t) &= \sum_{|I|\le k} ||\partial Z^I u(t,\cdot)||_2, \qquad & N_1(t) &= \sum_{|I|\le l} ||\partial Z^I u(t,\cdot)||_\infty,\\ M_2(t) &= \sum_{|I|\le k} ||Z^I u(t,\cdot)||_2, \qquad & N_2(t) &= \sum_{|I|\le l} ||Z^I u(t,\cdot)||_\infty,\\ M_3(t) &= M_1(t)+M_2(t), \qquad & N_3(t) &= N_1(t)+N_2(t). \end{aligned} \tag{2.28}$$

As before it sufficies to prove that if the solution exists for $0\le t<T$ then there are constants M_i, N_i, $i=1,2,3$ and δ, which are independent of T such that if $\epsilon\log(1+T)\le\delta$ then
(2.29)
$$M_1(t)\le M_1\epsilon, \quad M_i(t)\le M_i\epsilon\sqrt{1+t} \quad \text{for } i=1,2, \quad (1+t)N_i(t)\le N_i\epsilon, \quad \text{for } i=1,2,3.$$

We know that there are constants M_i, N_i, $i=1,2,3$, such that (2.29) is true for small t.

From the discussion at the beginning of section 1 it follows that

$$\Box\, Z^I u = \tilde Z^I(\,\Box\, u) = Z^I(\,\Box\, u) + \sum_{|J|<|I|} d_J Z^J(\,\Box\, u), \tag{2.30}$$

$$Z^I\partial^\alpha = \partial^\alpha Z^I + \sum_{|J|<|I|,|\beta|=|\alpha|} d_{\beta J}\partial^\beta Z^J. \tag{2.31}$$

We can write

$$G(u,u',u'') = \sum_{i=0}^3 \partial_i G_i(u,u') + G_4(u',u'') + G_5(u,u',u''), \tag{2.32}$$

where G_i for $i=0,...,4$ are quadratic forms and G_5 is a smooth function vanishing to third order at the origin. In what follows I_1, I_2 and α_i, β_i for $i=1,2$ will always denote indices such that

$$|I_1|+|I_2|\le|I|, \quad |I_1|\le|I_2|, \quad \text{and} \quad |\alpha_i|\le 2, \quad |\beta_i|\le 1 \quad \text{for } i=1,2. \tag{2.33}$$

We start with the estimate for $||\partial Z^I u(t,\cdot)||_2$. If we use (2.30) and (2.31) we see that $\tilde Z^I\sum_{i=0}^3\partial_i G_i(u,u')$ and $\tilde Z^I G_4(u',u'')$ consist of terms

$$(\partial^{\alpha_1}Z^{I_1}u)(\partial^{\alpha_2}Z^{I_2}u), \quad \text{with } |\alpha_1|+|\alpha_2|>0. \tag{2.34}$$

If $|I|\le k$ then $|I_1|\le [k/2]\le l-1$ by assumption. If in addition $|I_2|+|\alpha_2|<k+2$ we claim that

$$||(\partial^{\alpha_1}Z^{I_1}u)\partial^{\alpha_2}Z^{I_2}u(t,\cdot)||_2\le CN_3(t)M_3(t).$$

For the proof recall that ∂_j, $j=0,1,2,3$ belong to the family of operators Z^I, so if $|\alpha_2|>0$ this is obvious and if $|\alpha_2|=0$ then we can use Lemma 1.10. By (2.29) $|\partial^{\alpha_0}Z^{I_0}u|\le N_3\epsilon/(1+t)$ if $|\alpha_0|\le 2$ and $|I_0|\le l-1$ so a term in $\tilde{Z}^I G_5$ is either bounded by a term of the form (2.34) or bounded by

$$\frac{C\epsilon}{1+t}|(Z^{I_1}u)(Z^{I_2}u)|. \tag{2.35}$$

By Lemma 1.2 $||Z^{I_2}u(t,\cdot)||_2\le CM_1(t)(1+t)$ so the L^2 norms of these terms are also bounded by a constant times $N_3(t)M_1(t)$. Hence if $|I|<k$ it follows that

$$||\tilde{Z}^I G(u,u',u'')(t,\cdot)||_2\le CN_3(t)M_1(t),$$

which implies

$$||\partial Z^I u(t,\cdot)||_2\le ||\partial Z^I u(0,\cdot)||_2+C\int_0^t N_3(s)M_1(s)\,ds, \tag{2.36}$$

by Proposition 1.3. If $|I|=k$ then the L^2 norm of the terms in $\tilde{Z}^I G$ that contain $\partial^{\alpha_2}Z^{I_2}u$ as a factor with $|I_2|+|\alpha_2|=2+k$ can not be estimated by $CN_3(t)M_3(t)$. By (2.30) and (2.31) these terms are $\sum_{i,j=0}^3(\partial G/\partial u''_{ij})\partial_i\partial_j Z^I u$ and we have instead

$$||\big(\tilde{Z}^I G-\sum_{i,j=0}^3(\partial G/\partial u''_{ij})\partial_i\partial_j Z^I u\big)(t,\cdot)||_2\le CN_3(t)M_1(t).$$

To get an estimate for $||\partial Z^I u(t,\cdot)||_2$ in this case we are going to use Proposition 1.11, with $\gamma^{ij}=-(\partial G/\partial u''_{ij})$, instead of Proposition 1.3. Now if (2.29) is true then

$$\sum_{i,j=0}^3|\gamma^{ij}(u,u',u'')|\le C(N_1+N_2)\epsilon<1/2, \tag{2.37}$$

if ϵ is sufficiently small. Moreover, with $|\gamma'(t)|$ defined as in Proposition 1.11, we have

$$2\int_0^t|\gamma'(s)|\,ds\le CN_1\delta\le\log 2, \tag{2.38}$$

if $\epsilon\log(t+1)\le\delta$ and δ is sufficently small. It follows from Proposition 1.11 that (2.36) holds with an extra factor 4 on the right-hand side. Hence

$$M_1(t)\le C'\int_0^t N_3(s)M_1(s)\,ds+K_1\epsilon. \tag{2.39}$$

When estimating $||Z^I u(t,\cdot)||_2$ for $|I|\leq k$ we are going to use one of the three following estimates to treat the three kinds of terms in (2.32). By Proposition 1.3

$$||\partial_i E*f(t,\cdot)||_2 \leq C\int_0^t N_3(s)M_3(s)\,ds, \quad \text{if} \quad |f|\leq |(\partial^{\beta_1}Z^{J_1}u)\partial^{\beta_2}Z^{J_2}u|, \\ \text{where} \quad |J_1|\leq l,\ \ |J_2|\leq k, \quad \text{and} \quad |\beta_i|\leq 1 \ \text{ for } i=1,2. \tag{2.40}$$

(In fact $v=E*f$ is the solution of $\square\, v=f$ with initial data 0.) By Proposition 1.9 we have
(2.41)

$$||E*\big|(\partial_i Z^{J_1}u)\partial_j Z^{J_2}u\big|(t,\cdot)||_2 \leq C\int_0^t M_1(s)^2\frac{ds}{\sqrt{1+s}}, \quad \text{if} \quad |J_1|+2\leq k,\ \ |J_2|\leq k,$$

and
(2.42)

$$||E*\Big|\frac{(\partial^{\beta_1}Z^{J_1}u)\partial^{\beta_2}Z^{J_2}u}{1+s}\Big|(t,\cdot)||_2 \leq C\int_0^t M_3(s)^2\frac{ds}{(1+s)^{3/2}}, \quad \text{if} \quad \begin{cases} |J_1|+2\leq k,\ \ |J_2|\leq k, \\ |\beta_i|\leq 1 \ \text{ for } i=1,2 \end{cases},$$

where s in the convolution denotes the time variable t.

Let $|I|\leq k$. Now by (2.30) and (2.31) we can write

$$\tilde{Z}^I\sum_{i=0}^3\partial_i G_i = \sum_{j=0}^3\partial_j H_j, \quad \text{where} \quad H_j = \sum_{i=0,\dots,3,\ |J|\leq|I|} c_{ijJ}Z^J G_i.$$

We have

$$Z^I u = \sum_{i=0}^3 \partial_i E*H_i + E*(\tilde{Z}^I G_4) + E*(\tilde{Z}^I G_5) + v$$

where v is a solution of $\square\, v=0$ with the same initial data as $Z^I u-\partial_0 E*H_0$, and for this Lemma 1.4 gives an estimate $||v(t,\cdot)||_2\leq K\epsilon$.

The terms in H_i for $i=0,..,3$ are of the form

$$(\partial^{\beta_1}Z^{I_1}u)\partial^{\beta_2}Z^{I_2}u, \quad \text{with } |\beta_i|\leq 1, \quad i=1,2. \tag{2.43}$$

Since $|I_2|\leq k$ and $|I_1|\leq[k/2]\leq l$ we can use (2.40) to estimate $||\partial_j E*H_j(t,\cdot)||_2$ for $j=0,...,3$. The terms in $\tilde{Z}^I G_4$ are

$$(\partial^{\alpha_1}Z^{I_1}u)\partial^{\alpha_2}Z^{I_2}u \quad \text{with} \quad |\alpha_i|>0,\ \ i=1,2. \tag{2.44}$$

Here $|I_1|\leq[k/2]\leq k-3$ by assumption so if $|\alpha_2|+|I_2|<2+k$ we can use (2.41) to estimate these terms. By (2.29) $|\partial^{\alpha_0}Z^{I_0}u|\leq N_3\epsilon/(1+t)$ if $|\alpha_0|\leq 2$ and $|I_0|\leq l-1$ so the terms in $\tilde{Z}^I G_5$ are bounded by expressions of the form

$$\frac{C\epsilon}{1+t}|(\partial^{\alpha_1}Z^{I_1}u)\partial^{\alpha_2}Z^{I_2}u|, \tag{2.45}$$

and for these we can use (2.42) if $|\alpha_2|+|I_2|<2+k$.

If $|I| < k$ then $|\alpha_2| + |I_2| < 2 + k$ and the above estimates directely give an estimate

(2.46)
$$||Z^I u(t,\cdot)||_2 \le C\int_0^t N_3(s)M_3(s)\,ds + C\int_0^t M_1(s)^2\,\frac{ds}{\sqrt{1+s}} + C\int_0^t M_3(s)^2\,\frac{ds}{(1+s)^{3/2}} + K\epsilon.$$

When $|I| = k$ we must first subtract the terms in $\tilde{Z}^I G_m$, $m = 4,5$ which contain $\partial^{\alpha_2} Z^{I_2} u$, with $|\alpha_2| + |I_2| = 2 + |I|$, as a factor. By (2.30) and (2.31) these terms are $\sum_{i,j=0}^3 (\partial G_m/\partial u''_{ij})\partial_i\partial_j Z^I u$, $m = 4,5$, and we can write

$$(\partial G_m/\partial u''_{ij})\partial_i\partial_j Z^I u = \partial_i\big((\partial G_m/\partial u''_{ij})\partial_j Z^I u\big) - \sum_{|\alpha|\le 2} (\partial^2 G_m/\partial u^{(\alpha)}\partial u''_{ij})(\partial_i\partial^\alpha u)\partial_j Z^I u.$$

For $m = 4,5$ let

$$F_m = \tilde{Z}^I G_m - \sum_{i,j=0}^3 (\partial G_m/\partial u''_{ij})\partial_i\partial_j Z^I u - \sum_{i,j=0}^3\sum_{|\alpha|\le 2} (\partial^2 G_m/\partial u^{(\alpha)}\partial u''_{ij})(\partial_i\partial^\alpha u)\partial_j Z^I u,$$

and for $j = 0,..,3$ let

$$F_j = H_j + \sum_{m=4,5,\ i=0,\dots,3} (\partial G_m/\partial u''_{ij})\partial_i Z^I u. \tag{2.47}$$

Then

$$\Box\, Z^I u = \sum_{j=0}^3 \partial_j F_j + F_4 + F_5, \tag{2.48}$$

and hence

$$Z^I u = \sum_{j=0}^3 \partial_j E * F_j + E * F_4 + E * F_5 + v^I,$$

where v^I is the solution of $\Box\, v^I = 0$ with the same initial data as $Z^I u - \partial_0 E * F_0$. Now we can estimate $||E * F_m(t\cdot)||_2$ by (2.41) if $m = 4$ and (2.42) if $m = 5$. In fact in F_4 we have subtracted the terms which could not be estimated by (2.41) and add new terms which can be estimated by (2.41) since $|(\partial^2 G_4/\partial u^{(\alpha)}\partial u''_{ij})| \le C$. In the same way the terms in F_5 can be estimated by (2.42) since $|(\partial^2 G_5/\partial u^{(\alpha)}\partial u''_{ij})| \le C(|u|+|u'|+|u''|)$. $||\partial_j E * F_j(t,\cdot)||_2$ can still be estimated by (2.46) since $|(\partial G_m/\partial u''_{ij})| \le C(|u| + |u'| + |u''|)$, for $m = 4,5$. Since as before $||v^I(t,\cdot)||_2 \le K\epsilon$ it follows that (2.46) also holds for $|I| = k$ and hence

(2.49)
$$M_2(t) \le C\int_0^t N_3(s)M_3(s)\,ds + C\int_0^t M_1(s)^2\,\frac{ds}{\sqrt{1+s}} + C\int_0^t M_3(s)^2\,\frac{ds}{(1+s)^{3/2}} + K_2\epsilon.$$

When estimating $||Z^I u(t,\cdot)||_\infty$, for $|I| \le l$, we are going to use one of the three following estimates to estimate the three different sorts of terms in (2.32). By Proposition 1.8

(2.50)
$$(1+t)||E * \big(\partial_j\big((\partial^{\beta_1} Z^{J_1} u)\partial^{\beta_2} Z^{J_2} u\big)\big)(t,\cdot)||_\infty \le$$
$$C\Big(\int_0^t N_3(s)^2\,(1+s)ds + \int_0^t M_3(s)^2\,\frac{ds}{(1+s)^2}\Big), \quad \text{if} \begin{cases} |J_i| \le l, & |J_i| + 4 \le k, \\ |\beta_i| \le 1 & \text{for } i = 1,2 \end{cases}.$$

By Proposition 1.9 we have

$$(1+t)||E*\big|(\partial_i Z^{J_1}u)\partial_j Z^{J_2}u\big|(t,\cdot)||_\infty \le C\int_0^t M_1(s)^2\frac{ds}{1+s},\quad \text{if } |J_i|+2\le k,\ \text{ for } i=1,2, \tag{2.51}$$

and

$$(1+t)||E*\big|\frac{(\partial^{\beta_1}Z^{J_1}u)\partial^{\beta_2}Z^{J_2}u}{1+s}\big|(t,\cdot)||_\infty \le C\int_0^t M_3(s)^2\frac{ds}{(1+s)^2}, \tag{2.52}$$
$$\text{if}\quad |J_i|+2\le k,\quad \text{and}\quad |\beta_i|\le 1\ \text{ for } i=1,2,$$

where s in the convolution denotes the time variable t.

We have

$$Z^I u=\sum_{i=0}^{3} E*(\partial_i H_i)+E*(\tilde Z^I G_4)+E*(\tilde Z^I G_5)+w^I, \tag{2.53}$$

were w^I is the solution of $\Box\, w^I=0$ with the same initial data as $Z^I u$, and for this Lemma 1.4 gives

$$(1+t)||w^I(t,\cdot)||_\infty\le K\epsilon. \tag{2.54}$$

Let $|I|\le l$, where $l\le k-4$ by assumption. The terms in H_i are of the form (2.43) with $|I_1|\le|I_2|\le|I|$ so for the first terms in (2.53) we have the estimate (2.50). The remaining terms in (2.53) are either of the form (2.44) or of the form (2.45) with $|\alpha_i|\le 2$. Since $|I_1|\le|I_2|\le k-3$ and ∂_j, for $j=0,..,3$ are in the family of operators Z^I we can use (2.51) or (2.52) to estimate these terms. Hence we get an estimate for $(1+t)||Z^I u(t,\cdot)||_\infty$, for $|I|\le l\le k-4$ by adding the estimate (2.54) and the estimates (2.50)–(2.52). It follows that

$$(1+t)N_2(t)\le C'\Big(\int_0^t N_3(s)^2\,(1+s)ds+\int_0^t M_3(s)^2\,\frac{ds}{(1+s)^2}+\int_0^t M_1(s)^2\frac{ds}{1+s}\Big)+K_4\epsilon. \tag{2.55}$$

Assume that (2.29) holds. Then (2.37) and (2.38) are true if ϵ is sufficiently small so we obtain by (2.39), (2.49) and (2.55)

$$M_1(t)\le C'N_3M_1\epsilon^2\log{(t+1)}+K_1\epsilon, \tag{2.56}$$
$$M_2(t)\le C'(N_3M_3+M_1^2+M_3^2)\epsilon^2 2(\sqrt{1+t}-1)+K_2\epsilon, \tag{2.57}$$
$$(1+t)N_2(t)\le C'(N_3^2+M_3^2+M_1^2)\epsilon^2\log{(t+1)}+K_4\epsilon. \tag{2.58}$$

By Proposition 1.5 and (1.3) we also have

$$(1+t)N_i(t)\le CM_i(t),\quad \text{for}\quad i=1,2,3. \tag{2.59}$$

It follows that we can choose

$$M_1=2K_1,\quad M_2=2K_2,\quad N_2=2K_4,\quad N_1=CM_1,\quad M_3=M_1{+}M_2,\quad N_3=N_1{+}N_2.$$

Then (2.56)–(2.58) implies the estimates (2.29) with strict inequality as well as (2.37)–(2.38) for $\epsilon < \epsilon_0$ and $\epsilon \log(t+1) \le \delta$ if ϵ_0 and δ are sufficiently small.

In case $G''_{uu}(0) \neq 0$ then (2.39) and (2.49) remains true with $M_1(s)$ in the right-hand side replaced by $M_3(s)$. If we use (2.59) in (2.39) and (2.49) we obtain

$$M_3(t) \le C' \int_0^t M_3(s)^2 \frac{ds}{\sqrt{1+s}} + K_3\epsilon,$$

which proves that

$$M_3(t) \le 2K_3\epsilon, \quad \text{if} \quad C'K_3 8\epsilon\sqrt{1+t} \le 1$$

and ϵ is so small that (2.37) and (2.38) holds with $M_i = M_3$, $N_i = CM_3$ for $i = 1, 2$. $\square$

3. Appendix. Here we give a new proof, which is also due to L. Hörmander, of the first part of Proposition 1.6. Recall that E denotes the fundamental solution of $\square$.

Lemma 3.1. *Let $X = (t,x)$, $Y = (s,y) \in \mathbf{R}^{1+3}$ and let $L(X,Y) = ts - \langle x, y\rangle$. Assume that $f \in C^1([0,\infty) \times \mathbf{R}^3)$ and set $u = E * f$. Then if $L(X,X) > 0$ we have*

$$u(X) = \frac{1}{4\pi} \int_{\Lambda_X} (Z_{00}f + 3f)(Y) \frac{dY}{\sqrt{D(X,Y)}}, \tag{3.1}$$

where

$$D(X,Y) = L(X,Y)^2 - L(X,X)L(Y,Y) \ge 0,$$

$Z_{00} = t\partial_t + \sum_{i=0}^3 x_i\partial_i$ and Λ_X is the backward light cone, (with interior), from X.

Proof. Since $E(X) = \delta(L(X,X))H(t)/2\pi$, where $H(t) = 1$ when $t \ge 0$ and $H(t) = 0$ otherwise, we have

(3.2)

$$\begin{aligned} u(X) = \int_0^1 \frac{d}{d\tau}\tau u(\tau X)\, d\tau &= \int_0^1 (Z_{00}u + u)(\tau X)\, d\tau = \int_0^1 E * (Z_{00}f + 3f)(\tau X)\, d\tau \\ &= \int_0^1 \frac{1}{2\pi} \int \delta\big(L(\tau X - Y, \tau X - Y)\big) H(\tau t - s)(Z_{00}f + 3f)(Y)\, dY\, d\tau. \end{aligned}$$

Now

$$\begin{aligned} L(\tau X - Y, \tau X - Y) &= \tau^2 L(X,X) - 2\tau L(X,Y) + L(Y,Y) \\ &= L(X,X)\Big(\tau - \frac{L(X,Y) + \sqrt{D(X,Y)}}{L(X,X)}\Big)\Big(\tau - \frac{L(X,Y) - \sqrt{D(X,Y)}}{L(X,X)}\Big). \end{aligned} \tag{3.3}$$

Here $D(X,Y) \ge 0$ since $L(X,X) > 0$. (See Lemma 3.2 below.) The largest zero of (3.3) corresponds to Y being on the backward light cone from τX so

$$\int_0^1 \delta\big(L(\tau X - Y, \tau X - Y)\big) H(\tau t - s)\, d\tau = H\big(L(X-Y, X-Y)\big) H(t-s)/(2\sqrt{D(X,Y)}),$$

and the lemma follows if we change the order of integration in (3.2). $\square$

Lemma 3.2. *Let X, Y, $D(X,Y)$ and $L(X,Y)$ be as in Lemma 3.1 and set $|X|^2 = t^2+|x|^2$. Then if $L(X,X) \geq a|X|^2$, with $0 < a < 1$, we have*

$$D(X,Y) \geq a|X|^2|y - \frac{s}{t}x|^2 \tag{3.4}$$

and if we also have $L(Y,Y) \leq b|Y|^2$, with $-1 < b < a$, then

$$D(X,Y) \geq a(a-b)^2|X|^2|Y|^2/16 \tag{3.5}$$

Proof. For reasons of homogeneity we may assume that $s = t = 1$. Since the discriminant $D(X, Y + qX))$ is independent of q and since $L(X-Y, X-Y) = -|x-y|^2$ it follows that

$$D(X,Y) = D(X, Y-X) \geq L(X,X)|x-y|^2 \geq a|X|^2|x-y|^2,$$

which proves (3.4). If we subtract the inequalities

$$a(1+|x|^2) \leq 1 - |x|^2, \quad b(1+|y|^2) \geq 1-|y|^2,$$

which imply $|x| < |y|$ since $b < a$, and add $a(|y|^2 - |x|^2)$ to both sides, we obtain

$$(a-b)(1+|y|^2) \leq (1+a)(|y|^2-|x|^2) \leq 4|Y|(|y|-|x|).$$

Hence

$$|y-x| \geq |y|-|x| \geq (a-b)|Y|/4,$$

and (3.5) follows $\square$

Lemma 3.3. *If $g \in C_0^1(\mathbf{R}^3)$ then*

$$\int |g(y)|\, dy/|y| \leq \int |g'(y)|\, dy/2.$$

Proof. In polar coordinates this just means that

$$\int_0^\infty |g(r\omega)| r\, dr \leq \int_0^\infty |\partial_r g(r\omega)| r^2\, dr/2$$

which follows at once by a partial integration. $\square$

Lemma 3.4. *Let $f \in C^2([0,\infty) \times \mathbf{R}^3)$. Then*

$$|x||E * f(t,x)| \leq C \sum_{|I| \leq 2} \iint_{0<s<t} |Z^I f(s,y)|/|y|\, ds\, dy, \tag{3.6}$$

where we only have the vector fields of the Euclidean rotations in the sum.

Proof. By Sobolev's lemma

$$M(t,r) = \sup_\omega |f(t,r\omega)| \leq C \sum_{|I|\leq 2} \int_{|\omega|=1} |Z^I f(t,r\omega)|\, dS(\omega),$$

where we only have the vector fields of the Euclidean rotation in the sum. Hence in the right-hand side of (3.6) we have an estimate for $\iint_{0<s<t} M(s,\rho)\rho\, d\rho\, ds$. Replacing f by M we increase $|E * f|$, so it is enough to estimate $U = E * M$. Expressing $\square$ in polar coordinates we have

$$(\partial_t^2 - \partial_r^2) rU(t,r) = rM(t,r),$$

which implies that

$$rU(r,t) \leq \frac{1}{2} \iint_{0<s<t} M(s,\rho)\rho\, ds\, d\rho. \quad \square$$

Theorem 3.5. *Let $f \in C^2([0,\infty) \times \mathbf{R}^3)$. Then*

$$(1+t+|x|)|E * f(t,x)| \leq C \sum_{|J|\leq 2} \iint_{0<s<t} |Z^J f(s,y)|/(1+s+|y|)\, ds\, dy,$$

Proof. First we shall prove that

$$(t+|x|)|E * f(t,x)| \leq C \sum_{|J|\leq 2} \iint_{0<s<t} |Z^J f(s,y)|/(s+|y|)\, ds\, dy, \tag{3.7}$$

where we only use homogeneous Z's in the sum. In the proof we may assume that $f(t,x) = 0$ in a small neighborhood of $(t,x) = 0$. Let $\chi \in C^\infty(\mathbf{R})$, $\chi(q) \geq 0$, $\chi(q) = 0$ when $q \leq 1/4$ and $\chi(q) = 1$ when $q \geq 3/4$. Set $\psi(Y) = \chi(L(Y,Y)/|Y|^2)$. Then for the homogeneous Z^I's we have $|Z^I \psi(Y)| \leq C_I$ so writing $f = (1-\psi)f + \psi f$ we see that it is enough to prove (3.7) in the two cases i) $L(Y,Y) \geq |Y|^2/4$ in the support of $f(Y)$ and ii) $L(Y,Y) \leq 3|Y|^2/4$ in the support of $f(Y)$.

i) Assume that $L(Y,Y) \geq |Y|^2/4$ in the support of $f(Y)$. Then if $L(X,X) \leq |X|^2/8$ (3.7) follows from (3.1) if we use (3.5) but with X and Y interchanged. If instead $L(X,X) \geq |X|^2/8$ then by (3.4) $\sqrt{D(X,Y)} \geq \sqrt{1/8}|X||y - sx/t|$ so if we for fixed s take $z = y - sx/t$ as new variable in (3.1) and use Lemma 3.3 we get an estimate

$$(t+|x|)|E * f(t,x)| \leq C \sum_{i=1,2,3,\ j=0,1} \iint_{0<s<t} |\partial_i Z_{00}^j f(s,y)|\, ds\, dy.$$

Since $L(Y,Y) \geq |Y|^2/4$ in the support of $f(Y)$ (3.7) follows from (1.7) in this case.

ii) Assume that $L(Y,Y) \leq 3|Y|^2/4$ in the support of $f(Y)$. Then if $L(X,X) \geq 7|X|^2/8$ (3.7) follows from (3.1) if we use the estimate (3.5). If instead $L(X,X) \leq 7|X|^2/8$ then $|x| \geq |t|/\sqrt{15} > 0$. Since by assumption $|y| \geq s/\sqrt{7}$ in the support of $f(s,y)$, (3.7) follows from Lemma 3.4 in this case.

The theorem follows immediately from (3.7). In fact if $s + |y| \geq 1$ in the support of $f(s,y)$ this is obvious and if $s+|y| \leq 2$ in the support of $f(s,y)$ then (3.7) applied to $f(s, y_1+3, y_2, y_3)$ gives $(1+t)|E * f(t,x)| \leq C \sum_{|\alpha|\leq 2} \iint_{0<s<t} |\partial^\alpha f(s,y)|\, ds\, dy$. Hence the theorem follows from (3.7) by one more partition of unity. □

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