# Geometric Analysis of Hyperbolic Equations
# an introduction

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# Chapter 1

# Preface

The field of nonlinear hyperbolic partial differential equations has seen a tremendous development since the beginning of the eighties, following the pioneering works of F. John, D. Christodoulou, L. Hörmander, S. Klainerman and many others. On one hand, many papers were dedicated to understanding global existence and blowup for quasilinear wave equations or systems : Hörmander's book [21] offers a nice overview of the main results. On the other hand, Christodoulou and Klainerman [17] proved the stability of Minkowski space, and this was the starting point of many mathematical works in the framework of General Relativity. If we leave aside the papers about blowup, we observe essentially two main domains of interest : the study of *global* smooth solutions, and the study of *low regularity* solutions, both domains being obviously connected.

The striking fact is the *unity* of all the techniques and ideas used in these papers ; the emphasis is always put on good directions and good components, these components being taken relative to some null frame. In this way, papers have incorporated more and more tools of Riemannian geometry such as metrics, connexions, curvature, etc. They also borrowed concepts from general relativity books, such as energy-momentum tensors, deformation tensors, etc. There seems to be, however, some difficulties : most Riemannian geometry books do not include the specific Lorentzian tools ; most relativity books do not include a description of the relevant mathematical framework. Let us however point out two exceptions : the classical book by Hawking and Ellis [20] and the new book by Rendall [37].

We believe that the use of Lorentzian tools (null frames, etc.) in the mathematical study of nonlinear hyperbolic systems is going to intensify further, even in the aspects of the field not directly related with general relativity. This is what we call "geometric analysis of hyperbolic equations". It is true that there are nonlinear examples where one can get along with the geometry of the standard wave equation : these examples are striking, but the possibility of using standard fields seems to be related to the fact that one is considering only *small* solutions ; for large solutions, we believe that it wil be necessary to take into account the geometry of the linearized operator.

The goal of this booklet is twofold :

i) Give to PDE analysts a self-contained and elementary access to recent literature,

ii) Explain the fundamental ideas connected with the use of null frames.

This book is meant for students or researchers with an elementary background about PDE, specifically hyperbolic PDE. It can be read by students with five years of university training, or partial differential equations researchers, without any knowledge of differential geometry. Though the largest part of the text is about geometric concepts, this book is not a book about Lorentzian geometry : it only provides the geometric tools needed to understand the modern PDE literature. The author not being a geometer, we deliberatetly chose to give naive and self-contained proofs to all statements, which can be viewed as "do it yourself" exercises for the reader, without using sophisticated "well-known" facts. We hope that we will be forgiven for that.

Finally, we would like to thank S. Klainerman and F. Labourie for many helpful converstions.

# Chapter 2

# Introduction

The prototype of all hyperbolic equations is the d'Alembertian

$$\Box \equiv \partial_t^2 - \Delta_x$$

in $\mathbf{R}^4_{x,t}$. We first review briefly some properties of the solutions in the large (referring to [21] for proofs), in order to introduce the concepts and questions of this book.

**1.** We consider in $\mathbf{R}^4_{x,t}$ the Cauchy problem for the standard wave equation

$$\Box\phi = (\partial_t^2 - \Delta_x)\phi = 0,\ \phi(x,0) = \phi_0(x),\ (\partial_t\phi)(x,0) = \phi_1(x).$$

**a.** Suppose for simplicity $\phi_i \in C_0^\infty$, $\phi_i(x) = 0$ for $r = |x| \geq M$ : as a consequence of the classical solution formula, the function $\phi$ can be represented for $r \geq 1$ as

$$\phi(x,t) = r^{-1}F(r-t, \omega, r^{-1}), r = |x|, \omega = x/r,$$

for some $C^\infty$ function $F(\sigma, \omega, z)$ vanishing for $|\sigma| \geq M$ (as a consequence of the strong Huygens principle). Setting $\partial_r = \sum \omega i \partial_i$, we introduce the two fields

$$L = \partial_t + \partial_r,\ \underline{L} = \partial_t - \partial_r,$$

and define the rotation fields $R = x \wedge \partial$,

$$R_1 = x^2\partial_3 - x^3\partial_2,\ R_2 = x^3\partial_1 - x^1\partial_3,\ R_3 = x^1\partial_2 - x^2\partial_1.$$

Remark that $R_i(r) = 0$, and $\sum \omega^i R_i = 0$. Using the representation formula, we observe that

$$L\phi = O(r^{-2}),\ (R/r)\phi = O(r^{-2}),\ r \to +\infty,$$

while for instance $\underline{L}\phi$ has only the magnitude $r^{-1}$. Hence the special derivatives $L\phi$, $(R/r)\phi$ behave better at infinity that the other components of $\nabla\phi$.

**b.** We explain now an "energy method", which is an alternative approach to the preceding decay results, not using an explicit representation for $\phi$. We define the hyperbolic rotations $H = t\partial + x\partial_t$,

$$H_1 = t\partial_1 + x^1\partial_t,\ H_2 = t\partial_2 + x^2\partial_t,\ H_3 = t\partial_3 + x^3\partial_t,$$

and call **Lorentz fields** $Z$ all the fields

$$\partial_\alpha, S = t\partial_t + \sum x_i\partial_i = t\partial_t + r\partial_r, R = x \wedge \partial, H = t\partial + x\partial_t.$$

These fields are known to commute with $\Box$, except for $S$ which satifies $[\Box, S] = 2\Box$. In the situation of the preceding section, commuting any number of fields $Z$ with $\Box$ and using the standard energy inequality, we obtain the bound

$$\sum ||(\nabla Z^k\phi)(.,t)||_{L^2_x} \leq C.$$

Now, the following easy formula establish a connexion between the special derivatives $L$, $R/r$ and the $Z$ fields :

$$(r+t)L = S + \sum \omega_i H_i,\ (t-r)\underline{L} = S - \sum \omega_i H_i,\ R/r = t^{-1}\omega \wedge H.$$

Using these formula, we get for the special derivatives of $\nabla\phi$

$$||(L\nabla\phi)(.,t)||_{L^2} = O(t^{-1}),\ ||(R/r)(\nabla\phi)(.,t)||_{L^2} = O(t^{-1}),\ t \to +\infty.$$

Taking into account the support of $\phi$, we even obtain

$$||(L\phi)(.,t)||_{L^2} = O(t^{-1}),\ ||(R/r)\phi(.,t)||_{L^2} = O(t^{-1}),\ t \to +\infty.$$

Note the contrast with the information given by the standard energy inequality, which yields only boundedness of these quantities.

It is in fact possible to obtain pointwise estimates from the preceding $L^2$-estimates. For this, we use Klainerman inequality, valid for any smooth function $v$ sufficiently decaying at infinity

$$|v(x,t)|(1+t+r)(1+|t-r|)^{1/2} \leq C \sum_{k\leq 2} ||Z^k v(.,t)||_{L^2}.$$

We thus obtain again the pointwise bounds we had from the explicit representation formula

$$L\phi = O(t^{-2}), (R/r)\phi = O(t^{-2}).$$

Note however that this "energy method" is likely to work in variable coefficients situations (or nonlinear situations) where we do not know any representation formula.

If the data are not compactly supported but sufficiently decaying as $|x| \to +\infty$, this energy method still works, but the "interior" behavior of the solution (that is, away from the light cone $\{t = r\}$) is not as good as before.

**c.** There is another type of "energy approach" allowing to take notice of the better behavior of the special derivatives $L\phi$, $R/r\phi$ : we give two examples of this. First, one can prove the following improvement of the standard energy inequality : for all $\epsilon > 0$, there is some constant $C_\epsilon > 0$ such that, assuming $\Box\phi = 0$,

$$E_\phi(T)^{1/2} + \{\int_{0\leq t\leq T} < r-t >^{-1-\epsilon} [(L\phi)^2 + |(R/r)\phi|^2] dx dt\}^{1/2} \leq C_\epsilon E_\phi(0)^{1/2}.$$

This inequality is easily obtained in the same way as the usual energy inequality, using the multiplier $\partial_t$ and a weight $e^a$, where $a = a(r-t)$ is appropriately chosen (see [] for instance). This inequality is only useful in a region where $|r-t|$ is smaller that $t$, that is, close to the light cone : in the region $|r-t| \leq C$ for instance, the $L^2_x$ norm of the special derivatives $L\phi$, $(R/r)\phi$ is not just bounded, it is an $L^2$ function of $t$. We can thus identify the "good derivatives" of $\phi$ directly from the energy inequality, without commuting any fields with the equation.

The second example is the conformal energy inequality which gives, for $\Box\phi = 0$,

$$\tilde{E}_\phi(t)^{1/2} \leq C\tilde{E}_\phi(0)^{1/2},$$

where the conformal energy $\tilde{E}$ is

$$\tilde{E}_\phi(t) = (1/2)\int [(S\phi)^2 + |R\phi|^2 + |H\phi|^2 + \phi^2](x,t) dx.$$

This inequality is obtained in the usual way using the timelike multiplier $K_0$

$$K_0 = (r^2 + t^2)\partial_t + 2rt\partial_r.$$

As above, the direct bound of the quantities $||(Z\phi)(.,t)||_{L^2}$ provided by the inequality yields the bounds

$$||(L\phi)(.,t)||_{L^2} = O(t^{-1}),\ ||(R/r)\phi(.,t)||_{L^2} = O(t^{-1}).$$

Once again, we can identify the good derivatives of $\phi$ directly from the conformal energy inequality.

**2.** Consider now the "null frame"

$$e_1,\ e_2,\ e_3 = \underline{L} = \partial_t - \partial_r,\ e_4 = L = \partial_t + \partial_r,$$

where, at each point $(x_0, t_0)$, $(e_1,\ e_2)$ form an orthonormal basis of the tangent space to the sphere

$$\{(x,t),\ t = t_0,\ |x| = |x_0|.$$

Using spherical coordinates

$$x_1 = r\sin\theta\cos\phi,\ x_2 = r\sin\theta\sin\phi,\ x_3 = r\cos\theta,$$

we can take (away from the poles)

$$e_1 = r^{-1}\partial_\theta,\ e_2 = (r\sin\theta)^{-1}\partial_\phi.$$

The fields $e_1, e_2$ are related to the rotation fields by the formula

$$e_1 = -\sin\phi R_1/r + \cos\phi R_2/r,\ e_2 = (\sin\theta)^{-1}R_3/r.$$

Hence the "special derivatives" on which we insisted above are just, equivalently, the components of $d\phi$ on $e_1$, $e_2$ and $L$. Thus the good derivatives are part of a null frame, the only bad derivative being $\underline{L}$.

To understand the name "null frame", it is best to introduce on $\mathbf{R}^4$ the scalar product of special relativity. For two vectors $X = (X_0, X_1, X_2, X_3)$ and $Y = (Y_0, Y_1, Y_2, Y_3)$, we set

$$< X, Y >= -X_0Y_0 + \sum_{1\leq i\leq 3} X_iY_i.$$

We can then easily check the fundamental properties which define a null frame

$$(e_1, e_2)\perp(e_3, e_4),\ < L, L >= 0,\ < \underline{L}, \underline{L} >= 0, < L, \underline{L} >= -2.$$

The "gradient" $\tilde{\nabla} f$ of a function $f$ in the sense of this scalar product is defined by

$$\forall Y,\ < \tilde{\nabla} f, Y >= df(Y) = Y(f).$$

This gives immediately

$$\tilde{\nabla} f = (-\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f).$$

For instance,

$$\tilde{\nabla}(t-r) = -(1, x/r) = -L.$$

Since $L$ is "null", we also have, with $u = t - r$,

$$< \tilde{\nabla} u, \tilde{\nabla} u >= 0,$$

and we say that $u$ is an "optical" function. Remark that the null frame $(e_1, e_2, \underline{L}, L)$ is associated to the functions $u$ and $t$ in the sense that

i) the surfaces $\{t = t_0,\ u = u_0\}$ are the usual spheres,

ii) $L = -\tilde{\nabla} u$ and $(\underline{L}, L)$ are the two null vectors in the orthogonal space to these spheres.

We see in that way how null frames and optical functions are related. Of course, the function $\underline{u} = t + r$ is also an optical function, and $\underline{L} = -\tilde{\nabla}\underline{u}$. Remark that the level surfaces of $u$ are "outgoing cones", while level surfaces of $\underline{u}$ are "incoming cones" ; also, the good derivatives $(e_1, e_2, L)$ span, at each point, the tangent space to the outgoing cone through this point.

Let us mention to finish the nice relations between the fields $S$, $K_0$ (that we have already encountered) and $u, L, \underline{u}, \underline{L}$,

$$S = (1/2)(u\underline{L} + \underline{u}L),\ K_0 = (1/2)(u^2\underline{L} + \underline{u}^2L).$$

**3.** The aim of this book is to explain how one can extend the previously discussed concepts and results to a general framework. More precisely, suppose we have, instead of the "flat" Minkwoski metric $|X|^2 =< X, X >$ a more general metric $g$

$$g = g_{\alpha\beta} dx^\alpha dx^\beta, \; g(X, Y) \equiv < X, Y >= g_{\alpha\beta} X^\alpha Y^\beta.$$

We define the wave equation $\Box$ associated to this metric by

$$\Box_g \phi \equiv \Box \phi = |g|^{-1/2} \partial_\alpha (g^{\alpha\beta} |g|^{1/2} \partial_\beta \phi),$$

where $|g|$ is the determinant of the matrix $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ its inverse matrix. Our interest concentrates on this wave equation, and also on the associated Maxwell and Bianchi equations. From the considerations above for the "flat" case of the Minkowski metric, the following natural questions arise : for solutions of $\Box_g \phi = 0$,

i) Are there "good derivatives" of $\phi$ (in the sense of a better decay at infinity) ?

ii) How to pick up a null frame which would capture this "good derivatives" ?

iii) What is the relation between null frames and optical functions ?

iv) Can one prove energy inequalities where the good derivatives are singled out ?

v) Are there good substitute for the Lorentz fields $Z$ ?

vi) Can one commute these substitue with $\Box$ to obtain pointwise bounds for the solutions ?

In the case of systems of equations, the vector character of the unknown has to be taken into account : the new question of the good components of the unknown turns out to be crucial.

In quasilinear problems, the metric $g$ is a function of $\phi$ itself : among the most difficult examples are the Einstein equations, which can be written in special coordinates as the system

$$\Box_g g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g),$$

where $F$ is quadratic in $\nabla g$. In such problems, the properties of the solution "bootstrap" to imply properties of the metric, which in turn lead to better properties of the solution, etc.

The plan of the book follows from what we have said before, with the idea of introducing at each step only the necessary geometric machinery :

• In chapter 3, we discuss the notions of metric, optical functions and null frames, and give the basic examples found in the literature.

• The differential geometry aspects appear in chapter 4 where the metric connection is introduced, as a necessary tool to deal with frames ; we define then the frame coefficients and compute them on the basic examples.

• Chapter 5 is dedicated to the specific machinery used to prove energy inequalities : energy-momentum tensor, deformation tensor, etc. The idea is to compute in such a way that the energy and the additional "interior terms" can be easily expressed in the frame we are working with.

• The question about how to pick up a good frame and thus identify the good components of tensors is adressed in chapter 6, where we discuss extensions of the standard energy inequality and of the conformal energy inequality.

• The way to find substitute for the standard Lorentz fields and to commute them with $\Box$ is explained in chapter 7.

• The curvature tensor is introduced only in chapter 8, where we explain how to control optical functions and their associated null frames. We establish there the transport equations and elliptic systems (on (nonstandard) 2-spheres) which govern the frame coefficients.

• Finally, the last chapter is devoted to discuss a number of applications of the ideas of the previous chapters to nonlinear problems. Though it seems impossible to give complete proofs of very difficult results, we try to outline the constructions of frames, the inequalities used, etc., hoping to provide a guide for further reading.

# Chapter 3

# Metrics and Frames

## 3.1 Metrics, Duality

**a.** We will work in $\mathbf{R}^4$ or in a 4-dimensional manifold $M$. Local coordinates on $M$ are denoted by $x^\alpha$, $\alpha = 0, 1, 2, 3$. Sometimes, $x^0 = t$ is thought of as "the time", while $(x^1, x^2, x^3)$ are the "spatial coordinates", though this does not make much sense in the context of relativity theory. The corresponding partial derivatives are $\partial_\alpha = \partial / \partial x^\alpha$.

The **position of the indices** is essential : vector fields are indexed with a lower index, such as $\partial_\alpha$, 1-forms are indexed with an upper index, such as $dx^\alpha$. The components of a vector field $X$ are denoted by $X^\alpha$, since $X^\alpha = dx^\alpha(X)$, and the components of a 1-form $\omega$ are denoted by $\omega_\alpha$, since $\omega_\alpha = \omega(\partial_\alpha)$. Here and in the sequence, a repeated sum on an index in lower and upper position is *never* indicated ; for instance, we write in local coordinates a vector field $X = \sum X^\alpha \partial_\alpha = X^\alpha \partial_\alpha$, a 1-form $\omega = \sum \omega_\alpha dx^\alpha = \omega_\alpha dx^\alpha$. If $f$ is a function, we define $df = \sum (\partial_\alpha f) dx^\alpha = (\partial_\alpha f) dx^\alpha$, and $Xf = X^\alpha \partial_\alpha f$, etc.

**b.** A **metric** is the smooth assignment to each point $m$ of a symmetric bilinear form on $T_m M$. In local coordinates $x^\alpha$, the components of the metric are $g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta)$, which are supposed to be smooth. Hence $g$ can be locally identified with the symmetric $4 \times 4$ matrix $(g_{\alpha\beta})$. The elements of the inverse matrix are denoted by $g^{\alpha\beta}$, the determinant of $g$ by $|g|$. In the whole book, the signature of the quadratic form $g$ will be $(-1, +1, +1, +1)$ ; in other words, the metric is supposed to be Lorentzian (and non degenerate), in contrast with the Riemanian case where $g$ is assumed to be positive definite.

Using the same convention on repeated indices, the metric is sometimes written

$$g \equiv ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \; g(X, Y) = g_{\alpha\beta} X^\alpha Y^\beta.$$

The most common **examples of Lorentzian metrics** are the following :

i) The **Minkowski metric** (also called "flat" metric), given on $\mathbf{R}^4_{x,t}$ by

$$g = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Using spherical coordinates (see Introduction), we get

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2).$$

Setting $\underline{u} = t + r$, $u = t - r$, then $_tanp = \underline{u}$, $\tan q = u$, it is often convenient to compactify the whole of $\mathbf{R}^4$ by introducing new coordinates $t' = p + q$, $r' = p - q$, with

$$-\pi < t' + r' < \pi,\ -\pi < t' - r' < \pi,\ r' \geq 0.$$

In these coordinates, the metric is

$$g = [4\cos^2((1/2)(t' + r'))\cos^2((1/2)(t' - r'))]^{-1}\tilde{g},\ \tilde{g} = -dt'^2 + dr'^2 + \sin^2 r'(d\theta^2 + \sin^2\theta d\phi^2.$$

The corresponding drawing in two dimensions in the coordinates $(r', t')$ is called a *Penrose diagramm*. It allows a better understanding of "infinity" : the lines $\mathcal{I}^+ = \{r' + t' = \pi\}$ and $\mathcal{I}^- = \{t' - r' = -\pi\}$ are called respectively future and past null infinity, etc.

ii) Perturbations of the Minkowski metric such as

$$g = -dt^2 + \sum g_{ij} dx^i dx^j.$$

Note that in this context, latin indices run from 1 to 3 (while greek indices run from 0 to 3 !). The matrix $g_{ij}$ is positive definite. Considering the absence of "cross terms", we say that $g$ is **split**.

iii) The **Schwarzschild metric**

$$g = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Here, $m \geq 0$ is given, and $(r, \theta, \phi)$ are spherical coordinates on $\mathbf{R}^3$. When $m = 0$, this metric reduces to the Minkowski metric written in spherical coordinates. One can show that the surface $\{r = 2m\}$ is only an *apparent* singularity of the metric, and one can also construct Penrose diagramms for this metric (see [20] for details).

iv) The **Kerr metric**

$$g = -[(\Delta - a^2\sin^2\theta)/\Sigma]dt^2 + (\Sigma/\Delta)dr^2 - 4amr\sin^2\theta/\Sigma dt d\phi + A\sin^2\theta d\phi^2 + \Sigma d\theta^2,$$

with $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2mr$, $A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$. When $a = 0$, $g$ is the Schwarzschild metric. Again, it is explained in [20] how to construct Penrose diagramms for Kerr metrics.

**c.** The metric provides a bijection between vector fields and 1-forms according to the formula

$$\forall Y,\ < X, Y >= \omega(Y).$$

We say that $X$ and $\omega$ are **dual** of each other if the above relation is true for all fields $Y$. In local coordinates, this reads

$$X = X^\alpha \partial_\alpha,\ \omega = \omega_\alpha dx^\alpha,\ g_{\alpha\beta} X^\beta = \omega_\alpha.$$

We say that $\omega_\alpha$ is obtained from $X^\beta$ by "lowering" the index, and we just write $X_\alpha = g_{\alpha\beta} X^\beta$. Analogously, we write, "raising" the index, $\omega^\alpha = g^{\alpha\beta}\omega_\beta$. Hence we do not distinguish between $X$ and $\omega$, using the same letter for both. We do the same for more general tensors ; for instance, if $T$ is a 2-tensor acting on vector fields, with components $T_{\alpha\beta} = T(\partial_\alpha, \partial_\beta)$, we write

$$T_\alpha^{\ \beta} = g^{\beta\gamma} T_{\gamma\alpha},$$

and so on.

• Let $f$ be a $C^1$ function on $M$. The **gradient** $\nabla f$ is defined as the dual of $df$, with components

$$\nabla f^\alpha = \partial^\alpha f \equiv g^{\alpha\beta}\partial_\beta f.$$

Note that, by definition, $< \nabla f, X >= df(X) = Xf$, a very useful formula.

• If $e_\alpha$ is a basis, its **dual basis** is defined to be

$$e^\alpha = g^{\alpha\beta} e_\beta,$$

in such a way that $< e^\alpha, e_\beta >= \delta^\alpha_\beta$, $\delta^\beta_\alpha$ being one for $\alpha = \beta$ and 0 otherwise. For a two tensor $T_{\alpha\beta}$, we define its **trace** to be

$$trT = g^{\alpha\beta} T_{\alpha\beta} = T(e_\alpha, e^\alpha) = T(e^\alpha, e_\alpha).$$

The remarkable fact is that this trace is independent of the basis $e_\alpha$ chosen.

• Define locally the 4-form $\epsilon$ to be

$$\epsilon = |g|^{1/2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

From now on , we assume that we are working on $\mathbf{R}^4$ or on an *orientable* manifold $M$. One can then easily check that $\epsilon$ does not depend on the local coordinates, we call it the **volume form**. One should not confuse this volume form with the volume element used to integrate functions on the manifold, with is the positive measure $dv = |g|^{1/2} dx$.

• For any tensor $T_{\alpha\beta\dots\gamma}$, we define

$$|T|^2 = T_{\alpha\beta\dots\gamma} T^{\alpha\beta\dots\gamma}.$$

For instance, if $T$ is a vector field,

$$|T|^2 = T_\alpha T^\alpha = g_{\alpha\beta} T^\beta T^\alpha =< T, T >$$

as expected. Note that $|T|^2$ has no reason to be non negative.

## 3.2 Optical functions

A $C^1$ function $u$ is called an **optical function** if it satisfies the **eikonal equation**

$$g^{\alpha\beta}\partial_\alpha u \partial_\beta u = g_{\alpha\beta}\partial^\alpha u \partial^\beta u =< \nabla u, \nabla u >= |\nabla u|^2 = 0.$$

In PDE terms, this means that the level surfaces $\{u = C\}$ are characteristic surfaces for $\Box$. The classical examples for the Minkowski metric are the functions $u = t - r$ and $\underline{u} = t + r$ ; in this case, the level surfaces $\{u = C\}$ or $\{\underline{u} = C\}$ are respectively the outgoing light cones and the incoming light cones with vortices on the $t$-axis, and the integral curves of the fields $L = -\nabla u$ and $\underline{L} = -\nabla \underline{u}$ generate the cones. In general, note that if an optical function $u$ is constant on a surface $S$, then $\nabla u$ is both normal and tangent to $S$. Conversely, if $u$ is constant on a surface $S$ to which $\nabla u$ is tangent, this implies that $u$ is an optical function.

There are many ways of constructing optical functions for a given metric : one possibility is to solve a Cauchy problem with data on some hypersurface, using the classical method of characteristics for the eikonal equation. Another possibility is to define first outgoing and incoming (half-)cones with vortices on some line, and to take $u$ and $\underline{u}$ to be functions having these half-cones as level surfaces. We will come back to this in 4.3 and in the last chapter. Below, we show the role played by optical functions in constructing special basis called null frames.

## 3.3 Null Frames

In an euclidean space, orthonormal basis play an important role. The corresponding concept for a Lorentzian metric is that of **null frame**. A null frame is a basis $(e_1, e_2, e_3, e_4)$ given at each point (and depending smoothly on this point), such that

$$< e_1, e_1 >= 1, < e_2, e_2 >= 1, < e_1, e_2 >= 0,$$

$$< e_3, e_3 >= 0, < e_4, e_4 >= 0, < e_3, e_4 >= -2\mu,$$

and the subspace generated by $(e_1, e_2)$ is orthogonal to the subspace generated by $(e_3, e_4)$.

Note the formula which gives $\nabla f$ in a null frame :

$$\nabla f = e_1(f)e_1 + e_2(f)e_2 - (2\mu)^{-1}(e_4(f)e_3 + e_3(f)e_4).$$

The dual basis of a null frame is $(e_1, e_2, -(2\mu)^{-1}e_4, -(2\mu)^{-1}e_3)$, hence the trace of a symmetric 2-tensor $T$ will be

$$tr\, T = T(e_1, e_1) + T(e_2, e_2) - \mu^{-1}T(e_3, e_4).$$

The most classical example of null frame is, for the Minkowski metric, using spherical coordinates,

$$e_1 = r^{-1}\partial_\theta, e_2 = (\sin\theta)^{-1}\partial_\phi, e_3 = \partial_t - \partial_r, e_4 = \partial_t + \partial_r.$$

The first two vectors form an orthonormal basis on the spheres of $\mathbf{R}^3_x$ (for constant $t$), the last two a basis of the orthogonal space to the sphere, and $\mu = 1$. In general, for a given metric $g$, we wish to construct "good" null frames, that is null frames which make the computations as easy as possible (in particular, we try to arrange $\mu = 1$ in most cases). Here are a few examples :

**1. Quasiradial frame**

Let $g$ be a metric on $\mathbf{R}^4_{x,t}$ satisfying

$$g^{00} = -1,\ g^{0i}\omega_i = 0,\ \omega = x/r.$$

This will be in particular the case of a split metric. Set

$$T = -\nabla t = -g^{\alpha\beta}\partial_\alpha t \partial_\beta = -g^{0\beta}\partial_\beta = \partial_t - g^{0i}\partial_i.$$

We observe that $< T, T >= \nabla t(t) = -1$, and that $T$ is orthogonal to the surface

$$\Sigma_t = \{(x,t)\}.$$

We define then $N = \nabla r / |\nabla r|$, a unit vector orthogonal to the standard spheres ; moreover, $T$ and $N$ are orthogonal, since $\partial_i r = x_i/r = \omega_i$ and $|\nabla r| < T, N >= T(r) = -g^{0i}\omega_i = 0$. Hence, if we take $(e_1, e_2)$ to be an orthonormal basis (in the sense of $g$ !) on the standard spheres, the basis

$$e_1, e_2, e_3 = T - N, e_4 = T + N$$

is a null frame, with $< e_3, e_4 >= -2$.

The advantage of this choice is its explicit character and simplicity, since one uses only the foliation by the standard 2-spheres : we will see that it is sufficient for many applications. It turns out, however, that there can be good reasons to introduce nonstandard spheres, as we presently see.

**2. Null frame associated to one optical function**

Let $g = -dt^2 + g_{ij}dx^i dx^j$ be a split metric on $\mathbf{R}^4_{x,t}$ (close to the Minkowski metric) and $u$ an optical function for $g$ (close to $t - r$) ; this is for instance the framework of [24]. Using the coordinate $t$, we define the foliation $\Sigma_{t_0} = \{(x,t), t = t_0\}$, and using $u$, we define the foliation by nonstandard 2-spheres

$$S_{t_0,u_0} = \{(x,t), t = t_0, u(x,t) = u_0\}.$$

We set then $L = -\nabla u = (\partial_t u)\partial_t - (g^{ij}\partial_i u)\partial_j$. Since $\nabla u$ is orthogonal to $\{u = u_0\}$ and $\partial_t$ is orthogonal to $\Sigma_{t_0}$, the field $\tilde{N} = -(g^{ij}\partial_i u)\partial_j$ is an horizontal field orthogonal to $S_{t_0,u_0}$. Moreover,

$$< \tilde{N}, \tilde{N} >= g_{ij}(g^{ik}\partial_k u)(g^{jl}\partial_l u) = g^{kl}\partial_k u \partial_l u = (\partial_t u)^2.$$

We set $a = (\partial_t u)^{-1}$, $N = a\tilde{N}$. Then, if $(e_1, e_2)$ form an orthonormal basis on the nonstandard spheres, the frame

$$e_1, e_2, e_3 \equiv \underline{L} = a(\partial_t - N), e_4 \equiv L = a^{-1}(\partial_t + N)$$

is a null frame, with $< \underline{L}, L >= -2$.

**3. Null frame associated to two optical functions**

A more symmetric approachfor a general metric $g$ is to consider two optical functions $u$ and $\underline{u}$, and define the sphere foliation by

$$S_{u_0,\underline{u}_0} = \{(x,t), u(x,t) = u_0, \underline{u}(x,t) = \underline{u}_0\}.$$

One can think of $u$ and $\underline{u}$ as being close to $t - r$ and $t + r$, the spheres being nonstandard 2-spheres close to the usual ones. The advantage of doing so is that we do not have to consider any $t$-coordinate, which is more satifying in the context of relativity theory. We set then

$$L = -\nabla u, \ \underline{L} = -\nabla \underline{u}, \ 2\Omega^2 = - < L, \underline{L} >= -(g^{\alpha\beta}\partial_\alpha u \partial_\beta \underline{u})^{-1}.$$

The desired null frame is

$$e_1, e_2, e_3 = 2\Omega\underline{L}, e_4 = 2\Omega L, \ < e_3, e_4 >= -2,$$

if, as before, $(e_1, e_2)$ form an orthonormal basis of the spheres $S_{u_0,\underline{u}_0}$.

**4. Null frame associated to a sphere foliation**

More generally, following [25], we can start from a 2-sphere foliation chosen in such a way that the metric is positive definite on the tangent space to these 2-spheres. We choose then $(e_1, e_2)$ to be an orthonormal basis on the spheres, and $e_3$ and $e_4$ to be null vectors in the orthogonal space to the spheres. The quasiradial case, for instance (example 1), corresponds to choosing the standard spheres for this foliation, a simple choice which turns out to be sufficient for many applications. Working with non standard spheres as in examples 2 or 3 can be delicate.

In this construction, $e_4$ is orthogonal to the planes generated by $(e_1, e_2, e_4)$ ; if the distribution of these planes is integrable, that is, if there exists $u$ such that these planes are tangent to the hypersurfaces $\{u = C\}$, then $e_4$ is colinear to $\nabla u$, hence $u$ is an optical function as in example 2. This shows how optical functions appear naturally in this framework. Note that quasiradial frames are not integrable in general.

**5. Schwarzschild metric :** Since this metric is rotationally invariant, we also use the standard spheres, the restriction of the metric being then the standard euclidean metric. We take $(e_1, e_2)$ as usual, and

$$e_3 = \partial_t - (1 - 2m/r)\partial_r, \ e_4 = \partial_t + (1 - 2m/r)\partial_r.$$

**6. Kerr metric :** The case of the Kerr metrics is more delicate, since the metric is not rotationally invariant. According to [39], there are good *algebraic* reasons to set

$$X = \partial_t + a/(r^2 + a^2)\partial_\phi, Y = \Delta/(r^2 + a^2)\partial_r,$$

and to choose

$$e_1 = \Sigma^{-1/2}\partial_\theta, e_2 = (\Sigma^{1/2}\sin\theta)^{-1}(\partial_\phi + a\sin^2\theta\partial_t),$$
$$e_3 = X - Y, e_4 = X + Y, < e_3, e_4 >= -2\Sigma\Delta(r^2 + a^2)^{-2}.$$

Note that$(e_1, e_2)$ are not tangent to a sphere foliation, since

$$[e_1, e_2] = (r^2 + a^2)\Sigma^{-3/2}\cos\theta[a\Sigma^{-1/2}(e_3 + e_4) - (\sin\theta)^{-1}e_2]$$

is not generated by $(e_1, e_2)$.

# Chapter 4

# Computing with frames

Performing computations in a variable frame requires very often to take derivatives of components. A typical example will be an expression of the form $X < Y, Z >$, where $(X, Y, Z)$ are vector fields : the result involves both derivatives of the coefficients of the metric and derivatives of the coefficients of the fields $Y, Z$. Even such simple computations become quickly impossible if one does not use the appropriate geometric tool, the metric connexion.

## 4.1 Metric Connexion

**1.** A **connexion** is a derivation operator $D$ of one vector field by another, yielding a new vector field :

$$(X, Y) \mapsto D_X Y,$$

with the following properties :

i) For any *function* $f \in C^\infty$, $D_{fX}Y = fD_X Y$. We say that $D$ is "linear" in $X$ ; this implies in particular that $(D_X Y)(m)$ depends only on $X(m)$.

ii) For any function $f$, $D_X(fY) = fD_X Y + (Xf)Y$. This is similar to the usual derivation of a product.

iii) For any fields $X, Y$, $D_X Y - D_Y X = [X, Y] \equiv (XY - YX)$. We say that $D$ is "torsion free".

The fundamental theorem (see [19]) is that there exists a unique "metric" connexion, that is a connexion enjoying the additionnal property

$$X < Y, Z >=< D_X Y, Z > + < Y, D_X Z > .$$

This formula seems to ignore the derivatives of the coefficients of the metric, which are however present in $< Y, Z >$. This is because these derivatives are in fact part of the definition of $D$, as we see now. In local coordinates, set

$$D_{\partial_\alpha} \equiv D_\alpha,\ D_{\partial_\alpha}\partial_\beta \equiv D_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma,$$

thus defining the **Christoffel symbols** $\Gamma^\gamma_{\alpha\beta}$. The torsion free character of $D$ implies the symmetry $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$, and, using the properties in the definition of $D$, it is easy to obtain the formula (be careful that we lowered the first index !)

$$\Gamma_{\gamma\alpha\beta} \equiv g_{\gamma\nu}\Gamma^\nu_{\alpha\beta} =< D_\alpha\partial_\beta, \partial_\gamma >= (1/2)(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}).$$

The simplest example of connexion in $\mathbf{R}^4$ is

$$D_X(Y^\alpha\partial_\alpha) = X(Y^\alpha)\partial_\alpha,$$

which is the metric connexion corresponding to a constant coefficients metric. Why dont we take this in all cases ? just because it does not make sense : the formula is given in local coordinates, and not as an intrinsic formula ! In general, we have

$$D_X Y = X(Y^\beta)\partial_\beta + \Gamma^\gamma_{\alpha\beta}X^\alpha Y^\beta\partial_\gamma.$$

**2.** It is very useful to extend $D$ to derive any tensor field $T$. The natural way to do this is to generalize the product formula ; if $T$ acts on $p$ vectors, we set

$$X[T(Y_1, \ldots, Y_p)] = (D_X T)(Y_1, \ldots, Y_p) + T(D_X Y_1, Y_2, \ldots, Y_p) + \ldots + T(Y_1, \ldots, D_X Y_p).$$

For instance,

$$X[g(Y_1, Y_2)] = (D_X g)(Y_1, Y_2) + g(D_X Y_1, Y_2) + g(Y_1, D_X Y_2),$$

which gives, by comparison with the formula defining a metric connexion, $D_X g = 0$. Another instructive example is the computation of $D_X\omega$ for a 1-form $\omega$ : for any vector field $Y$,

$$X(\omega(Y)) = (D_X\omega)(Y) + \omega(D_X Y).$$

If $Z$ is the vector field dual to $\omega$, by the metric property,

$$< D_X Z, Y >= - < Z, D_X Y > +X(< Z, Y >) = -\omega(D_X Y) + X(\omega(Y)) = (D_X\omega)(Y).$$

In other words, $D_X Z$ is also dual to $D_X\omega$ : we said before we would not distinguish between $\omega$ and its dual $Z$ ; this holds also when we take derivatives : there is no need to know what object we derive (form or field), the result is the same.

**3.** The **divergence** of a vector field $X$ is defined as

$$div\, X = D_\alpha X^\alpha.$$

It is important to note that here and in the sequence, $D_\alpha X^\alpha$ *never* means that we take the derivative $\partial_\alpha$ of $X^\alpha$ : it means that we compute first $D_\alpha X$, and then take the $\alpha$-coordinate :

$$D_\alpha X^\alpha \equiv [D_\alpha X]^\alpha.$$

In local coordinates, using the above formula for the Christoffel symbols,

$$div\, X = \partial_\alpha(X^\alpha) + X^\beta \Gamma^\alpha_{\alpha\beta} = \partial_\alpha(X^\alpha) + (1/2) g^{\alpha\beta} X(g_{\alpha\beta}).$$

Using the formula $(\log det\, A)' = tr\,(A' A^{-1})$, we also obtain

$$\partial_\alpha |g| = |g| g^{\beta\gamma} \partial_\alpha g_{\beta\gamma},\; div\, X = |g|^{-1/2} \partial_\alpha (X^\alpha |g|^{1/2}).$$

**4.** The following useful emma is a consequence of $D_X g = 0$.

**Lemma.** *i) Let $T$ be a 2-tensor and $X$ any vector field. Then*

$$X(tr\, T) \equiv X(T^\alpha_\alpha) = tr D_X T \equiv D_X T^\alpha_\alpha.$$

*ii) Similarly, $X|T|^2 = 2 D_X T_{\alpha\beta} T^{\alpha\beta}$.*

To prove the first formula, we note that, in any frame $(e_\alpha)$, $tr\, T = T(e_\alpha, e^\alpha)$. Hence

$$X(tr\, T) = (D_X T)(e_\alpha, e^\alpha) + T(D_X e_\alpha, e^\alpha) + T(e_\alpha, D_X e^\alpha).$$

Since $< e^\alpha, e_\beta >= \delta^\alpha_\beta$, $D_X e^\alpha_\beta = -D_X e^\alpha_\beta$. This implies that the last two terms in the formula for $X(tr\, T)$ cancel out.

The proof of the second formula is similar :

$$X(T_{\alpha\beta} T^{\alpha\beta}) = X(T_{\alpha\beta}) T^{\alpha\beta} + T_{\alpha\beta} X(T^{\alpha\beta}) =$$

$$= 2(D_X T)_{\alpha\beta} T^{\alpha\beta} + T^{\alpha\beta}[T(D_X e_\alpha, e_\beta) + T(e_\alpha, D_X e_\beta)] +$$
$$+ T_{\alpha\beta}[T(D_X e^\alpha, e^\beta) + T(e^\alpha, D_X e^\beta)].$$

Using $D_X e^\alpha_\beta = -D_X e^\alpha_\beta$ as before, we see that the $T$ terms cancel out. ♢

## 4.2 Submanifolds

If $S \subset M$ is a submanifold of $M$, the restriction of $g$ to vectors tangent to $S$ gives a metric on $S$. If $S$ has codimension one with unit normal $N$, we define the bilinear **second form** $k$, acting on vector fields $X, Y$ tangent to $S$, by

$$k(X, Y) = - < D_X N, Y > .$$

Remark that $k$ is symmetric, since

$$k(X,Y) = -X < Y, N > + < N, D_X Y > = < N, [X,Y] + D_Y X > =$$

$$= < D_Y X, N > = - < D_Y N, X > = k(Y,X).$$

We have used here the torsion free character of $D$, the metric property and the fact that the Lie bracket $[X,Y]$ of $X$ and $Y$ is also tangent to $S$.

**Example 1 :** Let $g = -dt^2 + g_{ij}dx^i dx^j$ be a split metric ; the second form of $S = \Sigma_t$ is given by $k_{ij} = -(1/2)\partial_t g_{ij}$, since

$$N = \partial_t, < D_i \partial_t, \partial_j > = \Gamma_{ji0} = (1/2)\partial_t g_{ij}.$$

**Example 2 :** Consider in $\mathbf{R}^3_x$ the flat riemannian metric and let $S$ be the sphere of radius $R$ in $\mathbf{R}^3$ ; then

$$N = R^{-1}x^i\partial_i, \; D_X N = R^{-1}X(x^i)\partial_i = R^{-1}X$$

and $k(X,Y) = -R^{-1} < X, Y >$. Keep in mind, in particular, that the trace of $k$ is $-2/R$.

For vectors $X, Y$ tangent to the hypersurface $S$, we decompose $D_X Y$ into its tangential and normal parts

$$D_X Y = T(X,Y) + R(X,Y).$$

Since

$$< D_X Y, N > = < R(X,Y), N > = k(X,Y),$$

we have $R(X,Y) = k(X,Y)N$. It is then straightforward to check that $T(X,Y)$ enjoyes all the properties of a metric connexion on $S$ : by uniqueness, it is *the* metric connexion on $S$, denoted by $\not{D}_X Y$, and the formula reads

$$D_X Y = \not{D}_X Y + k(X,Y)N.$$

Finally, let us mention **Stokes formula** in this context : let $D$ be an open domain (we assume that there can be no confusion between the domain and the connexion !) of $\mathbf{R}^4$ with smooth boundary $\partial D$, and $X$ be a vector field on $D$. Then

$$\int_D div\, X dV = \int_{\partial D} < X, N > dv.$$

Here, the oriented unit normal $N$ is defined by $N = \nabla f / ||\nabla f||$, if $f$ defines $\partial D$ and $f < 0$ in $D$ ; the volume elements on $D$ and $\partial D$ are respectively $dV$ and $dv$.

## 4.3 Hessian and d'Alembertian

**1.** For a given $C^2$ function $f$, we define the **Hessian** $\nabla^2 f$ of $f$ as the bilinear form

$$\nabla^2 f(X,Y) = < D_X \nabla f, Y > .$$

More explicitly,

$$\nabla^2 f(X,Y) = X < \nabla f, Y > - < \nabla f, D_X Y >= X(Y(f)) - (D_X Y)f.$$

This formula gives in particular

$$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = (XY - YX)(f) - (D_X Y - D_Y X)(f) = [X,Y]f - [X,Y]f = 0,$$

that is, $\nabla^2 f$ is bilinear *symmetric*.

**2.** The **d'Alembertian** $\Box_g f = \Box f$ of $f$ is defined as the trace of $\nabla^2 f$ ; from the formula in section 3.1 we get the various representations

$$\Box f = (\nabla^2 f)^\alpha_\alpha =< D_\alpha \nabla f, \partial^\alpha >= div\, \nabla f = |g|^{-1/2}\partial_\alpha(g^{\alpha\beta}|g|^{1/2}\partial_\beta f),$$

$$\Box f = g^{\alpha\beta}(\nabla^2 f)(\partial_\alpha, \partial_\beta >= g^{\alpha\beta}[\partial^2_{\alpha\beta} f - (D_\alpha \partial_\beta)f] = \partial^\alpha\partial_\alpha f - (D^\alpha\partial_\alpha)f,$$

$$\Box f = g^{\alpha\beta}\partial^2_{\alpha\beta} f + [\partial_\alpha(g^{\alpha\beta}) + (1/2)g^{\lambda\mu}\partial^\beta g_{\lambda\mu}]\partial_\beta f.$$

Note that the principal symbol of $\Box$ is $p = g^{\alpha\beta}\xi_\alpha\xi_\beta$, but there are also lower order terms in $\Box$ !

In a null frame $(e_1, e_2, e_3, e_4)$, using the formula for the trace of a symmetric tensor, we get

$$\Box f = -\nabla^2 f(e_3, e_4) + \nabla^2 f(e_1, e_1) + \nabla^2 f(e_2, e_2) = -e_4 e_3 f + (e_1^2 + e_2^2)f + [D_4 e_3 - (D_1 e_1 + D_2 e_2)]f.$$

One has to be careful in interpreting this formula, as we can see in the flat case with the usual d'Alembertian. In this case,

$$e_4 e_3 = \partial_t^2 - \partial_r^2,\ D_4 e_3 = 0.$$

Using spherical coordinates and taking $e_1 = r^{-1}\partial_\theta$, $e_2 = (r\sin\theta)^{-1}\partial_\phi$, we have

$$D_1 e_1 = -r^{-1}\partial_r,\ D_2 e_2 = -(r\sin\theta)^{-2}(x^1\partial_1 + x^2\partial_2).$$

Hence, using the definition of the induced connexion on the sphere,

$$\not{D}_1 e_1 = 0, D_2 e_2 = -r^{-1}\partial_r + \not{D}_2 e_2,\ \not{D}_2 e_2 = -r^{-2}(\cos\theta/\sin\theta)\partial_\theta.$$

Gathering the terms,

$$D_1 e_1 + D_2 e_2 = -(2/r)\partial_r - r^{-2}(\cos\theta/\sin\theta)\partial_\theta.$$

Finally, we get

$$\Box = -\partial_t^2 + \partial_r^2 + (2/r)\partial_r + r^{-2}\Delta_S,\ \Delta_S = \partial_\theta^2 + (\sin\theta)^{-2}\partial_\phi^2 + (\cos\theta/\sin\theta)\partial_\theta.$$

We recognize the usual expression of the d'Alembertian in spherical coordinates, $\Delta_S$ being the Laplacian on the unit sphere.

**3. Geodesics, Bicharacteristics and Optical functions :**

**a.** To the metric $g$ is associated the symbol

$$p(x,\xi)=g^{\alpha\beta}\xi_\alpha\xi_\beta.$$

This is a well-defined function on the cotangent space to the manifold $M$, which is the principal symbol of the wave operator $\Box$. From a PDE point of view, it is important to consider (null) bicharacteristic curves of $\Box$ starting from a point $(x_0,\xi^0)$, which are defined by

$$(d/ds)(x^\alpha \equiv \dot{x}^\alpha=\partial_{\xi_\alpha}p, \dot{\xi}_\alpha=-\partial_\alpha p, x^\alpha(0)=x_0^\alpha, \xi_\alpha(0)=\xi^0_\alpha, p(x_0,\xi^0)=0.$$

If we start from the point $(x_0,\mu\xi^0)$, the solution is just $(x(\mu s),\mu\xi(\mu s))$. Differentiating the system once more, we get an autonomous differential equation for the coordinates $x^\alpha$

$$(d^2/ds^2)(x^\alpha)=\partial_\gamma g^{\alpha\beta}g_{\beta\mu}\dot{x}^\gamma\dot{x}^\mu-(1/2)\partial^\alpha g^{\lambda\mu}g_{\lambda\lambda'}g_{\mu\mu'}\dot{x}^{\lambda'}\dot{x}^{\mu'}=$$

$$=-g^{\alpha\beta}\partial_\gamma g_{\beta\mu}\dot{x}^\gamma\dot{x}^\mu+(1/2)\partial^\alpha g_{\lambda\lambda'}\dot{x}^\lambda\dot{x}^{\lambda'}=-\Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma.$$

We recognize the equation of a **geodesic curve**, which is, by definition, a curve such that $D_T T=0$, for $T=\dot{x}$ ; in fact,

$$D_T T=D_T\dot{x}=T(\dot{x}^\alpha)\partial_\alpha+\Gamma^\alpha_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma\partial_\alpha.$$

Since $Tf=(d/ds)(f(x(s)))$ for any $C^1$ function $f$, the claim is proved. The initial conditions for the geodesic curve which is the projection of the bicharacteristic are

$$x(0)=x_0, \dot{x}(0)^\alpha=2g^{\alpha\beta}\xi^0_\beta.$$

**b.** For a given point $x_0$, consider a non-zero future oriented null vector $\xi^0$ at this point : there is a unique bicharacteristic curve starting from $(x_0,\xi^0)$ ; the union of all such half-curves (for $s\geq 0$) starting from $x_0$ form the geodesic (half) cone with summit at $x_0$. Let $L$ be the vector field $\dot{x}$ on this cone. For each value of the parameter $s$, let $S_s$ be the locus of the points $x(s)$ for the various $\xi_0$. Choose a non zero field $X$ tangent to, say, $S_{s_0}$, and define $X$ on the cone to be $X$ extended by the action of the flow of $L$.

**Lemma.** *The vector $L$ is a null vector, which is orthogonal to the geodesic cone.*

First, $<L,L>=0$ since $L<L,L>=2<D_L L,L>=0$ and, for $s=0$,

$$<L,L>=g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=4p(x_0,\xi^0)=0.$$

Next, $[L,X]=0$ by construction. Then, for the induced connexion $\not{D}$ on the cone,

$$L<X,L>=<\not{D}_L X,L>+<X,\not{D}_L L>=<[L,X],L>+<\not{D}_X L,L>=(1/2)X<L,L>=0.$$

Since $<X,L>$ goes to zero when $s$ goes to zero, $<X,L>=0$ and the orthogonal to $L$ is the tangent plane to the cone. ♢

Consider now a one parameter family of geodesic cones such that there exists a function $u$ having the cones of this family as level sets : a typical case would be, for the Minkowski metric, the geodesic cones with vortices on the $t$-axis, corresponding to a function $u = F(t-r)$ (which is of course singular on the $t$-axis). Since $\nabla u$ is then normal to each geodesic cone, $\nabla u$ is colinear to $L$, hence $\nabla u$ is a null vector and $u$ an optical function.

**c.** This construction can be extended if, instead of starting from a single point $x_0$, we start from a spacelike 2-surface $S_0$. Choosing at each point $x_0$ of $S_0$ an outgoing future oriented null vector $\xi^0(x_0)$ orthogonal to $S_0$, we consider the union $\Sigma$ of all the geodesic curves issued from $(x_0, \xi^0(x_0))$. Defining $L$ and $X$ on $\Sigma$ as before, we obtain $L < X, L >= 0$ as before, hence $< X, L >= 0$ since this is true by construction for $s = 0$. Again, if we are given a one parameter family of such surfaces $\Sigma$, such that there exists a function $u$ having these $\Sigma$ as level sets, then $u$ is an optical function.

## 4.4 Frame coefficients

As explained above, working in a given frame $e_\alpha$ requires that we know the vectors $D_\alpha e_\beta$. In the case of local coordinates $x^\alpha$, $e_\alpha = \partial_\alpha$, $D_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta}\partial_\gamma$ and we have seen already the explicit formula for $\Gamma$.

**1. a.** Suppose we work with a null frame $(e_\alpha)$ for which we know the brackets $[e_\alpha, e_\beta]$. Using the various properties of the metric connexion, we can compute explicitly the vectors $D_\alpha e_\beta$. We give here a few examples of these manipulations :

i) $< D_1 e_1, e_1 >= 0$, since $e_1 < e_1, e_1 >= 0 = 2 < D_1 e_1, e_1 >$.

ii) $< D_1 e_1, e_2 >= - < e_1, D_1 e_2 >= - < e_1, D_2 e_1 + [e_1, e_2] >= - < e_1, [e_1, e_2] >$.

iii) For $j = 3, 4$,

$$< D_j e_1, e_2 >=< [e_j, e_1], e_2 > + < D_1 e_j, e_2 >,$$

$$< D_1 e_j, e_2 >= - < e_j, D_1 e_2 >= - < e_j, [e_1, e_2] > - < e_j, D_2 e_1 >,$$

$$< e_j, D_2 e_1 >= - < e_1, D_2 e_j >= - < e_1, [e_2, e_j] > + < e_2, D_j e_1 > .$$

Finally,

$$2 < D_j e_1, e_2 >=< [e_j, e_1], e_2 > - < e_j, [e_1, e_2] > + < e_1, [e_2, e_j] > .$$

**b.** Let us compute for instance the frame coefficients for the quasiradial null frame of a quasiradial situation, as described in chapter 3.3. We define the second form of the foliation $\Sigma_T = \{(x, t), t = T\}$ by $k(X, Y) = - < D_X T, Y >$. In local coordinates,

$$k_{ij} = - < D_i T, \partial_j >= (\nabla^2 t)_{ij} = -(D_i \partial_j)(t) = -\Gamma^0_{ij} = -(1/2) g^{0\alpha}(\partial_i g_{\alpha j} + \partial_j g_{\alpha i} - \partial_\alpha g_{ij}).$$

For vector fields $X, Y$, we define $\not D_X Y$ to be the orthogonal projection of $D_X Y$ onto the space generated by $(e_1, e_2)$

$$\not D_X Y =< D_X Y, e_1 > e_1 + < D_X Y, e_2 > e_2.$$

In the sequence, indices $a, b$ run from 1 to 2, corresponding to the basis on the spheres. We set

$$c = |\nabla r|,\ c^2 =< \nabla r, \nabla r >= g^{ij}\omega_i\omega_j.$$

**Theorem (Quasiradial case).** *The connexion $D$ satisfies*

$$D_T T = 0,\ D_N T = -k_{NN}N - k_{aN}e_a,\ D_a T = -k_{aN}N - k_{ab}e_b,$$

$$D_T N = k_{aN}e_a,\ D_N N = -k_{NN}T + e_a(c)/ce_a,\ D_a N = \not D_a N - k_{aN}T,$$
$$D_N e_a = \not D_N e_a + e_a(c)/N - k_{aN}T,\ D_T e_= \not D_T e_a - k_{aN}N.$$

• For any field $X$, by symmetry of the Hessian, $< D_T T, X >=< D_X T, T >= 0$, hence $D_T T = 0$. For tangential $X, Y$, $< D_X T, T >= 0$, $< D_X T, Y >= -k_(X, Y)$. This gives the first line.

• Since $g^{0i}\omega_i = 0$, using the decomposition formula of $\partial_i$ into its radial part and its rotation part

$$\partial_i = \omega_i\partial_r - (\omega \wedge R/r)_i,$$

we obtain

$$T = \partial_t - g^{0i}(\partial_i - \omega_i\partial_r) = \partial_t + g^{0i}(\omega \wedge R/r)_i,$$
$$N = c^{-1}g^{ij}\omega_i\partial_j = c\partial_r - c^{-1}g^{ij}\omega_i(\omega \wedge R/r)_j.$$

This implies

$$[T, N] = (Tc/c)N + \dots R.$$

We compute now the derivatives of $N = c^{-1}\nabla r$. First, $< D_T N, N >= 0$,

$$D_T N, T >=< [T, N], T > + < D_N T, T >= 0.$$

Next,

$$< D_T N, e_a >= c^{-1} < D_T \nabla r, e_a >= c^{-1} < D_a \nabla r, T >=< D_a N, T >= - < D_a T, N >= k_{aN}.$$

• Similarly, we know $D_N N$, since $< D_N N, N >= 0$,

$$< D_N N, T >= - < D_N T, N >= k_{NN},$$

$$< D_N N, e_a >= c^{-1} < D_N \nabla r, e_a >= c^{-1} < D_a \nabla r, N >= e_a(c)/c.$$

• Finally, $< D_a N, N >= 0$,

$$< D_a N, T >= c^{-1} < D_a \nabla r, T >= c^{-1} < D_T \nabla r, e_a >=< D_T N, e_a >= k_{aN}.$$

The quantities $< D_aN, e_b >=< \not{D}_aN, e_b >$ are the components of the second form of the standard spheres with respect to the induced metric $g_{ij}$ in $\Sigma_t$.

For any field $X$, $< D_Xe_a, T >= - < D_XT, e_a >$ and $< D_Xe_a, N >= - < D_XN, e_a >$ are already known. The other quantities $< D_Ne_a, e_b >$, $< D_Te_a, e_b >$ and $< D_ce_a, e_b >$ depend on the choice of $(e_1, e_2)$, and have to be computed using the expressions of the brackets as above. ♢

**2.** In the case of a frame associated to one or two optical functions (see 3.3), using the properties of the null frame and of the connexion, one can deduce all frame coefficients from some of them conveniently chosen :

i) Define first the analogues of the second form for the nonstandard spheres

$$\chi_{ab} =< D_aL, e_b >,\ \underline{\chi}_{ab} =< D_a\underline{L}, , e_b >,\ L = e_4,\ \underline{L} = e_3,\ a, b = 1, 2.$$

These two tensors are symmetric for the same reason as for the second form of a hypersurface, namely, because $[e_a, e_b]$ is tangent to the spheres.

ii) Next, we define for 1-forms on the spheres by

$$2\eta_a =< D_{\underline{L}}L, e_a >,\ 2\underline{\eta}_a =< D_L\underline{L}, e_a >,$$

$$2\xi_a =< D_LL, e_a >,\ 2\underline{\xi}_a =< D_{\underline{L}}\underline{L}, e_a > .$$

iii) Finally, the functions $\omega$ and $\underline{\omega}$ are

$$4\omega =< D_LL, \underline{L} >, 4\underline{\omega} =< D_{\underline{L}}\underline{L}, L > .$$

We check now that these quantities allow to recover all frame coefficients. Suppose the frame is associated to one optical function $u$, with $L = -\nabla u$ as explained in example 2 of 3.3.

**Theorem (Integrable case).** *The connexion $D$ satisfies the formula*

$$D_LL = 0,\ D_{\underline{L}}L = 2\eta_ae_a + 2\underline{\omega}L,\ D_aL = \chi_{ab}e_b - \eta_aL,$$

$$D_L\underline{L} = 2\underline{\eta}_ae_a,\ D_{\underline{L}}\underline{L} = 2\underline{\xi}_ae_a - 2\underline{\omega}\underline{L},\ D_a\underline{L} = \underline{\chi}_{ab}e_b + \eta_a\underline{L},$$

$$D_Le_a = \not{D}_Le_a + \underline{\eta}_aL,\ D_{\underline{L}}e_a = \not{D}_{\underline{L}}e_a + \underline{\xi}_aL + \eta_a\underline{L},$$

$$D_be_a = \not{D}_be_a + (1/2)\chi_{ab}\underline{L} + (1/2)\underline{\chi}_{ab}L,$$

$$\eta_a = e_a(a)/a + k_{aN}.$$

First, we note that $D_LL = 0$, since

$$< D_LL, X >= - < D_L\nabla u, X >= -\nabla^2u(L, X) =$$

$$= -\nabla^2 u(X,L) = - < D_X \nabla u, L >=< D_X L, L >= 0.$$

This means that the integral curves of $L$ along the outgoing cones are geodesics. Also, since $T \equiv \partial_t = -\nabla t$, $< D_T T, X >=< D_X T, T >= 0$, which shows that $D_T T = 0$. All other formula are easy, except the last one ; for this, we compute $e_a(< L,T >)$ in two different ways :

$$< L,T >= -1/a, e_a(< L,T >) = e_a(a)/a^2,$$

$$e_a(< L,T >) =< D_a L, T > + < L, D_a T > .$$

Now $\underline{L} = -a^2 L + 2aT$, which gives

$$< D_a L, T >= (2a)^{-1} < D_a L, \underline{L} + a^2 L >= \eta_a/a.$$

Also,

$$< L, D_a T >= a^{-1} < T + N, D_a T >= -a^{-1} k_{aN},$$

which proves the formula. ♢

We prove now that the underlined quantities can be recovered from the others and from the second form $k(X,Y) = - < D_X \partial_t, Y >$.

**Theorem.** *The following formula hold*

$$\underline{\chi}_{ab} = -a^2 \chi_{ab} - 2a k_{ab},$$

$$\underline{\xi}_a = -a^2 \eta_a + a^2 k_{aN},$$

$$\underline{\eta}_a = -k_{aN},$$

$$2\underline{\omega} = a k_{NN} - \underline{L}a/a.$$

For instance, since $\underline{L} = -a^2 L + 2a\partial_t$,

$$\underline{\chi}_{ab} = -a^2 \chi_{ab} - 2a k_{ab},$$

$$2\underline{\xi}_a =< D_{\underline{L}} \underline{L}, e_a >= -2a^2 \eta_a + 2a < D_{\underline{L}} \partial_t, e_a >,$$

$$< D_{\underline{L}} \partial_t, e_a >=< D_a \partial_t, \underline{L} >= -a < D_a \partial_t, N >= -a < D_N \partial_t, e_a >= a k_{aN},$$

$$2\underline{\eta}_a =< D_L(-a^2 L + aT, e_a >= 2a < D_L \partial_t, e_a >= 2 < D_N T, e_a >= -2k_{aN}.$$

Finally, we check $2\underline{\omega} = a k_{NN} - \underline{L}a/a$ : in fact,

$$< D_{\underline{L}} L, \underline{L} >= a^2 < D_{T-N}(a^{-1}(T+N)), T-N >=$$

$$= a^2[-2(T-N)(a^{-1}) + a^{-1} < D_{T-N}(T+N), T-N >,$$

$$< D_{T-N}(T+N), T-N >= -2k_{NN}.$$

♢

To conclude, note that the quantities $\chi$, $\eta$ are components of the Hessian $\nabla^2 u$ :

$$\chi_{ab} = -\nabla^2 u_{ab}, 2\eta_a = -\nabla^2 u_{a\underline{L}}.$$

Note in particular that, according to the trace formula of chapter 1,

$$\Box u = tr\,\nabla^2 u = \nabla^2 u_a^a - \nabla^2 u_{34} = -tr\,\chi + < D_{\underline{L}} L, L >= -tr\,\chi.$$

In chapter 7, we will discuss in more details how the tensors $\chi$, $\xi$, etc. can be estimated, in situations where they are not explicitly known.

# Chapter 5

# Energy Inequalities and Frames

To obtain an energy inequality for $\Box$, we proceed as usual by choosing a vector field $X$ (the "multiplier") and writing

$$(\Box\phi)(X\phi) = div\, P + Q.$$

Here, $P$ will be an appropriate field whose coefficients are quadratic forms in the components of $\nabla\phi$, while $Q$ is a quadratic form in these components with variable coefficients. Integrating $(\Box\phi)(X\phi)$ in some domain $D$, and using Stokes formula, we obtain boundary terms

$$\int_{\partial D} < P, N > dv$$

which yield the "energy" of $\phi$, and interior terms $\int_D Q dV$. In practice, since some derivatives of $\phi$ behave better than other, we must write these energies and interior terms in an appropriate frame, and *not* in the usual coordinates. We describe now the clever machinery which makes this possible and easy.

## 5.1 The Energy Momentum Tensor

**1.** Let $\phi$ be a fixed $C^1$ function, and define the **energy momentum tensor** $Q$ as a symmetric 2-tensor by

$$Q(X, Y) = (X\phi)(Y\phi) - (1/2) < X, Y > |\nabla\phi|^2,\ Q_{\alpha\beta} = (\partial_\alpha\phi)(\partial_\beta\phi) - (1/2)g_{\alpha\beta}|\nabla\phi|^2.$$

Consider a null frame $(e_1, e_2, e_3, e_4)$, with $< e_3, e_4 >= -2\mu$. Since $\nabla\phi = e_1(\phi)e_1 + e_2(\phi)e_2 - (2\mu)^{-1}(e_4(\phi)e_3 + e_3(\phi)e_4)$, we get

$$|\nabla\phi|^2 =< \nabla\phi, \nabla\phi >= e_1(\phi)^2 + e_2(\phi)^2 - \mu^{-1}e_3(\phi)e_4(\phi).$$

Note that $|\nabla\phi|^2$ is not a positive term ! Hence the components $Q_{\alpha\beta} = Q(e_\alpha, e_\beta)$ of $Q$ are, writing for simplicity $e_\alpha(\phi) = e_\alpha$,

$$Q_{11} = (1/2)(e_1^2 - e_2^2) + (2\mu)^{-1}e_3e_4, Q_{12} = e_1e_2, Q_{22} = -(1/2)(e_1^2 - e_2^2) + (2\mu)^{-1}e_3e_4,$$

$$Q_{13} = e_1e_3, Q_{14} = e_1e_4, Q_{23} = e_2e_3, Q_{24} = e_2e_4,$$
$$Q_{33} = e_3^2, Q_{34} = \mu(e_1^2 + e_2^2), Q_{44} = e_4^2.$$

In particular, the trace $tr\, Q = Q_\alpha^\alpha$ is, according to the formula of chapter 3,

$$tr\, Q = Q_{11} + Q_{22} - \mu^{-1}Q_{34} = -(e_1^2 + e_2^2) + \mu^{-1}e_3e_4 = -|\nabla\phi|^2.$$

**2.** The tensor $Q$ enjoyes a remarkable *positivity property* .

**Theorem (Positivity of $Q$).** *If $X$ and $Y$ are two timelike future oriented vectors, then*

$$Q(X, Y) \geq 0.$$

Recall the terminology for vectors : a vector $X$ is timelike, null or spacelike if $< X, X >< 0$, $< X, X >= 0$ or $< X, X >> 0$ respectively ; future oriented means that the $t$-component is positive. To prove the claim, let $e_3$ and $e_4$ be two *null* future oriented vectors in the plane generated by $(X, Y)$ : then, by convexity of the future timelike cone,

$$X = ae_3 + a'e_4,\ Y = be_3 + b'e_4,\ a, a', b, b' \geq 0.$$

Hence, setting $< e_3, e_4 >= -2\mu$, $\mu \geq 0$,

$$Q(X, Y) = abe_3(\phi)^2 + a'b'e_4(\phi)^2 + \mu(ab' + a'b)[e_1(\phi)^2 + e_2(\phi)^2] \geq 0.$$

◊

**3.** The energy-momentum tensor $Q$ is related to the d'Alembertian by the formula

$$D^\alpha Q_{\alpha\beta} = (\Box\phi)(\partial_\beta\phi).$$

To prove this formula, we write, using $D_X g = 0$,

$$Q = d\phi \otimes d\phi - (1/2)g|\nabla\phi|^2,$$

$$D_X Q = D_X d\phi \otimes d\phi + d\phi \otimes D_X d\phi - g < D_X\nabla\phi, \nabla\phi > .$$

Taking $X = \partial^\alpha$, and then the $\alpha\beta$ component, we obtain

$$D^\alpha Q_{\alpha\beta} = (\nabla^2\phi_\alpha^\alpha)(\partial_\beta\phi) + (\partial_\alpha\phi)\nabla^2\phi_\beta^\alpha - \nabla^2\phi(\partial_\beta, \nabla\phi).$$

In the right-hand side, the first term is by definition $(\Box\phi)(\partial_\beta\phi)$, the second is $\nabla^2\phi(\nabla\phi, \partial_\beta)$, which cancels out with the third. ◊

## 5.2 Deformation Tensor

**Definition.** *The deformation tensor of a given vector field $X$ is the symmetric 2-tensor ${}^{(X)}\pi$ defined by*

$$ {}^{(X)}\pi(Y,Z) \equiv \pi(Y,Z) =< D_Y X, Z > + < D_Z X, Y > . $$

In local coordinates (be careful about the place of the indices !),

$$ \pi_{\alpha\beta} = D_\alpha X_\beta + D_\beta X_\alpha. $$

• Let us digress shortly to explain a few things about **Lie derivatives** : for given $X$, we define $\mathcal{L}_X f = Xf$, $\mathcal{L}_X Y = [X,Y]$, and extend this to tensors by imitating the product formula, exactly as we have done for $D_X$ :

$$ X(T(Y_1,\ldots,Y_p)) = (\mathcal{L}_X T)(Y_1,\ldots,Y_p) + \sum_{1\le i\le p} T(Y_1,\ldots,[X,Y_i],\ldots,Y_p). $$

As is well known, the Lie derivative is defined using the flow of $X$, and $\mathcal{L}_X Y$ is *not* linear in $X$, in contrast with the covariant derivative $D_X Y$. Using this definition with $T = g$, we find

$$ X(g(Y,Z)) = (\mathcal{L}_X g)(Y,Z) + g([X,Y],Z) + g(Y,[X,Z]) = g(D_X Y, Z) + g(Y, D_X Z), $$

hence finally the important formula

$$ \pi = \mathcal{L}_X g. $$

This formula helps visualize what $\pi$ is ; in particular, $\pi$ vanishes if $g$ is invariant by the flow of $X$ : we call such a field a **Killing field**. If only $\mathcal{L}_X g = \lambda g$, the field is **conformal Killing**. For the Minkowski metric, the simplest examples of Killing fields are the derivations, the spatial rotations and the hyperbolic rotations $H_i = t\partial_i + x^i\partial_t$ ; note that among these fields, only $\partial_t$ is timelike. There are five conformal Killing fields (again, be aware of the position of the index $\mu$)

$$ S = x^\alpha \partial_\alpha, K_\mu = -2x_\mu S + |x|^2 \partial_\mu, $$

for which the corresponding deformation tensors are

$$ {}^{(S)}\pi = 2g, {}^{(K_\mu)}\pi = -4x_\mu g. $$

Among these, only $K_0 = (r^2 + t^2)\partial_t + 2rt\partial_r$ is timelike. For the Schwarzschild metric, $\partial_t$ and the spatial rotations are Killing fields ; for Kerr metric, $\partial_t$ and $\partial_\phi$ are Killing. For a general metric, there are no Killing fields, because the number of equations to be satisfied is 10, while there are only 4 unknowns.

• To compute explicitly ${}^X\pi$ in a frame, we need the frame coefficients. However, in local coordinates, we have the simple formula

$$ {}^{(X)}\pi^{\alpha\beta} = \partial^\alpha(X^\beta) + \partial^\beta(X^\alpha) - X(g^{\alpha\beta}). $$

In fact, $D_\alpha X_\beta =< D_\alpha X, \partial_\beta >= g_{\beta\gamma}\partial_\alpha(X^\gamma) + X^\gamma \Gamma_{\beta\alpha\gamma}$. Hence

$$\pi^{\alpha\beta} = g^{\alpha\alpha'} g^{\beta\beta'} [g_{\beta'\gamma}\partial_{\alpha'}(X^\gamma) + X^\gamma \Gamma_{\beta'\alpha'\gamma} + g_{\alpha'\gamma}\partial_{\beta'}(X^\gamma) + X^\gamma \Gamma_{\alpha'\beta'\gamma}] =$$

$$= \partial^\alpha(X^\beta) + \partial^\beta(X^\alpha) + X^\gamma g^{\alpha\alpha'} g^{\beta\beta'} [\Gamma_{\beta'\alpha'\gamma} + \Gamma_{\alpha'\beta'\gamma}].$$

Since, from the explicit formula, $\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = \partial_\gamma(g_{\alpha\beta})$, the last term is

$$g^{\alpha\alpha'} g^{\beta\beta'} X(g_{\alpha'\beta'}) = -X(g^{\alpha\beta}).$$

♢ • Finally, let us keep in mind the following formula, which will be useful later on :

$$^{(fX)}\pi(Y,Z) = f^{(X)}\pi(Y,Z) + (Yf) < X, Z > + (Zf) < X, Y >,$$

where $f \in C^1$ is an arbitrary function. We will see that the deformation tensors play a crucial role both in energy computations and in commutation formula.

## 5.3 Energy Inequality Formalism

**Theorem (Key formula).** *Let $\phi$ be a given $C^1$ funciton and $Q$ be the associated energy-momentum tensor. Let $X$ be a vector field, and set $P_\alpha = Q_{\alpha\beta}X^\beta$. Then*

$$div\, P \equiv D_\alpha P^\alpha = (\Box\phi)(X\phi) + (1/2) Q^{\alpha\beta(X)}\pi_{\alpha\beta}.$$

• This is the key formula for proving energy inequalities. Let us explain why : the formula is just a re-writing of $(\Box\phi)(X\phi)$ as the sum of a divergence of a field (the term $div\, P$) and a quadratic form $q$ in the components of $\nabla\phi$, as expected. The point is that $q$ is written as a double trace (summation on $\alpha$ and $\beta$), which can be computed using any null frame. More explicitly, we have for any null frame $(e_\alpha)$ , denoting as usual $e_\alpha(\phi) = e_\alpha$,

$$Q\pi = Q_{\alpha\beta}\pi^{\alpha\beta} = Q_{11}\pi_{11} + 2\pi_{12}e_1e_2 - (1/\mu)\pi_{14}e_1e_3 - (1/\mu)\pi_{13}e_1e_4 + \pi_{22}Q_{22}$$

$$-(1/\mu)\pi_{24}e_2e_3 - (1/\mu)\pi_{23}e_2e_4 + (1/(4\mu^2))\pi_{44}e_3^2 + (1/(4\mu^2))\pi_{33}e_4^2 + (1/(2\mu))\pi_{34}(e_1^2 + e_2^2) =$$

$$= (1/2)(e_1^2 - e_2^2)(\pi_{11} - \pi_{22}) + (1/(2\mu))\pi_{34}(e_1^2 + e_2^2) + 2\pi_{12}e_1e_2$$

$$-(1/\mu)[\pi_{14}e_1e_3 + \pi_{13}e_1e_4 + \pi_{24}e_2e_3 + \pi_{23}e_2e_4]$$

$$+(1/(4\mu^2))[\pi_{44}e_3^2 + \pi_{33}e_4^2 + 2\mu(\pi_{11} + \pi_{22})e_3e_4].$$

• To prove the key formula, using the definitions, we write

$$\partial^\alpha(P_\alpha) = \partial^\alpha(P(\partial_\alpha)) = D^\alpha P_\alpha + P(D^\alpha\partial_\alpha) =$$

$$= \partial^\alpha(Q(\partial_\alpha, X)) = (D^\alpha Q)(\partial_\alpha, X) + Q(D^\alpha\partial_\alpha, X) + Q(\partial_\alpha, D^\alpha X).$$

The terms $P(D^\alpha\partial_\alpha$ and $Q(D^\alpha\partial_\alpha, X)$ cancel out ; using the above formula $D^\alpha Q_{\alpha\beta} = (\Box\phi)(\partial_\beta\phi)$, we find the result. ♢

An energy inequality will be obtained by computing $\int_D (\Box\phi)(X\phi)dV$ for some domain $D$, using the key formula and Stokes formula : this yields boundary terms and interior terms, which we discuss now separately.

## 5.4 Energy

Suppose we compute the integral $\int_D (\Box\phi)(X\phi)dV$ for some domain $D$, using Stokes formula. The boundary terms that we obtain are

$$\int_{\partial D} Q(N,X)dv,$$

where $N$ is the unit outgoing normal to $\partial D$.

• The most common case is of course that of a split metric $g = -dt^2 + g_{ij}dx^i dx^j$, when $D = \{(x,t), 0 \leq t \leq T\}$ is a strip and $X = -\partial_t$ (we put a minus sign because, with the normalization of $g$, the wave equation is minus the usual one !) : the boundary integral is then equal to $E(T) - E(0)$, with

$$E(T) = \int_{\Sigma_T} Q(\partial_t, \partial_t)dv = (1/2)\int_{\Sigma_T} [(\partial_t\phi)^2 + g^{ij}\partial_i\phi\partial_j\phi]dv.$$

This defines the **"energy of $\phi$ at time $T$"**, which controls (with some weight due to the coefficients $g^{ij}$ and to $dv$) the $L^2$ norm of $\nabla\phi$ at time $T$.

• Suppose now that the domain $D$ is bounded by a portion $\Sigma_T \subset \{t = T\}$, a portion $\Sigma_0 \subset \{t = 0\}$ and some "lateral boundary" $\Lambda$ ; in this case, the boundary terms are

$$E(T) - E(0) - \int_\Lambda Q(N, \partial_t)dv,$$

the last term being always non negative as long as the lateral boundary of $D$ has a time-like past oriented normal $N$ : we recognize here the condition for $D$ to be a domain of determination for $\Box$ (see [1] for instance).

Usually, this last term is neglected in proving energy inequalities, but this is not necessarily a good idea. Suppose for instance that, for a given general metric $g$, we work with a null frame associated with two optical functions $u$ and $\underline{u}$ as in example 3 of 3.3. Let $D$ be the domain enclosed between two spacelike hypersurfaces $\Sigma_0$ and $\Sigma_T$ (playing here the role of horizontal planes) and the surface of an incoming light cone $\{\underline{u} = \underline{u}_0\}$ (with $\underline{u}_0 > 0$). On the lateral boundary $\Lambda$ of $D$, an exterior unit normal is $N = \alpha^{-1}\nabla\underline{u}$ (with $\alpha = ||\nabla\underline{u}||$) ; if we take $X = (1/2)(\nabla u + \nabla\underline{u})$ (which looks like $-\partial_t$), the energy density integrated on $\Lambda$ is

$$(2\alpha)^{-1}Q(\underline{L}, L + \underline{L}) = (8\alpha\Omega^2)^{-1}Q(e_3, e_3 + e_4) = (8\alpha\Omega^2)^{-1}[e_1(\phi)^2 + e_2(\phi)^2 + e_3(\phi)^2].$$

In other words, the energy yields a (weighted ) $L^2$ control of the derivatives of $\phi$ which are *tangential* to $\Lambda$. In the approach of [25] for instance, the authors do not introduce any $t$-coordinate, and never consider the "energy at time $t$" ; they integrate on domains bounded by incoming cones, and use the energy on these cones as we just explained.

• More generally, as long as $X$ is timelike past oriented and $D$ is a domain of determination of $\Box$ (that is, the exterior normal $N$ to the upper part of the boundary of $D$ is also timelike past oriented), the boundary integral on the upper part of $\partial D$ has a nonnegative energy density $Q(N, X)$. For the Minkowski metric for instance, the choice of the timelike past oriented $X$, $-X = K_0 = (r^2 + t^2)\partial_t + 2rt\partial_r$, leads to the well-known "conformal inequality" (to which we will return later).

## 5.5 Interior Terms and Positive Fields

In the previous section, we discussed only the sign of the *boundary terms* arising from the computation of $\int_D (\Box\phi)(X\phi)dV$ for some domain $D$, and the concept of (positive) energy. In general however, we also have to deal with the *interior terms*

$$\int_D Q_{\alpha\beta}{}^{(X)}\pi^{\alpha\beta} dV \equiv \int_D Q\pi dV.$$

There are basically two different strategies to deal with these terms :

i) Control by brute force,

ii) Discussion of the signs of the terms.

**1. Two examples of brute force control**

**a. Gronwall lemma**

This approach is to bound the interior integrand $Q\pi$ by the integrand of the energy ; in the simple case of a split metric and a domain $D = \{0 \leq t \leq T\}$, for instance, suppose that we can obtain the bound

$$\int_{\Sigma_t} |Q^{(\partial_t)}\pi| dv \leq 2C(t)E(t).$$

We will write our inequality

$$E(T) - E(0) \leq \int (\Box\phi)(X\phi)dV + \int_0^T C(t)E(t)dt,$$

and Gronwall lemma yields

$$E(T) \leq [E(0) + \int_D |(\Box\phi)(X\phi)|dV] \exp(\int_0^T C(t)dt).$$

The interior terms have thus disappeared from the energy inequality, but at the cost of the *amplification factor* $\exp(\int_0^T C(t)dt)$. When dealing with global in time problems, for example, this can be disastrous, if $C$ is not integrable.

**b. Weighted inequality**

An essentially equivalent approach is to replace $X$ by $fX$ for a well-chosen function $f$. Using the formula of section 4.2, we have then

$$Q^{(fX)}\pi = fQ\pi + 2Q(\nabla f, X).$$

A common choice is to take $f = e^{\lambda a}$ for some function $a$ and some real number $\lambda$ :

$$Q^{(fX)}\pi = e^{\lambda a}[Q\pi + 2\lambda Q(\nabla a, X)].$$

If $X$ and $\nabla a$ are both timelike past oriented (for instance, $a = t$), reasonable assumptions make it possible to obtain

$$Q\pi + 2\lambda Q(\nabla a, X) \geq 0$$

for large enough $\lambda$. In this case, $fX$ is positive. The drawback of this well-known approach is of course that one has to keep the weight $e^{\lambda a}$ in the formula for the energy.

**2. Sign control of interior terms**

In applications, the brute force strategy is generally too rough, so we discuss now the sign strategy . If the multiplier $X$ happens to be a Killing field, $\pi \equiv 0$ and the interior term is identically 0. This is the case for instance of the multiplier $X = \partial_t$ for the flat metric.

Leaving aside this trivial and miraculous case, we prove now the following formula.

**Theorem.** *For any $C^2$ function $R$, the following identity holds*

$$\int_D R|\nabla\phi|^2 dV = (1/2)\int_D \phi^2(\Box R)dV - \int_D R\phi(\Box\phi)dV$$

$$-(1/2)\int_{\partial D} \phi^2 < N, \nabla R > dv + \int_{\partial D} R\phi < N, \nabla\phi > dv.$$

To prove the above formula, observe that, for any two functions $f$ and $h$,

$$div\, f\nabla h = D_\alpha(f\nabla h)^\alpha = \partial_\alpha f \partial^\alpha h + f\Box h.$$

Using this formula with $f = R\phi$ and $h = \phi$, or $f = \phi^2$ and $h = R$, and integrating with the help of Stokes formula, we get the result. ♢

We give a first example of how this formula can be used.

**Example 1 :** Suppose the multiplier $X$ is conformal Killing ; then

$$Q_{\alpha\beta}\pi^{\alpha\beta} = \lambda Q_{\alpha\beta}g^{\alpha\beta} = \lambda tr\, Q = -\lambda|\nabla\phi|^2.$$

One should not be mislead by the expression $|\nabla\phi|^2$ : it is by no means a positive form ! Using the above formula, one can get rid of these bad terms. This transformation has however two drawbacks :

i) It produces new interior term $\int_D \phi^2(\Box\lambda)dV$, which must be nonnegative if we are to obtain an energy inequality,

ii) It produces additionnal boundary terms involving $\phi$ and $\nabla\phi$, which can spoil the positivity of the energy.

A typical example is the choice $-X = K_0$ of a multiplier for the flat metric in a domain $D = \{0 \leq t \leq T\}$. We have the identity

$$2(\Box\phi)(X\phi) = \partial_t[(r^2+t^2)((\partial_t\phi)^2 + \sum(\partial_i\phi)^2) + 4rt(\partial_t\phi)(\partial_r\phi)] +$$

$$+\sum\partial_i[\ldots] + 4t[(\partial_t\phi)^2 - \sum(\partial_i\phi)^2].$$

Transforming the interior term $-4t|\nabla\phi|^2$ yields the identity

$$\int_D (\Box\phi)(X\phi - 2t\phi)dxdt = \tilde{E}(T) - \tilde{E}(0),$$

where the modified energy $\tilde{E}$ is now

$$2\tilde{E}(T) = \int_{\Sigma_T} \{(r^2+t^2)[(\partial_t\phi)^2 + \sum(\partial_i\phi)^2] + 4rt(\partial_t\phi)(\partial_r\phi) + 4t\phi(\partial_t\phi) - 2\phi^2\}dx.$$

In this example, we dont have to worry about the term $\int \phi^2(\Box\lambda)dV$, since $\lambda = -4t$ and $\Box\lambda = 0$ ! It is a delicate task to prove that the modified energy is indeed positive (see [1] or [21] for instance). One finally obtains, for some constant $C > 0$,

$$C^{-1}\int[(S\phi)^2+|H\phi|^2+|R\phi|^2+\phi^2](x,t)dx \leq \tilde{E}(t) \leq C\int[(S\phi)^2+|H\phi|^2+|R\phi|^2+\phi^2](x,t)dx.$$

The formula on $R|\nabla\phi|^2$ can also be used to get rid a *some part* of the interior terms, as shown in the following example.

**Example 2 :** It can happen that the interior term $Q\pi$ is nonnegative, up to a multiple of $|\nabla\phi|^2$ ! The typical example is that of the Morawetz inequality for the flat metric : taking $X = -\partial_r$, we have the identity

$$2(\Box\phi)(X\phi) = \partial_t[\ldots] + \sum\partial_i[\ldots] + (2/r)[\sum(\partial_i\phi)^2 - (\partial_r\phi)^2] - (2/r)|\nabla\phi|^2.$$

In this case of course, the corresponding energy will not be positive, since $X$ is spacelike : the boundary terms will have to be controlled separately, using the standard energy inequality (corresponding to $X = -\partial_t$). Note that in this example, $\lambda = 2/r$, and $\Box(1/r)$, which is zero for $r > 0$, but singular at the origin : as a result, the new interior term $\int \phi^2(\Box\lambda)dV$ is $\int_D \phi^2\Box\lambda dV$ becomes $\int_0^T \phi^2(0,t)dt$ !

In the spirit of the preceding examples, we define a **positive field** $X$ to be a field such that, for some $R$,

$$I = Q_{\alpha\beta}{}^{(X)}\pi^{\alpha\beta} + R|\nabla\phi|^2$$

is a positive quadratic form in $\nabla\phi$.

**Lemma.** *The concept of positive field depends only on $X$ and the conformal class of the metric $g$.*

Set $\tilde{g} = e^{\lambda} g$, that is $\tilde{g}_{\alpha\beta} = e^{\lambda} g_{\alpha\beta}$. Then

$$\tilde{g}^{\alpha\beta} = e^{-\lambda} g^{\alpha\beta},\ \tilde{g}(\tilde{\nabla}\psi, \tilde{\nabla}\psi) = e^{-\lambda}|\nabla\psi|^2,$$

and consequently $\tilde{Q}_{\alpha\beta} = Q_{\alpha\beta}$. Now

$$^X\tilde{\pi} = \mathcal{L}_X \tilde{g} = e^{\lambda}[\mathcal{L}_X g + (X\lambda) g],$$

hence finally

$$\tilde{Q}^X \tilde{\pi} + \tilde{R}\tilde{g}(\tilde{\nabla}\psi, \tilde{\nabla}\psi) = e^{\lambda}[Q^X\pi + (X\lambda) tr\, Q + \tilde{R}e^{-2\lambda}|\nabla\psi|^2] =$$

$$= e^{\lambda}[Q^X\pi + R|\nabla\psi|^2 + |\nabla\psi|^2(-R + \tilde{R}e^{-2\lambda} - X\lambda)].$$

It is enough to choose $\tilde{R} = e^{2\lambda}(X\lambda + R)$ to show that $X$ is also positive for $\tilde{g}$. ♢

If $X$ is a positive field and we perform the above transformation on the term $R|\nabla\phi|^2$, we obtain for the interior terms

$$\int_D Q^{(X)}\pi dV = \int_D (I - R|\nabla\phi|^2) dV = \int_D (I - (1/2)\phi^2 \Box R) dV + \int_{\partial D} \ldots dv.$$

If $\Box R \leq 0$, the integrand in $D$ is positive and we have some hope to get in the end an energy inequality if the boundary terms are well behaved. If $\Box R$ has no special sign, we cannot say anything about the integrand in $D$, but it can happen that $\int_D I dV \geq \int_D (1/2)\phi^2 \Box R dV$ : an inequality of this type is called a Poincaré inequality (in France at least). A typical example occurs when trying to prove a Morawetz type inequality for the Schwarzschild metric.

**Example 3 :** Let us first introduce, in the exterior (of the black hole) region $r > 2m$, the coordinate

$$r^* = r + 2m\log(r - 2m) - 3m - 2m\log m,$$

where the funny normalization is meant to get $r^* = 0$ for $r = 3m$. With the coordinates $(t, r^*, \theta, \phi)$, the wave equation is

$$\Box\phi = (1 - 2m/r)^{-1}[-\partial_t^2\phi + r^{-2}\partial_{r^*}(r^2\partial_{r^*}\phi)] + r^{-2}\Delta_{S^2}\phi.$$

Consider now a domain $D = \{0 \leq t \leq T\}$, and choose $X = f(r^*)\partial_{r^*}$. A staightforward computation (see for instance [18]) gives

$$\int_D Q\pi dV = \int_0^T \int_{-\infty}^{+\infty} \int_{S^2} [(f'/(1-\mu))(\partial_{r^*}\phi)^2 + (2r)^{-1}(2 - 3\mu) f|\not\nabla\phi|^2 -$$

$$-(1/2)(f' + 2f(1-\mu)/r)|\nabla\phi|^2] r^2(1-\mu) dt dr^* d\sigma_{S^2}.$$

Here, $\mu = 2m/r$ and $\not\nabla\phi$ is the gradient of the restriction of $\phi$ to the spheres

$$|\not\nabla\phi|^2 = e_1(\phi)^2 + e_2(\phi)^2.$$

Since $2 - 3\mu = (2/r)(r - 3m)$, we see that if $f$ is increasing, vanishes at $r^* = 0$ and has the sign of $r^*$, the field $f\partial_{r^*}$ is positive. Getting rid of the $|\nabla\phi|^2$ term as above, we are left with the expression

$$I = \int_0^T \int_{-\infty}^{+\infty} \int_{S^2} [(f'/(1-\mu))(\partial_{r^*}\phi)^2 + (2r)^{-1}(2-3\mu)f|\not\nabla\phi|^2 -$$

$$-(1/4)(\Box(f' + 2f(1-\mu)/r))\phi^2]r^2(1-\mu)dtdr^*d\sigma_{S^2}.$$

Since

$$\Box(f' + 2f(1-\mu)/r) = (1-\mu)^{-1}f''' + (4/r)f'' - 8mr^{-2}(r-2m)^{-1}f' - 2mr^{-3}(3-\mu)f,$$

we easily see that $f$ cannot be chosen to ensure the negativity of this coefficient.

The only thing we can do is to try to use the strength of the first term $(f'/(1-\mu))(\partial_{r^*}\phi)^2$ to compensate for the bad sign of the $\phi^2$ term. The Poincaré inequality we use is the consequence of the following trivial identity, valid for any $C^1$ function $\alpha$,

$$\int f'(\partial_{r^*}\phi)^2 r^2 dr^* = \int f'(\partial_{r^*}\phi + \alpha\phi)^2 r^2 dr^* +$$

$$+ \int \phi^2[f''\alpha + f'(\alpha' - \alpha^2 + 2\alpha(1-\mu)/r]r^2 dr^*.$$

Dropping the first term of the right-hand side, we obtain a Poincaré inequality, depending on some unknown function $\alpha$ still to be chosen. It turns out that we also need to use the contribution from the $|\not\nabla\phi|^2$ term : for this, we decompose $\phi$ into spherical harmonics $\phi_l$. Finally, one can prove that there is some $l_0$, some function $f$ and some function $\alpha$ such that, if $\phi_l = 0$ for $l \leq l_0$,

$$I \geq \int < r^* >^{-3-0} \phi^2 dV.$$

**4. Interior terms and Poisson bracket**

Note that

$$(\Box\phi)(X\phi) = div\,(X\phi)\nabla\phi - div\,\phi\nabla(X\phi) + \phi[\Box, X]\phi + \phi X\Box\phi.$$

Integrating this identity in some domain $D$, we see that the interior term $Q\pi$ must correspond, modulo $|\nabla\phi|^2$, to the quadratic form $\int_D([\Box, X]\phi)\phi dV$. In fact, we have

$$Q\pi = \partial_\alpha\phi\partial_\beta\phi\pi^{\alpha\beta} - (1/2)|\nabla\phi|^2 tr\,\pi.$$

Using the expression of $\pi^{\alpha\beta}$ given in 5.2,

$$(\partial_\alpha\phi)(\partial_\beta\phi)\pi^{\alpha\beta} = (\partial_\alpha\phi)(\partial_\beta\phi)[2\partial^\alpha(X^\beta) - X(g^{\alpha\beta})].$$

This quadratic form corresponds to the operator with symbol $q = \xi_\alpha \xi_\beta [2\partial^\alpha(X^\beta) - X(g^{\alpha\beta})]$. We note that $q$ is precisely the **Poisson bracket**

$$q = \{g^{\alpha\beta}\xi_\alpha\xi_\beta, X^\gamma\xi_\gamma\}$$

of the symbol of $\Box$ with the symbol of $X$. Thus the formula of 5.2 for $\pi^{\alpha\beta}$ provides a connexin between deformation tensors and Poisson brackets.

## 5.6 Maxwell Equations

### a. Generalities

In the Maxwell system, the unknown object is a 2-form $F$, and Maxwell equations are

$$dF = 0,\ D^\alpha F_{\alpha\beta} = 0.$$

In the flat case, to connect this formulation with the usual formulation, set

$$E_i = -F_{0i}, i = 1, 2, 3,\ H^1 = -F_{23}, H^2 = -F_{31}, H^3 = -F_{12},$$

thus defining the **"electric field"** $E$ and the **"magnetic field"** $H$. Then

$$dF = 0 \Leftrightarrow div\, H = 0,\ \partial_t H - curl\, E = 0,$$

$$D^\alpha F_{\alpha\beta} = 0 \Leftrightarrow div\, E = 0,\ \partial_t E + curl\, H = 0.$$

• A convenient way of doing many computations is to introduce the **dual form**

$$*F_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta},$$

where $\epsilon$ is the volume form (see 3.1). This summation looks complicated, but it is enough to note that for given $(\mu, \nu)$, the are only two indices, say $(\underline{\alpha}, \underline{\beta})$ different from $\mu$ and from $\nu$, so that

$$*F_{\mu\nu} = \epsilon_{\mu\nu\underline{\alpha}\underline{\beta}}F^{\underline{\alpha}\underline{\beta}},$$

the right-hand side being in this case only one term (no sum !). Also,

$$**F_{\mu\nu} = (1/4)\epsilon_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = |\epsilon|F_{\mu\nu} = -F_{\mu\nu}.$$

In the flat case for instance,

$$*F_{01} = \epsilon_{0123}F^{23} = F_{23} = -H^1,$$

which shows that the electric field of $*F$ is just the magnetic field of $F$ ; also, in view of the relation above, the magnetic field of $*F$ is minus the electric field of $F$.

**Lemma.** *For any vector field $X$,*

$$D_X * F = *D_X F.$$

This may seem obvious, since $*$ is defined using the metric only, and $D_X g = 0$ ; we give however a self-contained proof. First, we observe $D_X \epsilon = 0$ : if $e_\alpha$ is an orthonormal basis,

$$X(\epsilon(e_1, e_2, e_3, e_4)) = 0 = (D_X\epsilon)(e_1, e_2, e_3, e_4) + \epsilon(D_X e_1, e_2, e_3, e_4) + \dots,$$

and the last three terms are zero since $D_X e_\alpha$ has no component on $e_\alpha$. Next, using the definitions, we get

$$X(*F_{\mu\nu}) = (D_X * F)_{\mu\nu} + (*F)(D_X\partial_\mu, \partial_\nu) + (*F)(\partial_\mu, D_X\partial_\nu) =$$

$$= [\epsilon(D_X\partial_\mu, \partial_\nu, \partial_\alpha, \partial_\beta) + \epsilon(\partial_\mu, D_X\partial_\nu, \partial_\alpha, \partial_\beta) + \epsilon(\partial_\mu, \partial_\nu, D_X\partial_\alpha, \partial_\beta) + \epsilon(\partial_\mu, \partial_\nu, \partial_\alpha, D_X\partial_\beta)]F^{\alpha\beta} +$$

$$+\epsilon_{\mu\nu\alpha\beta}[(D_X F)^{\alpha\beta} + F(D_X\partial^\alpha, \partial^\beta) + F(\partial^\alpha, D_X\partial^\beta)].$$

Cancelling the terms and using the symmetries, we obtain finally

$$[(D_X * F)_{\mu\nu} - (*D_X F)_{\mu\nu}] = 2\epsilon(\partial_\mu, \partial_\nu, D_X\partial_\alpha, \partial_\beta)F^{\alpha\beta} +$$

$$+2\epsilon_{\mu\nu\alpha\beta}F(D_X\partial^\alpha, \partial^\beta) = I + II.$$

Taking $X = \partial_\gamma$ and using the formula for the Christoffel symbols, we see that the first term $I$ is just

$$I = \epsilon_{\mu\nu\lambda\beta}g^{\lambda\lambda'}(-\partial_{\lambda'}g_{\alpha\gamma} + \partial_\alpha g_{\gamma\lambda'} + \partial_\gamma g_{\alpha\lambda'})F^{\alpha\beta}.$$

On the other hand,

$$D_\gamma\partial^\alpha = D_\gamma(g^{\alpha\beta}\partial_\beta) = \partial_\gamma g^{\alpha\beta}\partial_\beta + g^{\alpha\beta}\Gamma_{\lambda\gamma\beta}\partial^\lambda.$$

Hence, using $\partial_\gamma g^{\alpha\beta} = -g^{\alpha\alpha'}g^{\beta\beta'}\partial_\gamma g_{\alpha'\beta'}$,

$$2(D_\gamma\partial^\alpha)_\lambda = g^{\alpha\beta}(-\partial_\gamma g_{\lambda\beta} + \partial_\beta g_{\gamma\lambda} - \partial_\lambda g_{\gamma\beta}).$$

This gives us the expression for the second term $II$

$$II = \epsilon_{\mu\nu\alpha\beta}g^{\alpha\delta}(\partial_\delta g_{\gamma\lambda} - \partial_\lambda g_{\gamma\delta} - \partial_\gamma g_{\lambda\delta})F^{\lambda\beta}.$$

We see, changing the names of the indices, that each term in $I$ is the opposite of the corresponding term in $II$. ♢

• The duality makes it possible to rewrite Maxwell equations in the nice symmetric way

$$dF = 0,\ d * F = 0.$$

To prove this, let us admit the following formula for the exterior derivative of a 2-form

$$F = F_{\alpha\beta}dx^\alpha \wedge dx^\beta,\ dF = D_\gamma F_{\alpha\beta}dx^\gamma \wedge dx^\alpha \wedge dx^\beta.$$

Using this formula for $*F$ and the commuttion formula above, we obtain

$$d(*F) = (*D_\gamma F)_{\alpha\beta} dx^\gamma \wedge dx^\alpha \wedge dx^\beta =$$

$$= \epsilon_{\alpha\beta\mu\nu} D_\gamma F^{\mu\nu} dx^\gamma \wedge dx^\alpha \wedge dx^\beta.$$

Now fix the index $\delta$ and look at the sum $S$ of the terms involving (in some order) the three remaining indices $a < b < c$. We can write

$$S/2 = \epsilon_{ab\mu\nu} D_c F^{\mu\nu} dx^c \wedge dx^a \wedge dx^b +$$

$$+ \epsilon_{bc\mu\nu} D_a F^{\mu\nu} dx^a \wedge dx^b \wedge dx^c + \epsilon_{ac\mu\nu} D_b F^{\mu\nu} dx^b \wedge dx^a \wedge dx^c.$$

Consider the first term of the right-hand side : either $\mu$ or $\nu$ has to be $c$, and then the other index is necessarily $\delta$ ; hence this first term is $2\epsilon_{abc\delta} D_c F^{c\delta} dx^c \wedge dx^a \wedge d^b$. Handling similarly the two other terms, we get

$$S/4 = \epsilon_{abc\delta} dx^a \wedge dx^b \wedge dx^c [D_\alpha F^{\alpha\delta}],$$

which finishes the proof. ♢

**b. Energy formalism**

For Maxwell equations, we can also define an energy-momentum tensor

$$Q_{\alpha\beta} = 2F_{\alpha\gamma} F_\beta^\gamma - (1/2) g_{\alpha\beta} |F|^2, \; |F|^2 = F_{\lambda\mu} F^{\lambda\mu}.$$

If $F$ is a solution of the Maxwell equations, one can prove, in a fashion similar to what we have done in section 5.1,

$$\forall \beta, \; D^\alpha Q_{\alpha\beta} = 0.$$

Also, $Q$ enjoyes the same positivity property as in the case of the wave equation, since, in any null frame,

$$Q_{33} \geq 0, \; Q_{44} \geq 0, \; Q_{34} \geq 0.$$

These inequalities are proved in 6.2. The only difference with the case of the wave equation is that

$$tr\, Q = Q_\alpha^\alpha = 2F_{\alpha\gamma} F^{\alpha\gamma} - (1/2) \times 4 \times |F|^2 = 0.$$

The method for proving energy inequalities is exactly the same as before : we choose a timelike multiplier $X$, set $P_\alpha = Q_{\alpha\beta} X^\beta$. If $F$ is a solution of Maxwell equations, we have

$$D^\alpha P_\alpha = (1/2) Q_{\alpha\beta} \pi^{\alpha\beta}.$$

Integrating in some domain $D$ yields the identity

$$(1/2) \int_D Q\pi dV = \int_{\partial D} Q(N, X) dv.$$

All we have said about energy and interior terms extends also to this case, with the improvement that if $X$ is conformal Killing, the interior terms are identically zero.

In chapter 9, we will discuss briefly the analogous and more complicated case of Bianchi equations, for which an energy formalism also exists.

# Chapter 6

# The Good Components

## 6.1 The problem

Let $g$ be a given Lorentzian metric on $\mathbf{R}^4$, say, not too far from the flat Minkowski metric. Consider a solution $\phi$ of the wave equation $\Box_g \phi = 0$, with, smooth Cauchy data on $\{t = 0\}$ rapidly decaying at infinity. By analogy with the flat case (see the Introduction), we suspect that some derivatives of $\phi$ behave better than others, meaning here, they have better decay properties at infinity. These "good derivatives" of $\phi$ (again, "good" is meant here in the sense of having a better decay at infinity) are some of the components of $\nabla\phi$ in an appropriate null frame ; in the flat case, these good derivatives are $e_1(\phi), e_2(\phi), e_4(\phi)$. For a general metric $g$, the question we ask is the following :

*How to pick up a null frame $(e_\alpha)$ which would "capture" the good components of $\nabla\phi$ ? The same question arises when dealing with Maxwell equations : what frame is going to capture the good components of the 2-form $F$ ?*

There are, as far as we know, three approaches to this problem :

**1.** Weighted "standard" energy inequalities,

**2.** Conformal energy inequalities,

**3.** Commutation with modified Lorentz fields.

In the first approach, one establishes an improved version of the "standard" energy inequality (by this we mean the inequality corresponding to the multiplier $\partial_t$ in the flat case) ; such an inequality yields, besides the usual bound of the energy at time $t$, a bound of the (weighted) $L^2$ norm in both variables $x$ and $t$ of some special derivatives of $\phi$. In other words, the $L^2$ norm in $x$ of these special derivatives is not just bounded in $t$, it is an $L^2$ function of $t$ also.

This identifies these special derivatives as the "good derivatives" of our problem.

In the second approach, one generalizes the conformal inequality of the flat case. Recall that, in the flat case, the conformal inequality gives a bound of the modified energy $\tilde{E}$, which is equivalent to the sum of the $L^2$ norms of the special fields $S$, $R_i$, $H_i$,

$$\tilde{E}(t) \sim \int_{\Sigma_t} [(S\phi)^2 + |R\phi|^2 + |H\phi|^2 + \phi^2]dx.$$

Through the identities

$$L = (r+t)^{-1}[S + \sum \omega_i H_i],\ \omega \wedge \partial = R/r = (1/t)\omega \wedge H,$$

one obtains a $1/t$ decay of the $L^2$ norms in $x$ of the special derivatives $L\phi$, $(R/r)\phi$ of $\phi$, a fact which identifies these derivatives as the "good derivatives". In the general case, there are generalization of the conformal inequality : the structure of the corresponding modified energy $\tilde{E}$ will display the good derivatives.

Finally, the third approach is to commute the Lorentz vector fields $Z = \partial_\alpha$, $S$, $R_i$, $H_i$ with $\Box$, and then to use the standard energy inequality : one obtains in this way a bound of the $L^2$ norms in $x$ of $\nabla Z\phi$ (or, equivalently, of $Z\nabla\phi$), and one proceeds as above to identify the good derivatives from the $Z$ fields. In the general case, one constructs modified Lorentz fields $\tilde{Z}$, for which the commutators $[\Box, \tilde{Z}]$ are small, and obtains the good derivatives from the fields $\tilde{Z}$.

We give some precise statements for the first two cases, postponing the discussion of commutators to the next chapter.

## 6.2 An important Remark

In this chapter, we put emphasis on the decay at infinity of global solutions. However, as we shall see in the last chapter, the use of appropriate null frames is not limited to problems involving global solutions and their decay at infinity. It shoud be considered as a universal method for all hyperbolic problems : this is beautifully explained by Christodoulou in the prologue of [14]. If decay is not the problem, what are the guidelines to choose a good null frame ? The basic principle of the construction of null hypersurfaces and optical functions has been sketched in 4.3. We will see in the last chapter concrete examples of such constructions ; the exact choices and the benefit thereof depend on the context.

## 6.3 Ghost Weights and Improved Standard Energy Inequalities

### 1. The key idea

We explain here the key idea to obtain a "good" energy inequality. Let $g$ be given as usual, and suppose we have chosen a null frame which is *suspected* to capture the good components of $\nabla\phi$, in the sense that $e_1(\phi), e_2(\phi), e_4(\phi)$, the "good" derivatives, decay better than $e_3(\phi)$. For a given timelike multiplier $X$, we have to handle the interior terms $\int_D Q^{(X)}\pi dV$, as explained in chapter 5, 5.3. Writing $Q\pi$ in our null frame, we see that all terms involve at least one "good" derivative, except $\pi_{44}e_3(\phi)^2$. Since we want $Q\pi$ to be as small as possible, we choose $X$ such that ${}^X\pi_{44} = 0$ :

**Key idea :** *Choose the multiplier $X$ such that ${}^X\pi_{44} = 0$.*

This does not tell us, however, how to guess the good null frame ! What we do is to reverse the problem : according to our geometric intuition, we choose a reasonable null frame, and then give conditions on $g$ which ensure that this null frame is indeed a good null frame. Though this sketch may sound a little strange, we will see on examples how this method works.

We start with two cases, giving complete proofs of the corresponding theorems.

### 2. The Wave Equation in the Quasiradial Case

We make the assumptions described in chapter 3 for the quasiradial case :

$$g^{00} = -1, g^{0i}\omega_i = 0.$$

We take our null frame to be

$$(e_1, e_2, e_3 = T - N, e_4 = T + N),$$

where $(e_1, e_2)$ form an orthonormal basis (for $g$) on the standard spheres of $\mathbf{R}^3$ (for constant $t$), and

$$T = -\nabla t = \partial_t - g^{0i}\partial_i,\ N = \nabla r/||\nabla r||.$$

We set $c = ||\nabla r|| = (g^{ij}\omega_i\omega_j)^{1/2}$, and define the second fundamental form $k$ of $\Sigma_t$ by

$$k(X, Y) = - < D_X T, Y > .$$

Recall the formula for the components of $k$,

$$k_{ij} = -(1/2)g^{0\alpha}(\partial_i g_{\alpha j} + \partial_j g_{\alpha i} - \partial_\alpha g_{ij}).$$

Finally, we define the energy at time $t$ to be

$$E(t) = (1/2)\int_{\Sigma_t} [(T\phi)^2 + (N\phi)^2 + |\nabla\!\!\!/\phi|^2]dv,$$

recalling the notation $|\not\nabla\phi|^2 = e_1(\phi)^2 + e_2(\phi)^2$.

**Theorem.** *Assume that the components of $k$ satisfy, for some $\epsilon > 0$,*

*i)* $< t-r >^{1+\epsilon} [k_{1N}^2 + k_{2N}^2 + (k_{11}+k_{22})^2] \in L_t^1 L_x^\infty$,

*ii)* $< t-r >^{1+\epsilon} [|Tc/c| + |k_{1N}| + |k_{2N}| + |k_{11}| + |k_{12}| + |k_{22}|] \in L_{x,t}^\infty$.

*Then, for some constant $C = C_\epsilon$ and all $T \geq 0$,*

$$E(T) + \int_{0\leq t\leq T} < t-r >^{-1-\epsilon} [e_4(\phi)^2 + |\not\nabla\phi|^2] dtdv \leq$$

$$\leq CE(0) + C\int_{0\leq t\leq T} |\Box\phi||T\phi| dtdv + C\int_0^T A(t)E(t)dt.$$

*The amplification factor $A$ is*

$$A(t) = ||Tc/c||_{L_x^\infty} + || < t-r >^{-1} (c-1)||_{L_x^\infty}.$$

*In particular, if $A \in L_t^1$, $e_1(\phi), e_2(\phi), e_4(\phi)$ are the good components of $\nabla\phi$.*

Note that the assumptions i) and ii) do not require the derivatives of the metric to be integrable in time. The proof of the theorem is an application of the key idea above, combined with the use of the weight $e^a$, $a = a(t-r)$ :

• We choose the multiplier $X = e^a T$, and set $\pi = {}^T\pi$ : from the formula of chapter 5, 5.3,

$$Q^X\pi = e^a[Q\pi + 2Q(\nabla a, T)].$$

Since

$$\nabla a = a'\nabla(t-r) = -a'(T+cN) = -a'(T+N+(c-1)N),$$

and $2Q(T,T) = (T\phi)^2 + (N\phi)^2 + |\not\nabla\phi|^2$, we can compute the additionnal terms due to the weight

$$Q(T+N,T) = (1/2)(e_4(\phi)^2 + |\not\nabla\phi|^2),$$

$$-Q^X\pi = e^a[-Q\pi + a'(e_4(\phi)^2 + |\not\nabla\phi|^2) + 2a'(c-1)(T\phi)(N\phi)].$$

The idea of the "ghost weight" is to choose $a'(s) = A < s >^{-1-\epsilon}$ : in this way, $a$ is bounded and the weight $e^a$ *disappears* from the inequality (leaving only constants depending on $\epsilon$) ; on the other hand, choosing $A$ big enough will give us plenty of the good derivatives.

• We now dispatch all terms of $Q\pi$ into three categories :

i) The terms containing two good derivatives, which have the coefficients $\pi_{34}$, $\pi_{33}$, $\pi_{a3}$, $\pi_{ab}$ $(1 \leq a, b \leq 2)$,

ii) The terms containing only one good derivative, which have the coefficients $\pi_{a4}$, $\pi_{11}+\pi_{22}$,

iii) The bad term $\pi_{44} e_3(\phi)^2$.

To handle the terms in the first two categories, it is enough to assume respectively

$$< t - r >^{1+\epsilon} [|\pi_{34}| + |\pi_{33}| + \sum |\pi_{a3}| + \sum |\pi_{ab}|] \in L^\infty_{x,t},$$

$$< t - r >^{1+\epsilon} [\sum (\pi_{a4})^2 + (\pi_{11} + \pi_{22})^2] \in L^1_t L^\infty_x.$$

• To allow for the simple choice of our multiplier, we do not realize exactly $\pi_{44} = 0$ as announced in the "key idea", but $\pi_{44} = -2Tc/c$. In fact, since

$$T = \partial_t - g^{0i}(\partial_i - \omega_i \partial_r) = \partial_t + g^{0i}(\omega \wedge R/r)_i,$$

$$N c^{-1} g^{ij} \omega_i \partial_j = c\partial_r - c^{-1} g^{ij} \omega_i (\omega \wedge R/r)_j,$$

we obtain

$$[T, N] = (Tc/c)N + \ldots R, < [T, N], N >= Tc/c.$$

On the other hand, since $D_T T = 0$,

$$< [T, N], N >=< D_T N, N > - < D_N T, N >= k_{NN},$$

$$\pi_{44} = 2 < D_{T+N} T, T + N >= 2 < D_N T, T + N >= -2k_{NN}.$$

• Finally, it remains to compute the components of $\pi$ to translate the conditions above on $\pi$ into the conditions of the theorem on $k$. We obtain easily

$$\pi_{34} = 2k_{NN} = -\pi_{33}, \pi_{a4} = 2k_{aN} = -\pi_{a3}, \pi_{ab} = 2k_{ab}.$$

♢ The same method can be extended to generalize Morawetz type inequalities, see for instance [4], [7].

### 3. The Wave Equation in the General case

We present here a more geometric, but less explicit, result. Assume given an optical function $u$ to which we associate a null frame $(e_1, e_2, e_3 = \underline{L}, e_4 = L)$ as explained in chapter 3. Define the corresponding energy at time $T$ to be

$$E(T) = (1/2) \int_{\Sigma_T} [a(e_4(\phi))^2 + a^{-1}(e_3(\phi))^2 + (a + a^{-1})| \nabla\!\!\!/ \phi|^2] dv.$$

**Theorem.** *Let $T = (1/2)(L + \underline{L})$ and $\pi = {}^T\pi$. Assume that, for some $\epsilon > 0$, the components of $\pi$ satisfy the following estimates :*

$$< u >^{1+\epsilon} a[(\pi_{1L})^2 + (\pi_{2L})^2 + (\pi_{11} + \pi_{22})^2] \in L^1_t L^\infty_x,$$

$$< u >^{1+\epsilon} [|\pi_{11}| + |\pi_{22}| + |\pi_{12}| + |\pi_{1\underline{L}}| + |\pi_{2\underline{L}}| + |\underline{\omega}|] \in L^\infty_{x,t}.$$

*Then, for some $C = C_\epsilon$ and all $T$,*

$$E(T)^{1/2} + \{\int_{0\leq t\leq T} < u >^{-1-\epsilon} [(L\phi)^2 + | \not\nabla\phi|^2]dtdv\}^{1/2} \leq$$

$$\leq CE(0)^{1/2} + C\int_0^T ||(a^{1/2} + a^{-1/2})f||_{L^2(dv)}(t)dt.$$

The nice feature here is that there is *no amplification factor* at all ! In comparision with the preceding theorem, we see that the amplification factor $A$ there came from $Tc/c$ (the error in $\pi_{44}$) and $c-1$ (the error coming from taking $u = t - r$ instead of a *true* optical function).

In particular, in this theorem, the derivatives $e_1(\phi), e_2(\phi), e_4(\phi)$ are identified as the good derivatives. The components of $\pi$ can be computed explicitly in terms of the frame coefficients : following the "key idea", we have arranged

$$\pi_{LL} =< D_L(L + \underline{L}), L >= - < L + \underline{L}, D_L L >= 0.$$

The other components appearing in the assumptions of the theorem are

$$2\pi_{La} =< D_L(L + \underline{L}), e_a > + < D_a(L + \underline{L}), L >= 2\underline{\eta}_a - 2\eta_a,$$

$$2\pi_{\underline{L}a} =< D_{\underline{L}}(L + \underline{L}), e_a > + < D_a(L + \underline{L}), \underline{L} >= 2\eta_a + 2\underline{\xi}_a + 2\eta_a = 4\eta_a + 2\underline{\xi}_a,$$

$$\pi_{ab} = \chi_{ab} + \underline{\chi}_{ab}, 4\underline{\omega} = \nabla^2 u_{\underline{L}\underline{L}}.$$

The proof of the theorem follows the same lines as before. First, using the multiplier $T$ gives an energy density

$$Q(T, \partial_t) = (1/4)Q(L + \underline{L}, aL + a^{-1}\underline{L}) = (1/4)[a(L\phi)^2 + a^{-1}(\underline{L}\phi)^2 + (a + a^{-1})| \not\nabla\phi|^2].$$

Taking $X = e^a T$ as a multiplier xith $a = a(u)$, we get from the weight additional interior terms

$$-a'(u)Q(L, T) = (-a'(u)/2)[(L\phi)^2 + | \not\nabla\phi|^2].$$

We choose $a'(s) = A < s >^{-1-\epsilon}$ with $A$ big enough, and finish the proof exactly as before (see [4] for details). ♢

A variation on this theme of improved standard inequality appears also in [35] (see chapter 9 for a statement).

**4. Maxwell equations in the general case**

Let us work again with a null frame associated with one optical function, but consider now Maxwell equations. Instead of defining the electric and magnetic fields as in the flat case (which means altogether six functions), let us define the six components of $F$ in our null frame $(e_1, e_2, \underline{L}, L)$ as

$$\alpha_a = F(e_a, L), \underline{\alpha}_a = F(e_a, \underline{L}), \rho = (1/2)F(\underline{L}, L), \sigma = F(e_1, e_2).$$

Remark that $\alpha$ and $\underline{\alpha}$ are 1-forms on the (nonstandard) 2-spheres, and that $\sigma$ does not depend on the chosen orthonormal frames on these spheres, since for another frame

$$\tilde{e}_1 = \cos\theta e_1 - \sin\theta e_2,\ \tilde{e}_2 = \sin\theta e_1 + \cos\theta e_2,$$

we would find

$$\tilde{\sigma} = F(\tilde{e}_1, \tilde{e}_2) = -(\sin^2\theta)F(e_2, e_1) + (\cos^2\theta)F(e_1, e_2) = F(e_1, e_2) = \sigma.$$

First, we note that, computing the double trace $|F|^2$ in our null frame and taking into account the symmetries of $F$,

$$|F|^2 = 2F_{12}^2 - 2F_{13}F_{14} - 2F_{23}F_{24} - (1/2)F_{34}^2 = 2\sigma^2 - 2\rho^2 - 2\alpha\underline{\alpha}.$$

From the definition of the energy momentum tensor in this case, we can write

$$Q(X, Y) = 2F(X, \partial_\gamma)F(Y, \partial^\gamma) - (1/2) < X, Y > |F|^2,$$

and we observe that the first term is a trace, which can be computed in any basis with its dual basis. For instance,

$$F(X, \partial_\gamma)F(y, \partial^\gamma) = F(X, e_1)F(Y, e_1) + F(X, e_2)F(Y, e_2) -$$

$$-(1/2)F(X, \underline{L})F(Y, L) - (1/2)F(X, L)F(Y, \underline{L}).$$

For the components of $Q$, we thus find

$$Q(L, L) = 2|\alpha|^2, Q(\underline{L}, \underline{L}) = 2|\underline{\alpha}|^2, Q(L, \underline{L}) = 2(\rho^2 + \sigma^2),$$

$$Q(L, e_1) = 2(-\sigma\alpha_2 + \rho\alpha_1), Q(\underline{L}, e_1) = 2(-\sigma\underline{\alpha}_2 - \rho\underline{\alpha}_1),$$

$$Q(e_1, e_1) = \sigma^2 + \rho^2 - \alpha_1\underline{\alpha}_1 + \alpha_2\underline{\alpha}_2, Q(e_1, e_2) = -(\alpha_1\underline{\alpha}_2 + \alpha_2\underline{\alpha}_1),$$

and similarly for the other components.

Finally, define the energy of $F$ at time $T$ to be

$$E(T) = (1/2)\int_{\Sigma_T} [a|\alpha|^2 + a^{-1}|\underline{\alpha}|^2 + (a + a^{-1})(\rho^2 + \sigma^2)]dv.$$

The following theorem is completely analogous to the corresponding theorem for $\Box$.

**Theorem.** *Let $T = (1/2)(L + \underline{L})$ and $\pi = {}^T\pi$. Assume that, for some $\epsilon > 0$, the components of $\pi$ satisfy the following estimates*

$$< u >^{1+\epsilon} a(\pi_{11}^2 + \pi_{12}^2 + \pi_{22}^2 + \pi_{1L}^2 + \pi_{2L}^2) \in L_t^1 L_x^\infty,$$

$$< u >^{1+\epsilon} (|\pi_{11}| + |\pi_{22}| + |\pi_{1\underline{L}}| + |\pi_{2\underline{L}}| + |\underline{\omega}|) \in L^\infty_{x,t}.$$

*Then, for some constant $C = C_\epsilon$, for all solutions $F$ of the Maxwell equations and all $T$,*

$$E(T) + \int_{0\leq t\leq T} < u >^{-1-\epsilon} (|\alpha|^2 + \rho^2 + \sigma^2) dtdv \leq CE(0).$$

*In particular, under these assumptions, the good components of $F$ are $\alpha, \rho, \sigma$.*

The proof of this theorem is practically the same as before : we use the multiplier $X = e^a T$ with $a = a(u)$. This gives in $Q\pi$ additional terms

$$a'(u)Q(L, L + \underline{L}) = 2a'(u)(|\alpha|^2 + \rho^2 + \sigma^2)$$

and identifies the good components. The corresponding energy density is

$$Q(X, \partial_t) = (1/2)e^a Q(T, aL + a^{-1}\underline{L}),$$

justifying the definition of $E$. The rest of the proof is the same as before (see [4] for details). ♢

## 6.4 Conformal Inequalities

• In the flat case, this quite miraculous inequality is obtained using the multiplier

$$K_0 = (r^2 + t^2)\partial_t + 2rt\partial_r.$$

Recall that $K_0$ is a timelike conformal Killing field. The question is now : for a general metric $g$ (close to the flat case), how to pick up a nice substitute for $K_0$ ? To motivate the answer, let us go back once more to the flat case, and set $u = t - r$, $\underline{u} = t + r$. We easily check the formula

$$S = t\partial_t + r\partial_r = (1/2)(\underline{u}L + u\underline{L}), K_0 = (1/2)(\underline{u}^2 L + u^2 \underline{L}).$$

Suppose now that we have a general split metric $g$ and an optical function $u$, and that we work in the associated null frame as explained in 3.3. Following the presentation of [], by analogy with the flat case, define $\underline{u} = 2t - u$ (though terribly ugly, this formula will do), and set

$$K_0 = (1/2)(\underline{u}^2 L + u^2 \underline{L}).$$

• For a certain function $\Omega$ to be chosen later, let us set

$$\pi = {}^{K_0}\pi,\ \tilde{\pi} = \pi - \Omega g.$$

Similarly to what we have done in chapter 5, we can modify the key formula for energy inequality by writing

$$D^\alpha \tilde{P}_\alpha = (1/2)Q_{\alpha\beta}\tilde{\pi}^{\alpha\beta} - (1/4)\phi^2 \Box\Omega + (K_0\phi + (\Omega/2)\phi)(\Box\phi),$$

with the modified $\tilde{P}$

$$\tilde{P}_\alpha = Q_{\alpha\beta} K_0^\beta + (\Omega/2)\phi\partial_\alpha\phi - (1/4)\phi^2\partial_\alpha\Omega.$$

We choose $\Omega = 4t$, which is the value for the flat case. By integrating in a slab $\{0 \leq t \leq T\}$, we then obtain the **conformal energy** at time $T$

$$\tilde{E}(T) = (1/4)\int_{\Sigma_T} \{a\underline{u}^2(L\phi)^2 + a^{-1}u^2(\underline{L}\phi)^2 +$$

$$+(a^{-1}u^2 + a\underline{u}^2)|\not\nabla\phi|^2 + 8t\phi\partial_t\phi - 4\phi^2\}dv.$$

Recall here the notation

$$a = (\partial_t u)^{-1},\ N = -(\partial_t u)^{-1} g^{ij}\partial_i u\partial_j.$$

We prove now the following theorem.

**Theorem.** *Assume* $|a-1| \leq 1/10$ *and* $(div\, N)(\underline{u} - u) = 4 + \epsilon$ *with* $|\epsilon|$ *small enough. Then, for some constant* $C > 0$,

$$\tilde{E}(T) \geq C\int_{\Sigma_T} \{\underline{u}^2(L\phi)^2 + (u^2 + \underline{u}^2)|\not\nabla\phi|^2 + u^2(\underline{L}\phi)^2 + \phi^2\}dv.$$

This theorem identifies $L\phi$ and $\not\nabla\phi$ as the good derivatives of $\phi$. The additional control of $u\underline{L}\phi$ is not useless (as we shall see in 7.1) : it is a weak form of a control of $Z\phi$ ($Z$ denoting a Lorentz field a usual), since

$$\underline{u}L + u\underline{L} = 2S,\ \underline{u}L - u\underline{L} = 2\sum \omega_i H_i.$$

Once again, note that we have followed the "key idea", since

$$\tilde{\pi}_{LL} = \pi_{LL} = 2 < D_L K_0, L > = < D_L(\underline{u}^2 L + u^2\underline{L}), L > =$$

$$= L(\underline{u}^2) < L, L > + \underline{u}^2 < D_L L, L > + L(u^2) < \underline{L}, L > - u^2 < D_L L, \underline{L} > = 0.$$

Of course, the theorem provides only an analysis on the conformal energy : it should be complemented with a thorough analysis of the interior terms $\int_D Q\pi dV$ in the spirit of 5.5. We refer to [24] or to Hörmander book [21] ( in a non geometric setting) for such an analysis;

The idea of the proof is to transform the term $t\partial_t\phi$ of $\tilde{E}$, using two auxiliary fields $S$ and $\underline{S}$, into a term involving a tangential derivative, for which we can perform an integration by parts on $\Sigma_T$.

**1.** We set

$$S = (1/2)(a\underline{u}L + a^{-1}u\underline{L}),\ \underline{S} = (1/2)(a\underline{u}L - a^{-1}u\underline{L}).$$

Then

$$t\partial_t\phi = S\phi - (1/2)(\underline{u} - u)N\phi, t\partial_t\phi = [(2t)/(\underline{u} - u)]\underline{S}\phi - [(2t^2)/(\underline{u} - u)]N\phi.$$

For some number $0 < \lambda < 1$ to be determined, we split $t\partial_t\phi = \lambda t\partial_t\phi + (1-\lambda)t\partial_t\phi$ and use the expression with $S$ for the first term, and the expression with $\underline{S}$ for the second.

**2.** To integrate by parts in $\Sigma_T$, we need some formula. For any field $X$ and any two functions $f, h$, we have

$$div\,(fgX) = fg\,div\,X + X(fg).$$

We will use this with $X = N$, noting that

$$\nu \equiv div\,N = \sum < D_a N, e_a > = < D_a(aL - T), e_a > = atr\chi - k_a^a.$$

**3.** Using the formula of 2., we obtain

$$\int t\phi\partial_t\phi dv = \int \phi S\phi dv - (1/4)\int (\underline{u}-u)N(\phi^2)dv = \int \phi S\phi dv + (1/4)\int [N(\underline{u}-u)+\nu(\underline{u}-u)]\phi^2 dv.$$

Similarly, we get

$$\int t\phi\partial_t\phi dv = \int [(2t)/(\underline{u}-u)]\phi\underline{S}\phi dv + \int [N(t^2/(\underline{u}-u)) + \nu t^2/(\underline{u}-u)]\phi^2 dv.$$

Now, using the assumptions of the theorem, we find for the $\phi^2$-coefficient in the first expression

$$N(\underline{u}-u) = 2N(t-u) = -2Nu = 2 < N, L > = 2,\ N(\underline{u}-u) + \nu(\underline{u}-u) = 6 + \epsilon,$$

and for the $\phi^2$-coefficient in the second

$$N(t^2/(\underline{u}-u)) + \nu t^2/(\underline{u}-u) = (t^2/(\underline{u}-u)^2)[-2 + \nu(\underline{u}-u)] = (2+\epsilon)t^2/(\underline{u}-u)^2.$$

**4.** We observe now

$$a^2\underline{u}^2(L\phi)^2 + +a^{-2}u^2(\underline{L}\phi)^2 = 2[(S\phi)^2 + (\underline{S}\phi)^2].$$

Since $|a-1| \leq 1/10$, $d \leq a$ and $d \leq a^{-1}$ for $d = 9/10$. The integrand of $4\tilde{E}$ is bigger than $(a^{-1}u^2 + a\underline{u}^2)|\,\nabla\!\!\!/\phi|^2 + F$, with

$$F = 2d[(S\phi)^2 + (\underline{S}\phi)^2] + 8\lambda t\phi\partial_t\phi + 8(1-\lambda)t\phi\partial_t\phi - 4\phi^2.$$

Integrating on $\Sigma_T$, we get

$$\int F dv = \int \{2d(S\phi + 2\lambda\phi/d)^2 + \phi^2[(12+2\epsilon)\lambda - 4 - 8\lambda^2/d] +$$

$$+2d[(\underline{S}\phi + 4t(1-\lambda)/(d(\underline{u}-u))\phi)^2 + (t^2/(\underline{u}-u)^2)\phi^2(4(1-\lambda)(2+\epsilon)/d - 16(1-\lambda)^2/d^2)]\}dv.$$

Taking $\epsilon = 0$, we need to impose the conditions

$$\lambda > 1 - d/2,\ 2\lambda^2/d - 3\lambda + 1 < 0.$$

Since $d = 9/10$, this reads

$$\lambda > 0.55,\ 0.6 < \lambda < 0.75,$$

which is certainly possible. Thus, choosing $\lambda$ is also possible for $|\epsilon|$ small enough. ♢

# Chapter 7

# Pointwise Estimates and Commutations

In the previous chapters, we put emphasis on energy estimates, since nothing is possible without them : they provide the basic control of the solutions, and allow one to identify the "good components", as explained in chapter 6. However, for the sake of completeness or for applications to nonlinear equations, one generally needs more that (weighted) $L^2$ estimates, one also needs pointwise estimates, displaying the rate of decay and the qualitative behavior of the solutions.

The basic tool in this direction is **Klainerman inequality**

$$(1+|t|+r)^2(1+|r-t|)|v(x,t)|^2 \leq C \sum ||Z^k v||^2_{L^2_x},$$

where the fields $Z$ are the standard Lorentz fields $Z = \partial_\alpha$, $S$, $R_i$, $H_i$ and the sum is extended to all products $Z^k$ of $k$ such fields, $k \leq 2$ (see for instance [21] for a proof). **"Klainerman method"** is the fundamental strategy to obtain pointwise estimates. It consists of the following three steps :

i) Prove an energy inequality for $\Box$,

ii) Commute products of $Z$ fields with $\Box$ to obtain equations $\Box Z^k \phi = f$, and apply the energy inequality to these equations,

iii) Use Klainerman inequality to obtain the qualitative behavior of $\nabla\phi$.

The only problem with this strategy is that the *standard* Lorentz fields $Z$ have no reason to commute reasonably with $\Box = \Box_g$ in general ! Hence there are three possibilities :

**1.** We use modified $Z$ fields but do not commute them with $\Box$ : this is made possible by using (generalized) conformal inequalities, which, as we have seen in 6.3, yield directly a bound for the $L^2_x$ norms of

$$\underline{u}L\phi,\ u\underline{L}\phi,\ \underline{u}\ \nabla\!\!\!/\phi.$$

These fields can be considered good substitutes for the $Z$ fields. In this case, however, we control only *one* $Z$ field, and not *products* of such fields ! We will see that this is enough to get some qualitative information, though not as good as that from Klainerman inequality.

**2.** We modify the *standard* $Z$ into *deformed* fields $\tilde{Z}$ which commute better with $\Box$ : this is a rather difficult geometric construction, and we will see two aspects of it.

**3.** We nevertheless use the standard Lorentz fields $Z$ : rather unexpectedly, this approach turns out to be efficient in many nonlinear problems. We postpone this discussion to the last chapter.

## 7.1 Pointwise Decay and Conformal Inequalities

We have seen in chapter 6, 6.3 the expression of the modified energy $\tilde{E}$ which arises when establishing a conformal inequality. If we can bound $\tilde{E}$, we can bound in particular the spatial $L^2$ norm of $r\ \nabla\!\!\!/\phi$.

The article [24] is written using a null frame associated with one optical function $u$, as explained in chapter 1. Th strategy of [24] is the following :

• One does not try to commute the $Z$ fields with $\Box$ ; one commutes only $\partial_t$, or more precisely, $T_0 = (1/2)(L+\underline{L})$. We have already seen in 6.2 that ${}^{(T_0)}\pi_{LL} = 0$, which is, as we shall see in the next section, a condition ensuring cancellation of the bad terms in the commutator $[T_0, \Box]$. Using the equation, the control of the $T_0$ derivatives yields a control of all derivatives ; for instance, in the flat case,

$$||\partial^2\phi||_{L^2} \leq ||\Delta\phi||_{L^2} \leq (||\partial_t^2 u||_{L^2} + ||\Box u||_{L^2}).$$

• One defines a higher order energy $\tilde{E}_{k+1}$ by the formula

$$\tilde{E}_{k+1} = \sum_{|\alpha|\leq k} \tilde{E}(\partial^\alpha\phi).$$

The following proposition gives the required pointwise bound.

**Proposition.** *Let $\phi$ be smooth and sufficiently decaying as $|x| \to +\infty$. For $p > 2$,*

$$|\partial\phi(x,t)| \leq C_p(1+t)^{-2/p}\tilde{E}_3(t).$$

The drawback of this strategy is obviously that, even in the flat case, it can never give the good decay rate $t^{-1}$, since $p > 2$. However, it has the advantage of commuting only "ordinary" derivatives with $\Box$, instead of $Z$ fields.

We outline briefly the proof in the flat case, since we can see then the full strength of the conformal energy. First, we admit the following lemma.

**Lemma.** it For any smooth function $v$ on $\mathbf{R}^3_x$ sufficiently decaying at infinity, and $p > 2$, $s \geq 5/2 - 3/p$,

$$|v(x)| \leq C|x|^{-2/p}(||r\ \nabla\!\!\!/ v||_{H^s} + ||v||_{H^s}).$$

Considering a solution $\phi(x,t)$ of $\Box\phi = 0$, this lemma yields the inequality of the proposition only for $|x| \geq t/2$. For $|x| \leq t/2$, we just note that

$$\tilde{E} \geq Ct^2 \int_{|x|\leq t/2} |\partial\phi|^2 dx.$$

This is where the term $u\underline{L}\phi$ is used. The control of $\underline{u}L\phi$, $u\underline{L}\phi$, $r\ \nabla\!\!\!/\phi$ yields in fact the control of the hyperbolic rotations $H_i\phi$, since

$$H_0 \equiv \sum \omega_i H_i = t\partial_r + r\partial_t = (1/2)(\underline{u}L - u\underline{L}),$$

$$H_i = \omega_i H_0 + t(\partial_i - \omega_i\partial_r).$$

## 7.2 Commuting fields in the scalar case

**1.** Before discussing strategy 2 above, we establish a general commutation formula.

**Theorem (Commutation formula).** *For any field $X$ with deformation tensor $\pi = {}^X\pi$,*

$$[\Box, X]\phi = \pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta} + D_\alpha\pi^{\alpha\beta}\partial_\beta\phi - (1/2)\partial^\alpha(tr\,\pi)\partial_\alpha\phi.$$

In particular, $X$ is a Killing field if and only if $[\Box, X] = 0$ ; this is an easy way in practice to identify a Killing field : for instance, if $g$ is the Kerr metric, we see immediately that $\partial_t$ and $\partial_\phi$ commute with $\Box$ ! If $X$ is conformal Killing, that is, $\pi = \lambda g$, we have $D^\alpha\pi_{\alpha\beta} = \partial_\beta\lambda$, hence

$$[\Box, X] = \lambda\Box\phi - \partial^\alpha\lambda\partial_\alpha\phi.$$

For the flat metric for instance, we get $[\Box, S] = 2\Box$, but $[\Box, K_\mu] \neq -4x_\mu\Box\phi$. We say that $S$ commutes well with $\Box$, since if $\Box\phi = f$ is known, so is $\Box S\phi = Sf + 2f$ ; but this is not the case for $K_\mu$.

We give here a (pedestrian ) self-contained proof of the theorem, using the formula for $\pi^{\alpha\beta}$ given in chapter 5 :

**a.** We write $\Box\phi = \partial^\alpha\partial_\alpha\phi - (D^\alpha\partial_\alpha)\phi$, hence

$$X\Box\phi = [X, \partial^\alpha]\partial_\alpha\phi + \partial^\alpha[X, \partial_\alpha]\phi - [X, D^\alpha\partial_\alpha]\phi + \Box X\phi.$$

Now

$$[X, \partial_\alpha] = -(\partial_\alpha X^\mu)\partial_\mu, [X, \partial^\alpha] = X(g^{\alpha\beta})\partial_\beta - (\partial^\alpha X^\mu)\partial_\mu.$$

Gathering the terms, we get for a first order differential term $E$

$$[\Box, X]\phi = \pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta} + E,$$

$$E = \pi^{\alpha\beta}(D_\alpha\partial_\beta)\phi + (\partial^\alpha g_{\alpha\gamma})(\partial^\gamma X^\beta)\partial_\beta\phi + \partial_\alpha(\partial^\alpha X^\beta)\partial_\beta\phi + [X, D^\alpha\partial_\alpha]\phi.$$

**b.** From the definition of $D\pi$, we get

$$D^\alpha\pi_{\alpha\beta} =< D^\alpha D_\alpha X, \partial_\beta > + < D^\alpha D_\beta X, \partial_\alpha > -$$

$$- < D_{D^\alpha\partial_\alpha} X, \partial_\beta > - < D_{D^\alpha\partial_\beta} X, \partial_\alpha > .$$

Also,

$$(1/2)\partial_\beta tr\,\pi =< D_\beta D_\alpha X, \partial^\alpha > + < D_\alpha X, D_\beta\partial^\alpha > .$$

Hence

$$D^\alpha\pi_{\alpha\beta} - (1/2)\partial_\beta tr\,\pi = I + II + III,$$

where

$$III =< D_{[\partial^\alpha, X]}\partial_\alpha, \partial_\beta > + < D^\alpha[\partial_\alpha, X], \partial_\beta > +$$
$$+ < [X, D^\alpha\partial_\alpha], \partial_\beta > -\pi(D^\alpha\partial_\beta, \partial_\alpha) - (\partial_\beta g^{\alpha\gamma}) < D_\alpha X, \partial_\gamma > .$$

Here, we have introduced on purpose the terms

$$I =< D_\alpha D_\beta X - D_\beta D_\alpha X, \partial^\alpha >,$$

$$II =< D^\alpha D_X\partial_\alpha - D_X D^\alpha\partial_\alpha - D_{[\partial^\alpha, X]}\partial_\alpha, \partial_\beta > .$$

These terms are "curvature terms", denoted

$$I = R^\alpha_{X\alpha\beta},\ II = R^\alpha_{\beta\alpha X}.$$

For simplicity, we chose to introduce the curvature tensor $R$ only in the next chapter, so we have to admit at this point the symmetries of this tensor

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\gamma\delta\alpha\beta}.$$

This being admitted, we obtain $I + II = 0$, and

$$III = \pi^{\alpha\gamma} < D_\gamma\partial_\alpha, \partial_\beta > -\pi^{\alpha\gamma} < D_\alpha\partial_\beta, \partial_\gamma > +$$

$$+\partial^\alpha(\partial_\alpha X^\gamma)g_{\gamma\beta} + < [X, D^\alpha\partial_\alpha], \partial_\beta > -(\partial_\beta g^{\alpha\gamma}) < D_\alpha X, \partial_\gamma > .$$

**c.** Finally,

$$(D^\alpha \pi_{\alpha\beta} - (1/2)\partial_\beta tr\, \pi)\partial^\beta \phi - E =$$
$$= (1/2)(\nabla\phi)(g_{\alpha\beta})\pi^{\alpha\beta} - (1/2)(\nabla\phi)(g_{\alpha\gamma}\pi^{\alpha\gamma} = 0,$$

since

$$\pi^{\alpha\gamma} < D_\alpha \partial_\beta, \partial_\gamma >= \pi^{\alpha\gamma} < D_\beta \partial_\alpha, \partial_\gamma >= (1/2)\pi^{\alpha\gamma}\partial_\beta g_{\alpha\gamma}.$$

♢ **2.** The commutation formula has the advantage, like such formula, to be written as multiple traces : the term $\pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta}$ is a double trace, analogous to the term $\pi^{\alpha\beta}Q_{\alpha\beta}$ of the energy inequalities ; the other terms are

$$D_\alpha \pi^{\alpha\beta}\partial_\beta \phi - (1/2)\partial^\beta (tr\, \pi)\partial_\beta \phi = D_\alpha \pi^\alpha_{\nabla\phi} - (1/2)(\nabla\phi)(tr\, \pi).$$

## 7.3 Modified Lorentz fields

In practice, we cannot expect to find $X$ commuting exactly with $\Box$, and we choose fields $X$ which look close to the standard Lorentz fields *and* can be nicely expressed in the null frame we are working with. We start with a simple remark.

### 1. Good commutation condition

If one is mainly concerned with decay at infinity, one can use the concept of "good derivative" to construct modified fields. In chapter 5, when looking for a good multiplier $X$ to obtain an energy inequality, we found the condition ${}^X\pi_{LL} = 0$ ; this condition ensured that all terms among the interior terms $\pi^{\alpha\beta}Q_{\alpha\beta}$ would contain at least one "good derivative" $e_1, e_2$ or $L$. Similarly, we can sketch what could be a "good commuting field" $X$ for $\Box$ : it is a field $X$ such that the higher order terms $\pi^{\alpha\beta}\nabla^2\phi_{\alpha\beta}$ (given by theorem 7.2) involve only good derivatives of $\phi$, that is, second order derivatives containing at least one good derivative. Since, with this definition, the only bad second order derivative is $\underline{L}^2\phi$, the required condition is again ${}^X\pi_{LL} = 0$.

**Good commutation condition :** ${}^X\pi_{LL} = 0.$

### 2. The difficulty with the hyperbolic rotations

Consider the equation $\partial_t^2 - c^2\Delta_x$ instead of the standard $\Box$ ; for the hyperbolic rotations, we have to take $H_i = (x^i/c)\partial_t + ct\partial_i$. In other words, $H_i$ depends on the speed $c$, while $S$ and $R_i$ do not. It turns out that, in more general situations, good substitutes for $H_i$ are not known.

This being admitted, in view of strategy 2, one has to explain how to replace Klainerman inequality when the fields $H_i$ are missing : the idea is to use only the fields $\Gamma = \partial_\alpha$, $S$, $R$ and the operator itself $\Box$. For the flat case, one can prove the following inequalities (see [33]), which can be viewed as a substitute for Klainerman inequality :

**Proposition.** *Let $E(\phi)$ be the standard energy, and define a higher order energy $E_{k+1}$ by*

$$E_{k+1}(\phi) = \sum_{|\alpha|\leq k} E(\Gamma^\alpha \phi).$$

*Then, $\partial_i$ $(i = 1, 2, 3)$ being the spatial derivatives,*

$$(1 + r) < t - r > |\partial_i \nabla\phi| \leq C(E_4^{1/2} + t||\Box\phi||_{L^2}).$$

It turns out that this type of inequality is also available for more general situations than the flat case (see for instance [6]). Thus Klainerman method can be extended to non flat geometric situations, dropping the $H_i$ and using appropriately defined $S$ and $\tilde{R}_i$.

### 3. The fully geometric approach

This is for instance the approach of [25], where the null frame is associated with two optical functions $u$ and $\underline{u}$ as explained in chapter 3. In such a geometric framework, one is looking for substitutes for the standard Lorentz fields $S$, $R_i$, $H_i$. We have already mentioned that it was necessary and possible to forget about the hyperbolic rotations $H_i$. The generalization for $S$ follows easily from the formula of chapter 6 : we just take $S = (1/2)(u\underline{L} + \underline{u}L)$. For $R_i$, we take fields tangent to the (nonstandard) 2-spheres of the foliation. The actual construction in [25] is rather delicate, and we only sketch it here : first, we consider in $\Sigma_0 = \{t = 0\}$ a specific sphere foliation, with unitary normal field $N$. The flow of $N$ and the asymptotic properties of the metric at infinity allow one to pull back to a given sphere the standard rotations $R_i$ at infinity (which are homogeneous of degree zero). Once this is done, we push forward these rotation fields by the flow of $L$ along an outgoing cone. In this way, wwe obtain rotations ${}^iO$ satisfying good commutation relations

$$[{}^iO, {}^jO] = \epsilon_{ijk}{}^kO, \; [L, {}^iO] = 0.$$

The drawback of this definition is of course its global character.

### 4. A simplified approach

Suppose for instance that we choose a quasiradial frame and that our assumptions on $g$ allow us to identify $e_1, e_2$ and $L$ as the good derivatives (see chapter 5). The idea to construct a good commuting modified field $\tilde{Z}$ is to try $\tilde{Z} = Z + aT$, since the other terms in the perturbation of $Z$ would involve only good derivatives and would probably play a negligible role (note that we take here $T$ rather than $\underline{L}$ since $T$ has smooth coefficients everywhere). Then

$${}^{\tilde{Z}}\pi_{LL} = {}^Z\pi_{LL} + a^T\pi_{LL} - 2La = 2 < [L, Z], L > -2a(Tc/c) - 2La.$$

For the reasons already explained above, we forget about $H_i$ and will use and modify only $Z = R_i$ and $Z = S$. Since $[R_i, \partial_r] = 0$, $[S, R_i] = 0$,

$$[R, T] = [R, \partial_t] + \ldots R, \; [R, N] = (Rc/c)N + \ldots R, [S, T] = [S, \partial_t] + \ldots R,$$

$$[S,N] = (Sc/c)N + c[S,\partial_r] + \dots R, [S,L] = -L + (Sc/c)N + \dots R,$$

we get

$$< [L,R_i], L >= -Rc/c, < [L,S], L >= -Sc/c.$$

Finally, we define $a = {}^Z a$ by

$$La + aTc/c + Zc/c = 0,$$

and use the fields

$$\tilde{Z} = Z + {}^Z aT.$$

This causes of course a certain number of technical difficulties : control of $a$ simultaneously with the solution $\phi$, obtention of a pointwise estimate from a bound of $\nabla\tilde{Z}^k\phi$ in $L^2$ norm, etc. The advantage by comparison with the fully geometric approach more simplicity in the computations. We refer to [6] for details.

## 7.4 Commuting fields for Maxwell equations

In the scalar case of the previous section, our aim was to control $X\phi$, or more generally $X^k\phi$ for some collection of fields $X$, $\phi$ being a solution of $\Box\phi = 0$. For Maxwell equations, the unknown is a 2-form $F$, and $XF$ does not make sense : it has to be replaced by $\mathcal{L}_X F$, the Lie derivative that we briefly discussed in chapter 5. Since $d$ commutes with mappings, we have $[\mathcal{L}_X, d] = 0$. Thus, for a solution of Maxwell equations,

$$\mathcal{L}_X(dF) = 0 = d(\mathcal{L}_X F),$$

$$\mathcal{L}_X(d * F) = 0 = d(\mathcal{L}_X * F).$$

To compute the commutation defect coming from the second equation, one can use the following formula, where $\pi$ stands for the deformation tensor of $X$ :

$$\mathcal{L}_X * F_{\mu\nu} = *\mathcal{L}_X F_{\mu\nu} + *F_\mu^\rho \pi_{\rho\nu} + *F_\nu^\rho \pi_{\mu\rho} - (1/2)tr\,\pi * F_{\mu\nu}.$$

Using this formula, one obtains again the set of Maxwell equations for $\mathcal{L}_X F$, perturbed by first order derivatives of $F$. Let us point out that there is an analogous formalism for the Bianchi identities, that we will very briefly discuss in the last chapter. We refer to [25] for more details about commutators in these cases.

# Chapter 8

# Frames and Curvature

In this chapter, we always assume for simplicity that the metric $g$ is split

$$g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + g_{ij}dx^i dx^j,$$

that the coefficients $g_{ij}$ and $g^{ij}$ are bounded, and that, for some $C > 0$ and all points,

$$g_{ij}\xi^i\xi^j \geq C|\xi|^2.$$

We wish to examine in more details how one can control the frame coefficients of a given null frame. At this point, one needs to introduce the curvature tensor $R$.

## 8.1 The Curvature Tensor

• The three fields $X$, $Y$, $Z$ being given, the field $R(X,Y)Z$ is defined by the formula

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

Hence $R(X,Y)Z$ measures the commutation defect of $D_X$ and $D_Y$, when applied to $Z$. The remarkable feature here is that this expression is *linear* in all three fields $X, Y, Z$ ! For instance,

$$R(fX,Y)Z = fD_X D_Y Z - D_Y(fD_X Z) - D_{f[X,Y]-(Yf)X}Z = fR(X,Y)Z,$$

$$R(X,Y)(fZ) = D_X((Yf)Z + fD_Y Z) - D_Y((Xf)Z + fD_X Z) - D_{[X,Y]}(fZ) =$$

$$= (XYf)Z + (Yf)D_X Z + (Xf)D_Y Z - (YXf)Z -$$

$$-(Xf)D_Y Z - (Yf)D_X Z - ([X,Y]f)Z + fR(X,Y)Z = fR(X,Y)Z.$$

Remark also that, by construction,

$$R(Y,X)Z = -R(X,Y)Z.$$

• The **curvature tensor** is just

$$R(W,Z,X,Y) =< R(X,Y)Z, W > .$$

The point of introducing all four arguments $X, Y, Z, W$ lies in the remarkable symmetries of $R$ :

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma},$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}.$$

The symmetry in $\gamma\delta$ has already been shown ; the symmetry in $\alpha\beta$ is some "integration by parts" formula, since the derivatives acting on $Z = \partial_\beta$ are transferred to act on $W = \partial_\alpha$. The symmetry of the second line is more mysterious, and we refer to standard textbooks for its proof (see for instance [19]). Finally, we also have the "circular permutation" formula

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0.$$

Using the definition, one can write down an explicit formula for the components of $R$

$$R^{\delta}_{\alpha\beta\gamma} = \partial_\alpha \Gamma^{\delta}_{\beta\gamma} - \partial_\beta \Gamma^{\delta}_{\alpha\gamma} + \Gamma^{\delta}_{\alpha\mu}\Gamma^{\mu}_{\beta\gamma} - \Gamma^{\delta}_{\beta\mu}\Gamma^{\mu}_{\alpha\gamma}.$$

Thus $R$ is expressed using second order derivatives of the metric $g$. Generally speaking, whenever a computation involves second order derivatives of $g$, one can expect to see $R$ appearing ; we will follow this path in many proofs of this chapter...

• The **Ricci tensor** is a trace taken on $R$ (because of the symmetries of $R$, there are not many traces to be taken)

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu}^{\ \ \ \ \alpha},$$

and the scalar curvature is the trace of the Ricci tensor $R = R^{\alpha}_{\alpha}$. Remark that, due to the symmetries of $R$, the Ricci tensor is symmetric

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta} = g^{\alpha\beta} R_{\nu\beta\mu\alpha} = g^{\beta\alpha} R_{\nu\beta\mu\alpha} = R_{\nu\mu}.$$

From the explicit formula for $R$ we easily get

$$R_{\mu\nu} = (1/2)\partial^\alpha \partial_\mu g_{\alpha\nu} + (1/2)\partial^\alpha \partial_\nu g_{\alpha\mu} - (1/2) g^{\alpha\beta} \partial^2_{\mu\nu} g_{\alpha\beta} - (1/2) g^{\alpha\beta} \partial^2_{\alpha\beta} g_{\mu\nu}$$

$$+ g^{\alpha\beta} g_{\gamma\delta} (\Gamma^{\gamma}_{\mu\beta} \Gamma^{\delta}_{\alpha\nu} - \Gamma^{\gamma}_{\mu\nu} \Gamma^{\delta}_{\alpha\beta}).$$

## 8.2 Optical functions and curvature

### 1. New normalization of the null frame

From now on, to better fit the reference paper [26], we are going to change slightly the choice of the null frame associated to the optical function $u$ (see 4.3). In this new normalization, we set $L' = -\nabla u$, and take, still noting $a = (\partial_t u)^{-1}$,

$$L = aL' = \partial_t + N,\ \underline{L} = \partial_t - N.$$

Thus, as usual,

$$< L, L >= 0, < \underline{L}, \underline{L} >= 0, < L, \underline{L} >= -2.$$

With these new definitions, the formula for the frame coefficients have to be slightly modified. We give only the results, the proofs being analogous to that of section 4.4.

**Theorem (Frame coefficients).** *The frame coefficients are given by the following formula*

$$D_a L = \chi_{ab} e_b - k_{aN} L,\ D_a \underline{L} = \underline{\chi}_{ab} e_b + k_{aN} \underline{L},$$

$$D_L L = -k_{NN} L,\ D_L \underline{L} = 2\underline{\eta}_a e_a + k_{NN} \underline{L},$$

$$D_{\underline{L}} L = 2\eta_a e_a + k_{NN} L,\ D_{\underline{L}} \underline{L} = 2\underline{\xi}_a e_a - k_{NN} \underline{L},$$

$$D_L e_a = \not{D}_L e_a + \underline{\eta}_a L,\ D_{\underline{L}} e_a = \not{D}_{\underline{L}} e_a + \eta_a \underline{L} + \underline{\xi}_a L,$$

$$D_b e_a = \not{D}_b e_a + (1/2)\chi_{ab} \underline{L} + (1/2)\underline{\chi}_{ab} L.$$

The formula expressing $\underline{\chi}$, $\underline{\xi}$, $\underline{\eta}$ in terms of $\chi$, $\xi$, $\eta$, given in 4.4 have also to be slightly modified and read now

$$\underline{\chi}_{ab} = -\chi_{ab} - 2k_{ab},$$

$$\underline{\xi}_a = -\eta_a + k_{aN},\ \underline{\eta}_a = -k_{aN}, \eta_a = e_a(a)/a + k_{aN}.$$

An important fact which will be useful later is the following.

**Lemma.** *With the new normalization, the coordinates of $L$ are bounded .*

This is clear for the $t$-coordinate. Since $N = -a\partial^i u \partial_i$, the $i$-coordinate of $L$ is $-g^{ij}\partial_j u/\partial_t u$ : taking into account the eikonal equation $(\partial_t u)^2 = g^{ij}\partial_i u \partial_j u$ and the assumptions on the metric, the claim is proved $\diamondsuit$.

### 2. Transport equation for $u$

Let $u$ be an optical function, that is $< \nabla u, \nabla u >= 0$.

**Theorem.** *The second order derivatives of* $u$ *are related to the curvature tensor through the transport equation*

$$D_L \nabla^2 u_{\alpha\beta} - a \nabla^2 u_{\alpha\gamma} \nabla^2 u_\beta^\gamma = a^{-1} R_{\beta L \alpha L}.$$

Using the definitions, we get

$$D_L \nabla^2 u_{\alpha\beta} = L(\nabla^2 u_{\alpha\beta}) - \nabla^2 u(D_L \partial_\alpha, \partial_\beta) - \nabla^2 u(\partial_\alpha, D_L \partial_\beta),$$

$$L(\nabla^2 u_{\alpha\beta}) = L < D_\alpha \nabla u, \partial_\beta >=< D_L D_\alpha \nabla u, \partial_\beta > + < D_\alpha \nabla u, D_L \partial_\beta > .$$

Since $D_L \nabla u = -a D_{\nabla u} \nabla u = 0$, using the symmetry of the Hessian, we obtain

$$D_L \nabla^2 u_{\alpha\beta} =< D_L D_\alpha \nabla u, \partial_\beta > - < D_\alpha D_L \nabla u, \partial_\beta > - < D_{[L,\partial_\alpha]} \nabla u, \partial_\beta > +$$

$$+ < D_\beta \nabla u, [L, \partial_\alpha] > - < D_\beta \nabla u, D_L \partial_\alpha > .$$

Taking into account $[L, \partial_\alpha] = D_L \partial_\alpha + D_\alpha(a \nabla u)$, the formula is proved. $\diamondsuit$

The above formula can be viewed as a system of differential equations on the unknowns $\nabla^2 u_{\alpha\beta}$, along the integral curves of $L$. If we assume given the components $R_{\alpha L \beta L}$ of the curvature tensor, we can use the above proposition to compute the components of $\nabla^2 u$, by solving the differential equations along $L$, with initial data either on $\{t = 0\}$ or on the $t$-axis. To understand why this is not the best strategy, one has to imagine that the metric $g$ is not smooth, and that we wish to pay the greatest attention to the regularity of the various objects at hand, counting the derivatives, etc. In this context (which will be discussed shortly in the last chapter), the components of $R$ have two derivatives less than $g$, and integrating along $L$ does not gain anything (except of course one derivative...along $L$ !).

In particular, consider the components $\chi_{ab} = -a \nabla^2 u_{ab}$ : it turns out that we have to split $\chi$ and $\underline{\chi}$ into their traces and their traceless parts

$$tr\,\chi = \chi_a^a, \; tr\,\underline{\chi} = \underline{\chi}_a^a,$$

$$\chi = \hat{\chi} + (1/2)(tr\,\chi) g, \; \underline{\chi} = \hat{\underline{\chi}} + (1/2)(tr\,\underline{\chi}) g.$$

The traces will be controlled by integration along $L$, while the traceless parts will be controlled through an elliptic system on the sphere foliation.

## 8.3 Transport equations

**Theorem (Transport equations).** *The quantities* $a$ *and* $tr\,\chi$ *satisfy*

$$La = -a k_{NN},$$

$$L(tr\,\chi) + (1/2)(tr\,\chi)^2 = -|\hat{\chi}|^2 - k_{NN} tr\,\chi - R_{LL}.$$

• We first prove the second formula. Since $\chi_{ab} = -a\nabla^2 u_{ab}$, $tr\,\chi = -a\nabla^2 u_a^a$. In order to obtain $L(tr\,\chi)$ from the transport equation on $u$, we observe that $L$ commutes with the partial trace

$$L(\nabla^2 u_a^a) = D_L \nabla^2 u_a^a.$$

In fact,

$$L(\nabla^2 u_a^a) = D_L \nabla^2 u_a^a + 2\nabla^2 u(D_L e_a, e_a).$$

Now, since $< D_L e_a, e_a >= 0$, $< D_L e_a, L >= 0$ and $< D_L e_1, e_2 >= - < D_L e_2, e_1 >$, we have for some coefficients $\alpha, \beta_a$,

$$D_L e_1 = \alpha e_2 + \beta_1 L,\ D_L e_2 = -\alpha e_1 + \beta_2 L.$$

Hence

$$\nabla^2 u(D_L e_1, e_1) + \nabla^2 u(D_L e_2, e_2) = \nabla^2 u(e_1, e_2)(\alpha - \alpha) + \beta_a \nabla^2 u(L, e_a).$$

Since $D_L \nabla u = 0$, $\nabla^2 u(L, e_a) = 0$ and the claim is proved. We thus obtain

$$L(tr\,\chi) = (La/a) tr\,\chi - a D_L \nabla^2 u_a^a.$$

From the transport equations on $u$, we immediately get

$$D_L \nabla^2 u_a^a = a\nabla^2 u_{a\gamma} \nabla^2 u_a^\gamma + a^{-1} \sum R_{aLaL}.$$

From the definition of the Ricci tensor $R$, using the symmetries of $R$,

$$R_{LL} = \sum R_{aLaL} - (1/2) R_{\underline{L}LLL} - (1/2) R_{LL\underline{L}L} = \sum R_{aLaL}.$$

As we already observed that $\nabla^2 u_{La} = 0$, the trace in the above formula is just

$$\nabla^2 u_{a\gamma} \nabla^2 u_a^\gamma = \sum \nabla^2 u_{ab} \nabla^2 u_{ab} = a^{-2} |\chi|^2.$$

Summarizing, the formula is proved, since $|\chi|^2 = |\hat{\chi}|^2 + (1/2)(tr\,\chi)^2$. ♢

• To prove the first formula, we observe that

$$< D_L L, \underline{L} >= -2La/a,$$

since $D_L L = D_L(-a\nabla u) = (La/a)L$. On the other hand, since $T$ and $N$ are orthogonal unit vectors with $D_T T = 0$,

$$< D_L L, \underline{L} >=< D_{T+N}(T+N), T-N >=< D_T T, T-N > +$$

$$+ < D_T N, T-N > + < D_N T, T-N > + < D_N N, T-N >= 2k_{NN}.$$

♢

The point of these formula is that they involve only $k$ (first order derivatives of $g$), $\hat{\chi}$ (which will be separately controlled later on by an elliptic system) and $R_{LL}$. We have to show now what is so special about $R_{LL}$ !

**Theorem (Special structure of $R_{LL}$).** *Let $z = L^\nu g^{\alpha\beta}\partial_\beta g_{\alpha\nu} - (1/2)g^{\alpha\beta}L(g_{\alpha\beta})$. Then*

$$R_{LL} = Lz - (1/2)L^\mu L^\nu \Box g_{\mu\nu} + E,$$

*where, for some constant $C$, $|E| \leq C|\partial g|^2$.*

The proof is by brute force, using the explicit formula given for $R_{\mu\nu}$. Observe first that the quadratic terms in $\Gamma$ in the formula for $R_{\mu\nu}$ can be put into $E$. Next,

$$g^{\alpha\beta}\partial^2_{\alpha\beta}g_{\mu\nu} = \Box g_{\mu\nu} + E',$$

where $E'$ can be put into $E$. We are left with the first three terms of the formula, which can be handled similarly, so we handle only the first one $g^{\alpha\beta}L^\mu L^\nu\partial^2_{\mu\beta}g_{\alpha\nu}$. We just write it

$$L(L^\nu g^{\alpha\beta}\partial_\beta g_{\alpha\nu}) - L(L^\nu)\partial^\alpha g_{\alpha\nu} - L^\nu L(g^{\alpha\beta})\partial_\beta g_{\alpha\nu},$$

and observe that the last term can be put into $E$, while the first one enters into $z$. It remains to examine $L(L^\nu)$ : since $L = -a\nabla u$, $L^\nu = -ag^{\nu\mu}\partial_\mu u$. Hence

$$L(L^\nu) = (La/a)L^\nu + L(g^{\nu\mu}) < L, \partial_\mu > + g^{\nu\mu} < L, D_L\partial_\mu > .$$

Taking into account that $La/a = -k_{NN}$, all three terms are linear combinations, with bounded coefficients, of first order derivatives of $g$. Thus the term $L(L^\nu)\partial^\alpha g_{\alpha\nu}$ can be put into $E$, and this finishes the proof. $\diamondsuit$

## 8.4 Elliptic Systems

These systems will control the traceless part of $\chi$ on one hand, and $\eta$ on the other hand. Recall that $\chi$ and $\eta$ are tensors on the spheres : we will thus consider the induced connexion $\not{D}$ on the spheres to take their derivatives.

Recall that, if $X$ or $Y$ are not tangent to the spheres, we have defined $\not{D}_X Y$ as the orthogonal projection of $D_X Y$ on the spheres ; we can extend this definition to tensors by the usual formula

$$XT(Y,Z) = (\not{D}_X T)(Y,Z) + T(\not{D}_X Y, Z) + T(Y, \not{D}_X Z).$$

This extension will be used in the proofs.

**1.** We first establish the system on $\hat{\chi}$.

**Theorem (Codazzi equation).** *The tensor* $\hat{\chi}$ *satifies*

$$div\,\hat{\chi}_a + \hat{\chi}_{ab}k_{bN} = (1/2)(e_a(tr\,\chi) + k_{aN}tr\,\chi) + R_{bLba}.$$

Recall that $\hat{\chi}$ is a symmetric 2-tensor on the sphere foliation, and *div* means the trace with respect to one argument

$$div\,\hat{\chi}_a = \not{D}_b\hat{\chi}_a^b.$$

One should be careful, as usual, that we consider first $\not{D}_b\hat{\chi}$, and then take the $ab$ component and sum. This is different from taking the divergence of the 1-form $\hat{\chi}(e_a, .)$ !

• To prove the Codazzi equation, we first prove

$$\not{D}_c\chi_{ab} - \not{D}_a\chi_{bc} = R_{bLca} - k_{cN}\chi_{ab} + k_{aN}\chi_{cb}.$$

This follows smoothly from the definitions ; in fact,

$$\not{D}_c\chi_{ab} = e_c(\chi_{ab}) - \chi(\not{D}_c e_a, e_b) - \chi(e_a, \not{D}_c e_b) =$$

$$= < D_c D_a L, e_b > + < D_a L, D_c e_b > - < D_b L, \not{D}_c e_a > - < D_a L, \not{D}_c e_b > .$$

Since a second order derivative of $L$ appears, we wish to introduce the missing terms to see a curvature term. Thus we write

$$\not{D}_c\chi_{ab} = R_{bLca} + < D_a D_c L, e_b > + < D_{[e_c,e_a]}L, e_b > +$$

$$+ < D_a L, D_c e_b - \not{D}_c e_b > - < D_b L, \not{D}_c e_a > .$$

We transform terms according to the formula

$$< D_a D_c L, e_b > = e_a(\chi_{bc}) - < D_c L, D_a e_b >,$$

$$< D_{[e_c,e_a]}L, e_b > = < D_b L, [e_c, e_a] > = < D_b L, \not{D}_c e_a - \not{D}_a e_c >,$$

and we get

$$\not{D}_c\chi_{ab} = R_{bLca} + I + II,$$

$$I = e_a(\chi_{bc}) - < D_b L, \not{D}_a e_c > - < D_c L, \not{D}_a e_b >,$$

$$II = < D_a L, D_c e_b - \not{D}_c e_b > - < D_c L, D_a e_b - \not{D}_a e_b > .$$

Since

$$\not{D}_a\chi_{bc} = e_a(\chi_{bc}) - < D_c L, \not{D}_a e_b > - < D_b L, \not{D}_a e_c > = I,$$

we are left to deal with the terms II of the second line.

In section 4.2, in the case of a submanifold of codimension one, we introduced the second form $k$, and proved the formula $D_X Y = \not{D}_X Y + k(X, Y)N$. Here, we are dealing with the submanifold $S_{t,u}$ of codimension two, for which $\chi$ and $\underline{\chi}$ act as a pair of second forms. Just as in section 4.2, we can prove the formula

$$D_a e_b - \not{D}_a e_b = (1/2)\chi_{ab}\underline{L} + (1/2)\underline{\chi}_{ab}L.$$

In fact,

$$< D_a e_b, L >= e_a < e_b, L > - < e_b, D_a L >= -\chi_{ab}$$

gives the coefficient of $\underline{L}$, and similarly for $L$. Then

$$II = (1/2) < D_a L, \chi_{bc}\underline{L} > -(1/2) < D_c L, \chi_{ab}\underline{L} > .$$

Since

$$< D_a L, \underline{L} >=< D_a T, T > - < D_a T, N > +$$
$$+ < D_a N, T > - < D_a N, N >= 2k_{aN},$$

we find

$$II = k_{aN}\chi_{bc} - k_{cN}\chi_{ab}.$$

• In the previously obtained formula, lift the index $b$ and take $c = b$ to obtain

$$\not{D}_b \chi_a^b - \not{D}_a \chi_b^b = R_{bLba} - k_{bN}\chi_a^b + k_{aN} tr\,\chi.$$

We finally have to split $\chi$ in this formula. First,

$$e_a(tr\,\chi) = e_a(\chi_b^b) = \not{D}_a \chi_b^b + 2\chi(e_b, \not{D}_a e_b).$$

Since $< \not{D}_a e_b, e_b >= 0$ and $< \not{D}_1 e_1, e_2 >= - < \not{D}_1 e_2, e_1 >$, we find for instance

$$\chi(e_b, \not{D}_1 e_b) =< \not{D}_1 e_1, e_2 > (\chi(e_1, e_2) - \chi(e_2, e_1)) = 0,$$

and similarly for $\not{D}_2$. Next,

$$\not{D}_b \chi = \not{D}_b \hat{\chi} + (1/2)e_b(tr\,\chi)g,$$
$$\not{D}_b \chi_a^b = \not{D}_b \hat{\chi}_a^b + (1/2)e_a(tr\,\chi),$$

which yields the formula. ♢

**2.** We have to explain why this system is called "elliptic". Note that $\hat{\chi}$ depends only on two functions $\hat{\chi}_{11}$ and $\hat{\chi}_{12}$, and that we have two equations. More precisely, we know from the definition that

$$\not{D}_c \hat{\chi}_{ab} = e_c(\hat{\chi}_{ab}) + \ldots,$$

where the dots stand for zero order terms in $\hat{\chi}$. Hence the Codazzi equations are

$$e_1(\hat{\chi}_{11}) + e_2(\hat{\chi}_{12}) + \ldots = \ldots, \; e_1(\hat{\chi}_{12}) + e_2(\hat{\chi}_{22}) + \ldots = \ldots.$$

Since $\hat{\chi}$ is symmetric and traceless, this can be written as a first order $2 \times 2$-system on the unknowns $\hat{\chi}_{11}$, $\hat{\chi}_{12}$, with matrix

$$\begin{bmatrix} e_1 & e_2 \\ -e_2 & e_1 \end{bmatrix}$$

The operator matrix has for principal symbol the principal symbol of $e_1^2 + e_2^2$, that is the principal symbol of the Laplace operator on the spheres.

**3.** We turn now to the system on $\eta$.

**Theorem (div-curl system on $\eta$).** *The 1-form $\eta$ satisfies the following system*

$$div\,\eta = (1/2)\underline{L}(tr\,\chi) - (1/2)k_{NN}tr\,\chi - |\eta|^2 + (1/2)\hat{\chi}_{ab}\underline{\hat{\chi}}_{ab} + (3/4)(tr\,\chi)(tr\,\underline{\chi}) - (1/2)R^a_{\ L\underline{L}a},$$

$$curl\,\eta_{ab} = (1/2)(\hat{\chi}_{bc}\underline{\hat{\chi}}_{ac} - \hat{\chi}_{ac}\underline{\hat{\chi}}_{bc}) + (1/2)(-R_{aL\underline{L}b} + R_{bL\underline{L}a}).$$

• We first prove the formula

$$\not{D}_{\underline{L}}\chi_{ab} = 2\ \not{D}_a\eta_b + \chi_{ab}k_{NN} + 2\eta_a\eta_b - \underline{\chi}_{ac}\chi_{cb} + R_{bL\underline{L}a}.$$

Using the definitions,

$$\not{D}_{\underline{L}}\chi_{ab} = \underline{L}(\chi_{ab}) - \chi(\not{D}_{\underline{L}}e_a, e_b) - \chi(e_a, \not{D}_{\underline{L}}e_b) =$$

$$= < D_{\underline{L}}D_aL, e_b > + < D_aL, D_{\underline{L}}e_b > -\chi(\not{D}_{\underline{L}}e_a, e_b) - \chi(e_a, \not{D}_{\underline{L}}e_b).$$

Forcing the curvature term into the formula by adding and substracting terms, we get

$$\not{D}_{\underline{L}}\chi_{ab} = R_{bL\underline{L}a} + < D_aD_{\underline{L}}L, e_b > + < D_{[\underline{L},e_a]}L, e_b > +$$

$$+ < D_aL, D_{\underline{L}}e_b > -\chi(\not{D}_{\underline{L}}e_a, e_b) - \chi(e_a, \not{D}_{\underline{L}}e_b) =$$

$$= R_{bL\underline{L}a} + 2e_a(\eta_b) - < D_{\underline{L}}L, D_ae_b > +$$

$$+ < D_{[\underline{L},e_a]}L, e_b > + < D_aL, D_{\underline{L}}e_b > -\chi(\not{D}_{\underline{L}}e_a, e_b) - \chi(e_a, \not{D}_{\underline{L}}e_b).$$

The third, fourth and fifth term are handled by brute force, using the formula above for the frame coefficients :

$$< D_{\underline{L}}L, D_ae_b > = 2\eta(\not{D}_ae_b) - \chi_{ab}k_{NN},$$

$$[\underline{L}, e_a] = \not{D}_{\underline{L}}e_a - \underline{\chi}_{ac}e_c + (\eta_a - k_{aN})(\underline{L} - L),$$

$$< D_aL, D_{\underline{L}}e_b > = \chi(e_a, \not{D}_{\underline{L}}e_b) + 2k_{aN}\eta_b.$$

Substituing into the above formula, we see that the $\chi$ terms cancel out, while

$$2e_a(\eta_b) - 2\eta(\not{D}_ae_b) = 2\ \not{D}_a\eta_b.$$

This yields the formula.

• We use the above formula in two ways : first, we take $b = a$ and sum to obtain

$$\not{D}_{\underline{L}}\chi_a^a = 2div\,\eta + k_{NN}tr\,\chi + 2|\eta|^2 - \chi_{ab}\underline{\chi}_{ab} + R^a_{\ L\underline{L}a}.$$

As before, noting that $\not{D}_{\underline{L}}\chi_a^a = \underline{L}(tr\,\chi)$, and splitting $\chi$ and $\underline{\chi}$, we obtain the first formula of the theorem.

For the second, we substract the formula with $ba$ from the formula with $ab$ we have established, to get

$$2curl\,\eta_{ab} = \chi_{bc}\underline{\chi}_{ac} - \chi_{ac}\underline{\chi}_{bc} - R_{aL\underline{L}b} + R_{bL\underline{L}a}.$$

Splitting $\chi$ and $\underline{\chi}$ as usual, the $\chi, \underline{\chi}$ terms yield the same terms with $\chi$ replace by $\hat{\chi}$ and $\underline{\chi}$ by $\hat{\underline{\chi}}$, the other terms cancelling out by symmetry. $\diamondsuit$

Remark that the system on $\eta$ is also elliptic, since its matrix operator has the same principal symbol as the Laplace operator on the spheres.

## 8.5 Mixed transport-elliptic systems

If we examine the system on $tr\,\chi$, $\hat{\chi}$, $\eta$, we observe the presence of the term $\underline{L}(tr\,\chi)$ in the expression for $div\,\eta$, a term for which we have no control. Since we already know $L(tr\,\chi)$, we can compute $L(div\,\eta)$ and write $L(\underline{L}tr\,\chi) = [L, \underline{L}]tr\,\chi + \underline{L}(Ltr\,\chi)$. The result is summarized in the following theorem.

**Theorem.** *Set $\mu_1 = \underline{L}(tr\,\chi) - (1/2)(tr\,\chi)^2$. The quantity $\mu_1$ satisfies the following transport equation:*

$$L\mu_1 + (tr\chi)\mu_1 = -\underline{L}(R_{LL}) - 2\ \not{D}_{\underline{L}}\hat{\chi}_{ab}\hat{\chi}^{ab} + 2(\underline{\eta}_a - \eta_a)e_a(tr\,\chi) -$$

$$-\underline{L}(k_{NN})(tr\,\chi) - (k_{NN} + tr\,\chi)L(tr\,\chi) - (1/2)(tr\,\chi)^3,$$

*where the quantity $\not{D}_{\underline{L}}\hat{\chi}$ is given through*

$$\not{D}_{\underline{L}}\hat{\chi}_{ab} = 2\ \not{D}_a\eta_b - div\,\eta\delta_{ab} + k_{NN}\hat{\chi}_{ab} + 2(\eta_a\eta_b - |\eta|^2\delta_{ab}) -$$

$$-(1/2)(tr\,\underline{\chi})\hat{\chi}_{ab} - (1/2)(tr\,\chi)\hat{\underline{\chi}}_{ab} + R_{aL\underline{L}b}.$$

The formula about $\not{D}_{\underline{L}}\hat{\chi}$ follows from the formula in the proof of the theorem about $\eta$ just by splitting $\chi$ :

$$\not{D}_{\underline{L}}\chi = \not{D}_{\underline{L}}\hat{\chi} + (1/2)(\underline{L}(tr\,\chi)g + (tr\,\chi)\ \not{D}_{\underline{L}}g).$$

Proving as usual $\not{D}_{\underline{L}}g = 0$, and using the expression of $\underline{L}(tr\,\chi)$ in terms of $div\,\eta$ given in the theorem, we obtain the formula.

To prove the transport formula, we check first

$$L\mu_1 + (tr\,\chi)\mu_1 = L\underline{L}(tr\,\chi) + \underline{L}(tr\,\chi)(tr\,\chi) - L(tr\,\chi)(tr\,\chi) - (1/2)(tr\,\chi)^3.$$

On the other hand, using the formula for $L(tr\,\chi)$, we get

$$\underline{L}(L(tr\,\chi)) = -\underline{L}(R_{LL}) - 2\ \not{D}_{\underline{L}}\hat{\chi}_{ab}\hat{\chi}^{ab} -$$

$$-(tr\,\chi)[\underline{L}(tr\,\chi) + \underline{L}(k_{NN}] - k_{NN}\underline{L}(tr\,\chi).$$

Using the formula

$$[L, \underline{L}] = 2(\underline{\eta}_a - \eta_a)e_a + k_{NN}(\underline{L} - L),$$

we finish the proof. ♢

The way one uses these formula to obtain an actual control of all frame coefficients is far from being obvious :

i) First, we note the presence of terms $e_a(tr\,\chi)$ in the transport equation for $\mu_1$ : hence one has to establish also a transport equation for $\not{D}(tr\,\chi)$. The system of the transport equations on $tr\,\chi$, $e_a(tr\,\chi)$ and the elliptic Codazzi equations on $\hat{\chi}$ and $\not{D}\hat{\chi}$ is closed.

ii) To control $\eta$, we use the transport equation on $\mu_1$ along with the elliptic system for $\eta$, which form a closed system on $\eta$, $\mu_1$ and $\not{D}\eta$.

We refer to [26] for the actual implementation of this strategy.

# Chapter 9

# Applications to some Quasilinear Hyperbolic problems

As explained in the Introduction, it is not possible here to give complete proofs of delicate results, some of them being several hundred pages long. We just want to point out how the methods and ideas explained in the preceding chapters enter in an essential way in the proofs of these results. For each example, we give a very brief sketch of the method of proof ; we explain what is the frame or the optical functions used in the work, how it is constructed, and why this frame is supposed to be a good frame. The examples we have chosen to discuss do not of course represent the whole literature on the subject, but they seem to us recent and representative enough. In the following list, to facilitate an overview, we characterize the method in one line :

i) Global existence for small solutions of quasilinear wave equations

$$-\partial_t^2\phi + \Delta\phi + \sum g^{ij}(\partial\phi)\partial_{ij}^2\phi = 0$$

satisfying the null condition. The proof is by commuting standard Lorentz fields to get decay estimates.

ii) Global existence for small solutions of quasilinear wave equations

$$g^{\alpha\beta}(\phi)\partial_{\alpha\beta}^2\phi = 0.$$

Though the first proof used modified Lorentz fields, a simpler new proof uses only the standard Lorentz fields to get decay estimates.

iii) Low regularity well-posedness for quasilinear wave equations

$$-\partial_t^2\phi + \Delta\phi + \sum g^{ij}(\phi)\partial_{ij}^2\phi = N(\phi, \partial\phi).$$

The proof uses the full machinery of chapter 8 to obtain *decay* of solutions of some linear wave equation $\Box_{h_\lambda}$. Here, $h_\lambda$ is a smoothed rescaled version of $g$.

iv) Stability of Minkowski spacetime (first version) : the full machinery of chapter 8 is used to prove decay for the solutions of the Bianchi equations, a first order system on the curvature tensor $R$.

v) The $L^2$ conjecture for Einstein equations : the machinery of chapter 8 is used to obtain a control of the geometry and of the solutions of Bianchi equations in a context of very low regularity.

vi) Stability of Minkowski spacetime (second version) : just as in examples i) and ii), standard Lorentz fields are used to get decay estimates.

vii) Formation of Black Holes : the full machinery using the sphere foliation associated to two optical functions is used.

As shown by examples i), ii) and vi), it turns out that, surprisingly enough, for some nonlinear problems, one can ignore the geometry of the linearized operator and work with the standard Lorentz fields. For example ii), and even more for example vi), this came as a surprise. This is due to the specific nonlinear structure of the equations, and to the fact that we are dealing with small solutions.

In constrast with these examples, examples iii), iv), v) and vii) show that the geometric machinery explained in this book can be used in many different contexts

a) to prove *decay estimates* and global existence of solutions,

b) to prove low regularity results, counting carefully derivatives,

c) to prove formation of singularities.

## 9.1 Quasilinear Wave Equations satisfying the Null Condition

This is a classical result (see [13], [21] or [22] for instance). We present here the sketch of a new proof based on the results of chapter 6.

Consider the Cauchy problem with small data for a quasilinear wave equation

$$-\partial_t^2\phi + \Delta\phi + \sum_{1\leq i,j\leq 3} g^{ij}(\partial\phi)\partial^2_{ij}\phi = 0,\ \phi(x,0) = \epsilon\phi_0(x),\ (\partial_t\phi)(x,0) = \epsilon\phi_1(x).$$

We assume $\phi_i \in C_0^\infty$ $(i = 1, 2)$ ; for simplicity (cubic terms playing no role), we take

$$g^{ij}(\partial\phi) = \sum_{1\le k\le 3} g^{ijk}\partial_k\phi.$$

The equation is said to satisfy the *null condition* if, for all $\xi \in \mathbf{R}^3$,

$$g^{ijk}\xi_i\xi_j\xi_k = 0.$$

This condition can also be interpreted by saying that the function $u = r - t$ is closer to be an optical function than in the general case of a quasilinear wave equation. In fact,

$$< \nabla u, \nabla u >= g^{ij}(\partial\phi)\omega_i\omega_j = g^{ijk}\omega_i\omega_j(\partial_k\phi) = g^{ijk}\omega_i\omega_j(\partial_k\phi - \omega_k\partial_r\phi) = O(t^{-1}|Z\phi|).$$

This suggests that we can work with the standard frame, as will be seen in the proof of the following classical result.

**Theorem.** *There exists $\epsilon_0 >$ such that, for $\epsilon \le \epsilon_0$, the problem admits a unique global $C^\infty$ solution $\phi$.*

• We explain here two facts about the null condition. We first show how the null condition is related to the Lorentz fields $Z$ : recall the formula

$$\partial_i = \omega_i\partial_r - (\omega \wedge (R/r))_i,\ R/r = t^{-1}\omega \wedge H.$$

If, in the nonlinear expression $\sum g^{ijk}\partial_k u\partial_{ij}^2 v$, we replace $\partial_i$ using these formula, we obtain the pointwise estimate (see [21], lemma 6.6.4)

$$|\sum g^{ijk}\partial_k u\partial_{ij}^2 v| \le C(1+t)^{-1}(|Zu||\partial^2 v| + |\partial u||Z\partial v|),$$

since the main term

$$(\sum g^{ijk}\omega_i\omega_j\omega_k)(\partial_r u)(\partial_r^2 v)$$

vanishes.

The following algebraic fact (see Lemma 6.6.5 of [21]) is also useful : for all Lorentz fields $Z$,

$$Z(\sum g^{ijk}\partial_k\phi\partial_{ij}^2\phi) = \sum g^{ijk}\partial_k Z\phi\partial_{ij}^2\phi + \sum g^{ijk}\partial_k\phi\partial_{ij}^2 Z\phi + \sum \tilde{g}^{ijk}\partial_k\phi\partial_{ij}^2\phi,$$

where the new sum with the constant coefficients $\tilde{g}^{ijk}$ satisfies again the null condition.

• The idea which is common to the solutions of almost all nonlinear Cauchy problems is that of *induction on time* :

*Assume that the solution exists and belongs to $C^\infty$ for $0 \le t < T$ ; if moreover, for some $C$ and all $t < T$,*

$$|\partial^2\phi(x,t)| \le C,$$

*then for some $\eta > 0$, the solution exists and belongs to $C^\infty$ for $0 \le t < T + \eta$.*

This is the consequence of the standard local in time existence theorem (see for instance [36]). Using all Lorentz fields $Z$, we define the higher order energy $E_N$ from the standard energy $E$ by

$$E_N(t) = \sum_{k \le N} E(Z^k \phi)(t) = (1/2) \sum_{k \le N} \int |\partial Z^k \phi|^2 (x,t) dx.$$

With $C_0 > 0$ some constant to be fixed later, we make the following induction hypothesis.

**(Induction Hypothesis.)** *Assume that the solution $\phi$ exists for $0 \le t < T$ and satisfies there, for some big $N$ to be chosen later,*

$$E_N(t) \le C_0^2 \epsilon^2.$$

• Using Klainerman inequality, the induction hypothesis implies

$$|Z^l \partial \phi|(x,t) \le C C_0 \epsilon (1+t)^{-1} < r - t >^{-1/2},\ l \le N-2.$$

The idea is now to commute with the equation products $Z^l$ of standard Lorentz fields ($l \le N$). Defining the linearized operator $P$ by

$$P \equiv \Box + g^{ijk} \partial_k \phi \partial_{ij}^2 + g^{ijk} \partial_{ij}^2 \phi \partial_k$$

and using repeatedly the algebraic fact above, we obtain an equation

$$P Z^l \phi \equiv \Box Z^l \phi + \sum g^{ijk} \partial_k \phi \partial_{ij}^2 Z^l \phi +$$

$$+ \sum g^{ijk} \partial_k Z^l \phi \partial_{ij}^2 \phi = \sum_{p+q \le l-1} h_{pq}^{ijk} \partial_k Z^p \phi \partial_{ij}^2 Z^q \phi \equiv H,$$

where, for each $(p,q)$ and all $\xi \in \mathbf{R}^3$

$$\sum h_{pq}^{ijk} \xi_i \xi_j \xi_k = 0.$$

If $N \ge 3$, then $(l-1)/2 \le N-2$, and we can use the induction assumption to bound the factor in $H$ containing less $Z$-fields than the other, thus obtaining, for some integrable $h(t)$,

$$|H| \le C \epsilon h(t) \sum_{p \le l} |\partial Z^p \phi|.$$

• The *key* is this : There exists a standard energy inequality for $P$ with an integrable amplification factor. This inequality is very similar to that we proved in 6.2, 2. Here,

$$c = 1 + \sum g^{ijk} \omega_i \omega_j \partial_k \phi = 1 + \sum g^{ijk} \omega_i \omega_j (\partial_k \phi - \omega_k \partial_r \phi),$$

hence $|c-1| \leq C\epsilon(1+t)^{-2} < r-t >^{1/2}$. Similarly, $|\partial_t c| \leq C\epsilon(1+t)^{-2} < r-t >^{-1/2}$. As a consequence, the amplification factor is integrable, and its presence appears in the inequality only through a factor $e^{C_1\epsilon}$, $C_1$ depending of course of $C_0$.

Using the energy inequality for $P$, we obtain finally the estimate

$$E_N(t) \leq CE_N(0)e^{C_2\epsilon},$$

where $C$ is independent of $C_0$ while $C_2$ depends on $C_0$.

To finish the proof, it is enough to choose first $C_0^2 = 2CE_N(0)/\epsilon^2$, then $\epsilon$ small enough to obtain

$$E_N(t) \leq (2/3)C_0^2\epsilon^2.$$

This shows that the solution $\phi$ is global. ♢

## 9.2 Quasilinear Wave Equations

We consider the Cauchy problem for the quasilinear wave equation

$$g^{\alpha\beta}(\phi)\partial^2_{\alpha\beta}\phi = 0,\ \phi(x,0) = \epsilon\phi_0(x),\ (\partial_t\phi)(x,0) = \epsilon\phi_1(x).$$

The coefficients $g^{\alpha\beta}(s)$ are given (smooth enough) functions of one real variable $s$, with

$$g^{\alpha\beta}(0) = m^{\alpha\beta},$$

$m$ being here the Minkowsksi metric, that is, $m^{\alpha\beta}\partial^2_{\alpha\beta} = -\partial_t^2 + \Delta_x$. The data $\phi_0$, $\phi_1$ are fixed functions in $C_0^\infty$. The general result is the following.

**Theorem ([5], [34]).** *There exists $\epsilon_0 > 0$ such that, if $\epsilon \leq \epsilon_0$, there is a global $C^\infty$ smooth solution to the Cauchy problem.*

Remark that it makes a big difference whether the coefficients $g$ depend on $\phi$ or, as in the preceding example, on $\nabla\phi$ ; for instance, consider the model equation

$$-\partial_t^2\phi + c^2(\partial_t\phi)\Delta\phi = 0.$$

Taking the $t$-derivative and setting $\psi = \partial_t\phi$, we get

$$-\partial_t^2\psi + c^2(\psi)\Delta\psi = -2(c'/c)(\psi)(\partial_t\psi)^2.$$

In other words, the left-hand side of the equation is of the form considered in this section, but there is a *source term* in the right-hand side, which makes even the small solutions blowup in finite time (see [3]). The case of Einstein equations in "harmonic coordinates" is a system of such equations, for which the source terms display some sort of null condition : we will discuss it later in this chapter.

The theorem was first proved in [5] in the special case

$$-\partial_t^2\phi + c^2(\phi)\Delta\phi = 0,$$

using *modified Lorentz fields*. However, Lindblad [34] gave recently a simpler proof, the geometrical aspects of which we discuss now. The starting point of the proof of [34] is the following *bet*, which is far from being obvious :

*One can use the standard Lorentz fields $Z$ and commute them with the equation.*

Assuming this is true, it means that the good derivatives of $\phi$ will be the standard ones $L\phi = \partial_t\phi + \partial_r\phi$, $(R/r)\phi$, as explained in 6.1. This example shows that, for a given problem, it is not clear beforehand how to choose the geometry of the relevant fields. The induction hypothesis reads, for some $0 < \delta < 1$,

$$E_N(\phi) \leq 16N\epsilon^2(1+t)^\delta.$$

• One proceeds to express the linear operator

$$\tilde{\Box} \equiv g^{\alpha\beta}\partial^2_{\alpha\beta}$$

using the standard derivatives $L$, $\underline{L}$, $R/r$. One should be careful that we do not consider the metric $g_{\alpha\beta}$ (inverse matrix of $g^{\alpha\beta}$), but consider $g^{\alpha\beta}$ just as a symmetric 2-tensor on the background manifold $\mathbf{R}^4$ with the flat Minkowski metric $m$. In particular, we define

$$g_{\alpha\beta} = m_{\alpha\alpha'}m_{\beta\beta'}g^{\alpha'\beta'}.$$

To express $g^{\alpha\beta}$ in terms of the coordinates of the fields $L$, $\underline{L}$, $e_a$, we compute the double trace

$$g^{\alpha\beta} = m^{\alpha\alpha'}m^{\beta\beta'}g_{\alpha'\beta'}$$

in the standard null frame $(e_1, e_2, \underline{L}, L)$. We thus get

$$g^{\alpha\beta} = (1/4)g_{LL}m^\alpha_{\underline{L}}m^\beta_{\underline{L}} + (1/4)g_{\underline{L}\underline{L}}m^\alpha_L m^\beta_L +$$

$$+(1/4)g_{L\underline{L}}(m^\alpha_L m^\beta_{\underline{L}} + m^\alpha_{\underline{L}}m^\beta_L) - (1/2)g_{aL}(m^\alpha_{\underline{L}}m^\beta_a + m^\alpha_a m^\beta_{\underline{L}}) -$$

$$-(1/2)g_{\underline{L}a}(m^\alpha_L m^\beta_a + m^\alpha_a m^\beta_L) + g_{ab}m^{\alpha a}m^{\beta b}.$$

Since, for any field $X$, $m^\alpha_X = X^\alpha$, we obtain

$$\tilde{\Box} = (1/2)g_{L\underline{L}}L^\alpha\underline{L}^\beta\partial^2_{\alpha\beta} + (1/4)g_{LL}\underline{L}^\alpha\underline{L}^\beta\partial^2_{\alpha\beta} - g_{aL}\underline{L}^\alpha e^\beta_a\partial^2_{\alpha\beta} + \gamma^{\alpha\beta}\partial^2_{\alpha\beta},$$

with

$$\gamma^{\alpha\beta}\partial^2_{\alpha\beta} = -g_{a\underline{L}}L^\alpha e^\beta_a\partial^2_{\alpha\beta} + (1/4)g_{\underline{L}\underline{L}}L^\alpha L^\beta\partial^2_{\alpha\beta} + g_{ab}e^\alpha_a e^\beta_b\partial^2_{\alpha\beta}.$$

In other words, $\gamma^{\alpha\beta}\partial^2_{\alpha\beta}$ is the part of $\tilde{\Box}$ which is expressed with *two* good derivatives. Setting

$$L_1 = -(1/2)g_{L\underline{L}}L - (1/4)g_{LL}\underline{L} + g_{aL}e_a,$$

we obtain
$$\tilde{\Box} = -L_1^\alpha \underline{L}^\beta \partial^2_{\alpha\beta} + \gamma^{\alpha\beta}\partial^2_{\alpha\beta}.$$

• After some rough estimates, it turns out that one can discard a part of $L_1$ and define

$$L_2 = L - (1/4)g_{LL}\underline{L}$$

as a substitute for the standard $L$ in the transport equations. One also introduces, as a substitute for the optical function $u = r - t$, the function $\rho$ defined by

$$|r-t| \geq t/2 \Rightarrow \rho(x,t) = r - t,\ |r-t| \leq t/2 \Rightarrow L_2\rho = 0.$$

Introducing the coordinates $q = r - t, p = r + t$, the key step of the proof is to obtain, by integration along $L_2$, for some $0 < \nu < 1$, the estimates

$$|\nabla\phi| \leq C\epsilon(1+t)^{-1}(1+|\rho|)^{-\nu},\ |\nabla^2\phi| \leq C\epsilon(1+t)^{-1}(1+|\rho|)^{-1-\nu}|\partial_q\rho|.$$

• These estimates once obtained, the rest of the proof is by commuting products $Z^k$ to the equation, a procedure which requires serveral delicate arguments. As a result, one obtains, for some constant $C$,
$$E_N(t) \leq 8N\epsilon^2(1+t)^{C\epsilon}.$$
Taking $C\epsilon_0 \leq \delta$ finishes the proof by induction. ♢

## 9.3 Low Regularity results for Quasilinear Wave Equations

This is the paper [25] we used in chapter 8. The problem is the well-posedness of the (local) Cauchy problem with non smooth data

$$-\partial_t^2\phi + g^{ij}(\phi)\partial^2_{ij}\phi = N(\phi, \partial\phi),\ \phi(x,0) = \phi_0(x),\ (\partial_t\phi)(x,0) = \phi_1(x).$$

We assume $(\phi_0,\ \phi_1) \in H^s \times H^{s-1}$, and $N$ quadratic in $\partial\phi$. The result of [] (under some technical assumptions that we skip) is the following.

**Theorem.** *The Cauchy problem has a unique local solution for* $s > s_c = 2 + (2-\sqrt{3})/2$.

Let us recall that standard methods give the well-posedness for $s > 5/2$ (see [36] for instance). In contrast with the first two examples above, this is an example where one does not use the standard Lorentz fields $Z$, but develops the specific geometry of the problem.

• The first step of the proof, due to Bahouri and Chemin [10], [11] is to reduce this problem to the problem of the *time decay* of solutions of some linear wave equation. This linear wave equation is associated to a (split) metric $h_\lambda$, depending on some parameter $\lambda$, which is a smoothed rescaled version of $g$. The precise behavior of $h$ and its derivatives with respect to $\lambda$ reflects the smoothness assumptions on $g$. In the original approach of [10], the Strichartz type estimate was proved using parametrices ; the "vector field" approach of [26] is a refinement of an original idea of Klainerman [24].

• According to the reduction step above, the authors deal with the linear operator $\Box_{h_\lambda}$. They define a "canonical" optical function $u$ as being $t$ on the time axis and "having forward light cones with vertices on the time axis" as level surfaces. This refers to the construction explained in 3.2. There seems to be no special reason for this choice, except its natural character ; the normalization on the time axis reflects the will of imitating the flat case. The full machinery of chapter 8 is then developed, along with the use of conformal inequalities as explained in chapter 7, to prove the required *decay* estimates. Since we have presented this material in this book, we refer to [26] for the actual implementation.

## 9.4 Stability of Minkowski space-time (first version)

We refer here to the book by Klainerman and Nicolò "The Evolution Problem in General Relativity" [25], the previous book by Christodoulou and Klainerman [17] being more difficult to access. The goal is to solve Einstein equations with initial data close to the flat Minkowski metric. More precisely, we look for a metric $g$, close to the Minkowski metric, for which the Ricci tensor $R$ is identically zero (these are the simplest Einstein vacuum equations), and which extends for all $t$ the Cauchy data given on $\{t = 0\}$ (in a sense which has to be made precise).

There are many ideas in this long work, but two of them seem especially relevant in the context of this booklet :

*i) The authors never use the time variable. In the proof, they construct two optical functions $u$ and $\underline{u}$ (substitutes for the usual $u = t - r$ and $\underline{u} = r + t$ of the flat case), and use the frame associated to these two functions,*

*ii) The authors do not deal with the wave equation $\Box$, but with the Bianchi equations.*

We postponed the discussion of this case until now because of its complexity, though it presents some features analogous to the case of Maxwell equations.

**1. Bianchi equations**

The Bianchi equations (or second Bianchi identity ) are

$$D_{[\lambda}R_{\gamma\delta]\alpha\beta} = 0.$$

The bracket here means that we take the sum on the circular permutation of the indices. The idea is of course that these equations control directly the second order derivatives of

$g$ through $R$ : one has better chances to recover some induction hypothesis on $g$ without loosing derivatives...The drawback is that these equations are more difficult to handle than a wave equation, or even the Maxwell system. The general character of this strategy, which Christodoulou calls the “first method”, is sketched in the prologue of [14], refering to the book [15].

• If the metric $g$ satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$, the curvature tensor $R$ is traceless, by definition. Since we want to handle the Bianchi equations by the method of energy inequalities and commuting fields, we have to define a concept containing the curvature tensor $R$ and some of its Lie derivatives. Hence, generally, we define a Weyl field as a traceless 4-tensor $W$ with the symmetries of the curvature tensor

$$W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma} = W_{\gamma\delta\alpha\beta},$$

$$W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0,$$
$$g^{\alpha\gamma} W_{\alpha\beta\gamma\delta} = 0.$$

The tensor $W$ is said to satisfy the Bianchi equations if

$$D_{[\lambda} W_{\gamma\delta]\alpha\beta} = 0.$$

• Just as we did for Maxwell equations, one can define a dual tensor $*W$ by

$$*W_{\alpha\beta\gamma\delta} = (1/2)\epsilon_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta}.$$

Here, $\epsilon$ is the volume form, and $*(*W) = -W$ as in the case of Maxwell equations. As in chapter 5, there exists an energy machinery to prove energy inequalities for the Bianchi equations, that we explain here without proofs. We define the energy-momentum tensor (called here the Bel-Robinson tensor)

$$Q_{\alpha\beta\gamma\delta} = W_{\alpha\rho\gamma\sigma} W_{\beta\ \delta}^{\ \rho\ \sigma} + *W_{\alpha\rho\gamma\sigma} * W_{\beta\ \delta}^{\ \rho\ \sigma}.$$

If $W$ is a Weyl field satisfying the Bianchi equations, then

$$D^\alpha Q_{\alpha\beta\gamma\delta} = 0.$$

This is of course similar to the formula proved in chapter 5. Moreover, $Q$ enjoyes the *positiviy property* : if $X, Y, Z, T$ are non-spacelike future oriented fields,

$$Q(X, Y, Z, T) \geq 0.$$

To prove an energy inequality, one chooses *three* multipliers $X, Y, Z$ and sets

$$P_\alpha = Q_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta.$$

For a solution $W$ of the Bianchi equations, we have then

$$div\, P = (1/2) Q_{\alpha\beta\gamma\delta} ({}^{(X)}\pi^{\alpha\beta} Y^\gamma Z^\delta + {}^{(Y)}\pi^{\alpha\gamma} X^\beta Z^\delta + {}^{(Z)}\pi^{\alpha\delta} X^\beta Y^\gamma).$$

We see that we have now the choice of three multipliers, which gives a lot of flexibility. In [], many choices appear, where the multipliers are picked among the fields $T_0 = (1/2)(L + \underline{L})$ (standard choice), $K_0 = (1/2)(u^2\underline{L} + \underline{u}^2 L)$ (conformal choice), $L$ or $\underline{L}$.

• In [25], the authors also need to commute vector fields with the Bianchi equations. We already know from the case of Maxwell equations that the good way to do it is to consider $\mathcal{L}_X W$. There is however some technical difficulty, namely : this Lie derivative is no longer a Weyl field ! This forces us to define a modified Lie derivative $\hat{\mathcal{L}}_X W$, which is $\mathcal{L}_X W$ plus some linear combination of components of $W$. We will not go any further in this direction, and refer to [25].

**2. Optical functions**

Consider a bounded region $K$ of spacetime whose boundary is formed by

i) A portion of the spacelike hypersurface $\Sigma_0$,

ii) A portion of the null outgoing hypersurface $C_0$,

iii) A portion of the null incoming hypersurface $\underline{C}_*$.

We want to construct two optical functions $u$ and $\underline{u}$ such that $C_0$ is contained in a level set of $u$, and $\underline{C}_*$ is contained in a level set of $\underline{u}$. In fact, in the approach of [], the authors proceed by induction on the "last slice" $\underline{C}_*$, instead of proceeding by induction on time as in example 9.1 for instance.

**a.** To construct $\underline{u}$, we prescribe an initial foliation on $\Sigma_0$ given by the level sets of a function $w$. The natural frame associated with such a foliation is $(N, e_a)$, where

$$N = |\nabla w|^{-1}\nabla w$$

is the unit vector normal to the leaves of the foliation, and $e_a$ form an orthonormal basis of the 2-dimensional leaves. If $\theta$ denotes the second form of the foliation (in $\Sigma_0$), one can establish the equation (we keep the notation of [] for convenience)

$$N(tr\,\theta) + (1/2)(tr\,\theta)^2 = -(\not\Delta \log a + \rho) + [-|\not\nabla \log a|^2 - |\hat{\theta}|^2 + g(k)],$$

where

$$a = |\nabla w|^{-1},\ \rho = -(1/4)R_{3434},\ g(k) = k_{NN}^2 + \sum |k_{Na}|^2.$$

Here, the frame implicitly used in the notation $R_{3434}$ is

$$e_3 = N - T_0,\ e_4 = N + T_0,$$

where $T_0$ is a unit vector orthogonal to $\Sigma_0$. In order to save derivatives of $\theta$, we want $w$ to be constant on the trace of $\underline{C}_*$ on $\Sigma_0$, and $a$ to satisfy the elliptic equation on the leaves (the symbol "bar" denotes the mean value on the leaves)

$$\not\Delta \log a = -(\rho - \bar{\rho}),\ \overline{\log a} = 0.$$

The existence of such a function $w$ requires of course a proof. We refer to [24] for details : the point we want to make is that $\underline{u}$ is constructed in a very careful way, in accordance with smoothness requirements.

**b.** The optical function $u$ is the outgoing solution of the eikonal equation with initial condition $u = u_*$ on the last slice $\underline{C}_*$. The function $u_*$ is the solution of a highly non trivial system which we do not discuss here.

The careful construction of both optical functions $u$ and $\underline{u}$ and their associated frame makes it possible to obtain specific decay properties for the various components of $R$ on this frame : these are called the "peeling properties".

## 9.5 $L^2$ conjecture on the curvature

We refer here to two series of works

i) the papers [30]-[32] where the local well-posedness of the vacuum Einstein equations is proved with an initial curvature in $H^{+0}$,

ii) the papers [27]-[29] starting the proof of the same result with curvature only in $L^2$.

The general framework is very similar to that of the preceding example 9.4. The challenge is to control the geometry of null geodesic cones, and of the associated optical functions and frames, using only $L^2$ bounds on some components of the curvature. The control of this geometry will allow us to use the machinery of chapter 8 to obtain estimates on the curvature via the Bianchi equations. We sketch here the issue of the boundedness of $tr\,\chi$, the importance of which we first explain.

• Assume given an optical function $u$. Let $S_0$ be a fixed 2-"sphere" in an initial spacelike hypersurface $\Sigma_0$. Let $u$ be constant on $S_0$ and consider the hypersurface $H$ which is the union of the integral curves of $L = -\nabla u$ starting from $S_0$. Then $u$ is constant on $H$, and $H$ is a null hypersurface. Let $s$ be the function on $H$ defined by

$$Ls = 1,\, m \in S_0 \Rightarrow s(m) = 0.$$

The image of $S_0$ by the flow of $L$ at time $s_0$ is the level surface $S_{s_0}$ of $s$, and the 2-"spheres" $S_s$ form a foliation, called the "geodesic foliation", of $H$. The null frame we are working with is associated to this sphere foliation as explained in chapter 3.

We pick up on $S_0$ coordinates $\omega = (\omega^1, \omega^2)$ and define coordinates on $H$ following the trajectories of $L$ ; more precisely, if $m$ is the image of the point of $S_0$ of coordinates $\omega$ by the flow of $L$ at time $s$, the coordinates of $m$ are $(s, \omega)$. The importance of $tr\,\chi$ with respect to the foliation $S_s$ is displayed in the following theorem.

**Theorem.** *Let $dA_s$ be the area element on $S_s$, and $|S_s| = \int_{S_s} dA_s$ be the area of $S_s$. Then*

$$(d/ds)|S_s| = \int_{S_s} (tr\,\chi) dA_s.$$

Following [], we first prove that, with the coordinates we introduced on $H$,

$$\partial_s(g_{ab}) = 2\chi_{ab}.$$

This is due to the fact that $L = \partial_s$ :

$$L < \partial_a, \partial_b >=< D_L\partial_a, \partial_b > + < \partial_a, D_L\partial_b >=$$

$$=< D_aL, \partial_b > + < \partial_a, D_bL >= 2\chi_{ab}$$

since $[L, \partial_a] = 0$.

Denoting by $\gamma$ the restriction of $g$ to the spheres, this implies

$$\partial_s(|\gamma|^{1/2}) = |\gamma|^{1/2} tr\,\chi.$$

Now

$$|S_s| = \int |\gamma|^{1/2} d\omega^1 d\omega^2,$$

$$(d/ds)|S_s| = \int tr\,\chi |\gamma|^{1/2} d\omega^1 d\omega^2 = \int_{S_s} tr\,\chi dA_s. \diamondsuit$$

• Define some components of $R$ by

$$\beta_a = R_{La\underline{L}L},\ \rho = (1/4)R_{\underline{L}L\underline{L}L},\ \sigma = *R_{\underline{L}L\underline{L}L}.$$

Assume that these components are bounded in $L^2(H)$ by $R_0$. We want to use the machinery of chapter 7 to obtain a bound for $tr\,\chi$ in $L^\infty$, carefully counting derivatives.

For this, we come back to the transport equation on $tr\,\chi$, which is here, since the Ricci tensor is zero,

$$L(tr\,\chi) + (1/2)(tr\,\chi)^2 = -|\hat{\chi}|^2.$$

To obtain a $L^\infty$ control of $tr\,\chi$, we need to control $\int_\Gamma |\hat{\chi}|^2$ on each of the geodesics $\Gamma$ which foliate $H$. We turn then to the Codazzi equation on $\hat{\chi}$, which is here

$$div\,\hat{\chi} = -\beta + (1/2)\ \nabla\!\!\!\!/\, tr\,\chi + ...,$$

where the dots denote terms which are supposed to cause no problems. Denoting by $D^{-1}$ the pseudodifferential operator of order $-1$ which solves the elliptic system on $\hat{\chi}$, we have to bound

$$I_1 = \int_\Gamma |D^{-1}\beta|^2,\ I_2 = \int_\Gamma |D^{-1}\ \nabla\!\!\!\!/\, tr\,\chi|^2.$$

• The bound on $\beta$ implies that $D^{-1}\beta$ is bounded in $H^s(H)$ for $s = 1$, and we cannot consider its trace on the codimension 2 curve $\Gamma$, since this would require $s > 2/2 = 1$ ! In a way which is analogous to what has been done with the special component $R_{44}$ in 6.3, we investigate now the *special structure* of $\beta$. To this aim, we write the Bianchi equations in our frame, and obtain

$$div\,\beta = D_L\rho + \dots,\ curl\,\beta = -D_L\sigma + \dots,$$

where the dots as usual are supposed to be harmless terms. In short, we write the solution of this elliptic system $\beta = D^{-1}(L(\rho), L(\sigma))$, thus obtaining

$$D^{-1}\beta = \nabla_L Q + \dots,\ Q = D^{-2}(\rho, \sigma).$$

The integral $I_1$ is bounded by $||Q|\Gamma||^2_{H^1}$, which is itself bounded by

$$||Q||^2_{H^2(H)} \leq C||(\rho, \sigma)||^2_{L^2(H)} \leq CR_0^2.$$

• To bound $||tr\,\chi||_{L^\infty}$ using the transport equation, we also have to bound $I_2$ with this same norm. The difficulty here is that $D^{-1}\ \not\nabla$ is a zero order pseudodifferential operator on the spheres, which does not act on $L^\infty$. This forces the authors to work in Besov spaces, and leads to considerable developments which are beyond the scope of this introduction. A related approach in progress, based on a parametrix construction, has been exposed by Szeftel in his Cours Peccot [38].

## 9.6 Stability of Minkowski spacetime (second version)

For quite a long time, it has been believed that working in harmonic coordinates for the Einstein equations could only lead to local (in time) existence results, see for instance the work of Choquet-Bruhat [12]. The idea to prove global results was then to avoid coordinates altogether, as was the case for the first version mentioned in example 9.4. In this second version [35] however , just as in the example ii) above, it turns out that one can use the standard Lorentz fields to handle the problem of small solutions.

• Let us explain first the use of "harmonic coordinates" : this means that we are working on $\mathbf{R}^4$ with coordinates $x^\alpha$, each one of them being a solution of the wave equation $\Box_g x^\alpha = 0$. From the formula of 2.3, this means, for each $\mu$,

$$\partial^\alpha g_{\alpha\mu} = (1/2)g^{\alpha\beta}\partial_\mu g_{\alpha\beta}.$$

It also means that the lower order terms in $\Box$ are identically zero

$$\Box = g^{\alpha\beta}\partial^2_{\alpha\beta}.$$

If we take the $\nu$ derivative of the above formula, we get

$$\partial^\alpha\partial_{nu}g_{\alpha\mu} = (1/2)g^{\alpha\beta}\partial^2_{\mu\nu}g_{\alpha\beta} + q_{\mu\nu},$$

where $q_{\mu\nu}$ is a quadratic expression in the first order derivatives of $g$

$$q_{\mu\nu} = (1/2)\partial_\nu(g^{\alpha\beta})\partial_\mu g_{\alpha\beta} - (\partial_\nu g^{\alpha\beta})\partial_\beta g_{\alpha\mu}.$$

Exchanging $\mu$ and $\nu$ and summing, we obtain

$$\partial^\alpha\partial_\nu g_{\alpha\mu} + \partial^\alpha\partial_\mu g_{\alpha\nu} - g^{\alpha\beta}\partial^2_{\mu\nu}g_{\alpha\beta} + q_{\mu\nu} + q_{\nu\mu}.$$

Using the explicit formula from 6.1 for the Ricci tensor, we observe that, in harmonic coordinates, the first three terms cancel out, modulo quadratic terms in first order derivatives of $g$. Hence the vacuum Einstein equations can be written

$$\Box g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g),$$

for some appropriate expressions $F_{\mu\nu}$, quadratic in $\partial g$. This is the only known way to display the *hyperbolic character* of Einstein equations.

Besides reducing Einstein equations to a hyperbolic system, the point of harmonic coordinates is this : *suppose $g$ satisfies the system*

$$g^{\alpha\beta}\partial^2_{\alpha\beta}g_{\mu\nu} = F_{\mu\nu}(g)(\partial g, \partial g)$$

*with initial values $(g, \partial_t g) = (g_0, g_1)$ on $\{t = 0\}$ satisfying the harmonic coordinates relations ; then the harmonic coordinates relations are true for all $t$, and $g$ is in fact a solution of Einstein equations.*

We refer to [19] for a proof of this well-known fact.

• From the preceding point, we see that we have to work with a *diagonal* (in its principal part) system of wave equations, *coupled* with the first order condition of harmonic coordinates. In the *scalar* case of example ii), we concentrated on the good derivatives of the solution $\phi$ ; since we work here with a system on the tensor $g$, we will not only concentrate on the good derivatives of single components $g_{\alpha\beta}$, but also on ordinary derivatives of good components of $g$. The harmonic coordinates relation provides precisely a *link between good derivatives and good components* : with the notation of [35], let $T$ be one good derivative $L, e_1, e_2$ : defining the perturbation $h$ of the Minkowski metric by $g^{\alpha\beta} = m^{\alpha\beta} + h^{\alpha\beta}$, we have

$$T^\mu\partial^\alpha h_{\alpha\mu} = T(h^\alpha_\alpha) + O(h\partial h).$$

On the left-hand side, we have for fixed $\mu$, modulo $O(h\partial h$ terms, the trace of the tensor

$$(X, Y) \mapsto D_X h(Y, \partial_\mu).$$

In the null frame $(e_1, e_2, \underline{L}, L)$, this trace is

$$D_a h(e_a, \partial_\mu) - (1/2)D_L h(\underline{L}, \partial_\mu) - (1/2)D_{\underline{L}} h(L, \partial_\mu).$$

This implies the relation

$$|(\partial h)_{LT}| \leq |Th| + O(h\partial h).$$

In words, the good component $LT$ of any derivative of $h$ is controlled, modulo harmless terms, by all components of a good derivative of $h$ : this is the *"duality"* specific of this system.

• As in ii), we start with the induction hypothesis

$$E_N(t) \leq 64\epsilon^2(1+t)^{2\delta}.$$

The above estimate and similar other estimates following the same duality principle and using the harmonic coordinates condition gives, for some $\gamma > 0$,

$$|(\partial h)_{LT}| + |(\partial Zh)_{LL}| \leq C\epsilon(1+t+r)^{-1-2\gamma},\ |h_{LT}| + |(Zh)_{LL}| \leq C\epsilon(1+t+r)^{-1} < r-t > .$$

Now, the authors establish the following improved energy inequality, which is very close to the one in chapter 6.

**Theorem.** *Assume that the metric $g$ satisfies the decay estimates*

$$< r-t >^{-1} |h_{LL}| + |(\partial h)_{LL}| + |Th| \leq C\epsilon(1+t)^{-1},$$

$$< r-t >^{-1} |h| + |\partial h| \leq C\epsilon(1+t)^{-1/2}(< r-t >^{-1/2-\gamma} .$$

*Then, for any $0 < \gamma \leq 1/2$, there exists $\epsilon_0 > 0$ such that, for $\epsilon \leq \epsilon_0$,*

$$E_\phi(T) + (1/2)\gamma \int_{0\leq t\leq T} < r-t >^{-1-2\gamma} \sum |T\phi|^2 dxdt \leq$$

$$8E_\phi(0) + C\epsilon \int_{0\leq t\leq T} (1+t)^{-1}|\partial\phi|^2 dxdt + 16 \int_{0\leq t\leq T} |\Box\phi||\partial_t\phi| dxdt.$$

Remark that this inequality gives the classical improved energy inequality with an amplification factor $(1+t)^{C\epsilon}$. The estimates already proved on $h$ fit with the assumptions of the theorem, and this is the key to the proof.

There are of course many other difficult problems to settle : estimates for the right-hand sides $F_{\mu\nu}$, estimates of the commutators $[\Box, Z^k]$, etc. We will not go into this, refering to the well-written introduction of [35] for details.

This work shows that, in the study of systems of wave equations or hyperbolic symmetric systems with an unknown $u \in \mathbf{R}^N$, it is an essential step to understand the duality between the good derivatives of $u$ and its good components. Note that in the present case, as well as in the case of the Bianchi equations briefly discussed in example iii), the *same* null frame is used to identify the good derivatives $(e_1(h), e_2(h), L(h))$ and the good components $h_{LT}, \partial h_{LT}$. In a more general situation, it could occur that one has to construct a null frame in the physical space to identify the good derivatives of $u$, and another frame (with which properties ?) in $\mathbf{R}^N$ to capture the good components of $u$. An example of this situation is given in [9] : following the guidelines given by a nonlinear geometrical optics approximation of the solution (see the "weak null " condition discussed in [35]), we identify good components of $\partial u$, which play an essential role in the proof.

## 9.7 The Formation of Black Holes

We refer here to the monography [14] by Christodoulou. We will not discuss here the heart of the book which is what the author calls the "short pulse method" or "third method", the first two methods being working with Bianchi identities and null frames, as explained in example 9.4. We only want to point out the construction of the optical functions, which is very close to that of example 9.3. The author constructs first a timelike geodesic line $\Gamma_0$, and considers the outgoing future null geodesic cones with vertices on $\Gamma_0$ : the optical function $u$ is then taken to have these cones $C_u$ as level surfaces. For $\underline{u}$, its level surfaces are assumed to be the past incoming geodesic cones $\underline{C}_{\underline{u}}$ with vertices on $\Gamma_0$. The exact values of $u$ and $\underline{u}$ on $\Gamma_0$ depend only on two functions of one variable, which leaves much less flexibility than in example 9.4, where $u$ and $\underline{u}$ were depending on the choices of two functions of three variables. Despite this fact, it turns out that these choices of $u$ and $\underline{u}$ are relevant, since in the end some sphere of the foliation

$$S_{u\underline{u}} = C_u \cap \underline{C}_{\underline{u}}$$

turns out to be the desired "trapped surface". We will not explain this term, let us only say that a trapped surface announces a singularity, as explained in [20] for instance.

# Chapter 10

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