A priori estimates for the motion of a self-gravitating incompressible liquid with free surface boundary. Preliminary Revised Version

Hans Lindblad and Karl Håkan Nordgren

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Abstract

In this paper, we prove *a priori* estimates in Lagrangian coordinates for the equations of motion of an incompressible, inviscid, self-gravitating fluid with free boundary. The estimates show that on a finite time interval we control five derivatives of the fluid velocity and five and a half derivatives of the coordinates of the moving domain.

1   Introduction.

Let $\Omega_t \subseteq \mathbf{R}^n$ be the domain occupied by a fluid at time $t \in [0, T]$ and suppose that the fluid has velocity $v(t, x)$ and pressure $p(t, x)$ at a point x in Ω_t. For an inviscid, self-gravitating fluid these two quantities are related by Euler's equation

$$(1.1) \qquad \left(\partial_t + v^i \partial_i\right) v_j = -\partial_j p - \partial_j \phi$$

in Ω_t, where $\partial_i = \frac{\partial}{\partial x^i}$ and $v^i = \delta^{ij} v_j$ and where ϕ is the Newtonian gravity-potential defined by

$$(1.2) \qquad \phi(t, x) = -\chi_{\Omega_t} * \Phi(x)$$

on Ω_t, where χ_{Ω_t} is a function which takes the value 1 on Ω_t and the value 0 on the complement of Ω_t and where Φ is the fundamental solution to the Laplacean. Thus ϕ satisfies $\Delta \phi = -1$ on Ω_t. We can impose the condition that the fluid be incompressible by requiring that the fluid-velocity be divergence-free:

$$(1.3) \qquad \operatorname{div} v = \partial_i v^i = 0 \text{ in } \Omega_t.$$

The absence of surface-tension is imposed with the following boundary condition:

$$(1.4) \qquad p = 0 \text{ on } \partial\Omega_t$$

and to make the free-boundary move with the fluid-velocity, we have

$$(1.5) \qquad \partial_t + v^i \partial_i \text{ is in the tangent-space of } \cup_{t \in [0,T]} [\Omega_t \times \{t\}].$$

The problem is, then, to prove *a priori* estimates for v satisfying (1.1) - (1.5) in some interval $[0, T]$, given the initial-conditions

$$(1.6) \qquad v = v_0 \text{ on } \Omega_0,$$

where v_0 and Ω_0 are known. We will also assume that initially there is a constant c_0 such that

$$(1.7) \qquad \nabla p \cdot N \leq -c_0 < 0 \text{ on } \partial\Omega_0$$

where N is the exterior unit normal to $\partial\Omega$, (1.7) is a natural physical condition since the pressure of a fluid has to be positive and the problem is ill-posed if this is not satisfied, see Ebin [1]. This condition is related to Rayleigh-Taylor instability.

We will assume for simplicity that n, the number of space-dimensions, is 2. We will also assume that there is a volume-preserving diffeomorphism $x_0 : \Omega \to \Omega_0$, where $\Omega = \{y \in \mathbf{R}^2 : |y| < 1\}$, which will allow us describe the derivatives which are tangential to Ω_t in a particularly simple way, and it will also allow fractional derivatives to be defined by using Fourier series without recourse to partitions of unity. That part of the argument can be used, with minor modifications, in the case of arbitrary space dimension. The dimension will also allow simpler energy estimates because the curl of the velocity is conserved.

Suppose now that v satisfies (1.3). We define Lagrangian coordinates as follows: Define x by

$$(1.8) \qquad \frac{dx}{dt}(t,y) = v(t, x(t,y)) \text{ and } x(0,y) = x_0(y)$$

for y in Ω and for t in some time interval $[0, T]$. Since v satisfies (1.3) and this means that

$$(1.9) \qquad \partial_t \det\left(\frac{\partial x}{\partial y}\right)(t,y) = \operatorname{div} v \circ x(t,y) = 0.$$

And since $\det\left(\frac{\partial x}{\partial y}\right)(0,y) = 1$ we therefore have $\det\left(\frac{\partial x}{\partial y}\right) = 1$ in Ω. We will prove the following theorem:

THEOREM 1.1 *Let v satisfy (1.1) and (1.3) and let p satisfy (1.4) and (1.7). Let the flow x of v be defined by (1.8) and define $V(t,y) = v(t, x(t,y))$. Define*

$$E(t) = \sup_{[0,t]} \left[\|V\|_5 + \|x\|_{5.5} + \|curl(v)\|_{H^{4.5}(\Omega_t)} \right].$$

Then there is $T > 0$ such that $E(T) \le P\big[E(0)\big]$ where P is a polynomial.

1.1 Background.

Past progress has been made in three situations: The first progress was made on the water-wave problem under the assumption that the fluid be irrotational $-$ that is, the curl of the fluid-velocity is zero $-$, incompressible and that the free-boundary not be subject to surface-tension. Notable results in this area are Wu's papers [2] and [3] where she uses Clifford analysis to show well-posedness in two and then three dimensions in an infinitely deep fluid; and also Lannes' paper [4] where the Nash-Moser technique is used to prove well-posedness in arbitrary space-dimesions for a fluid of finite depth.

In [5], Christodoulou and Lindblad proved *a priori* estimates for the incompressible Euler's equation, without the assumption of irrotationality. They were not sufficient to obtain the existence result, however, because no approximation-schemes was discovered which did not destroy the structure in the equations on which the estimates relied. In [6] Lindblad proved that the linearized equations are well-posed. In [7] Lindblad then used the Nash-Moser approximation scheme to obtain the full well-posedness. Well-posedness was also proved by Coutand and Shkoller in [8], using a fixed-point argument which relies on smoothing the fluid-velocity only $-$ crucially $-$ in the direction tangential to the boundary. This is followed by energy estimates which we will discuss in detail in section 4. Also, in [9], Shatah and Zeng prove *a priori* estimates under these conditions by considering Euler's equation as the geodesic equation on the group of volume-preserving diffeomorphisms. The latter two papers also consider the case of positive surface-tension.

2 Preliminaries.

We will let x be coordinates on Ω_t defined by (1.8) and let $\partial_i, \partial_j, \partial_k, \dots$ be derivatives on Ω_t; and we will let y be coordinates on Ω and let $\partial_a, \partial_b, \partial_c, \dots$ be derivatives on Ω. Also, we will let ∇ denote an arbitrary derivative on Ω_t and ∂ be an arbitrary derivative on Ω.

2.1   Change of variables.

Let $A_i^a(t,y) = \frac{\partial y^a}{\partial x_i} \circ x(t,y)$ and let $B_a^i(t,y) = \frac{\partial x_i}{\partial y^a}(t,y)$. By (1.9) we see that $\det(B) = 1$ and hence $\det(A) = 1$ as well in a time interval $[0,T]$. This means that $x(t,\cdot) : \Omega \to \Omega_t$ is a change of variables.

2.2   Norms.

Let $f : \Omega \to \mathbf{R}^2$. Define $\|f\|^2 = \int_\Omega \delta_{ij} f^i(y) f^j(y) dy$ and for an integer $k \geq 0$ we define

$$\|f\|_k^2 = \sum_{i=0}^k \|\nabla^i[f]\|^2$$

where $\nabla = \left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right)$. We define the intermediate spaces by interpolation, see for instance [10] and [11]. For a function $g : \Omega_t \to \mathbf{R}^2$, define $\|g\|_{L^2(\Omega_t)}^2 = \int_{\Omega_t} \delta_{ij} g^i(x) g^j(x) dx$ and for an integer $k \geq 0$ define

$$\|g\|_{H^k(\Omega_t)}^2 = \sum_{i=0}^k \|\nabla^i[g]\|_{L^2(\Omega_t)}^2$$

where $\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right)$. Again, we define the intermediate spaces by interpolation. For a function $h : \partial\Omega_t \to \mathbf{R}^2$, define $\|h\|_{L^2(\partial\Omega_t)}^2 = \int_{\partial\Omega_t} \delta_{ij} h^i(x) h^j(x) dS(x)$

2.3   Special derivatives.

In this paper, we will make use of two special derivatives. First we have derivatives which are tangential to the boundary:
$$y^a \partial_b - y^b \partial_a \text{ for } a,b = 1,2$$

where the summation convention is *not* employed. We will abuse notation and denote these derivatives and their push-forwards on Ω_t with ∂_θ. The second type of special derivative is defined as follows: Let $f : \Omega \to \mathbf{R}^2$ and let's abuse notation and write $f(\rho,\theta)$ to mean the polar representation of f. Let $\sum f_k(\rho) e^{ik\theta}$ be the tangential Fourier expansion of $f(\rho,\theta)$. We now define a tangential Sobolev-type-derivative $\langle\partial_\theta\rangle^s$ to be an operator which sends

$$\sum f_k(\rho) e^{ik\theta} \text{ to } \sum \langle k \rangle^s f_k(\rho) e^{ik\theta},$$

where $\langle k \rangle = \left[1 + |k|^2 \right]^{\frac{1}{2}}$. For a function g on Ω_t, define $\|\langle\partial_\theta\rangle^s[g]\|_{L^2(\Omega_t)} = \|\langle\partial_\theta\rangle^s[g \circ x]\|_{L^2(\Omega)}$. For integral s the operator $\langle\partial_\theta\rangle^s$ is equivalent (in the $L^2(\Omega)$- and $L^2(\partial\Omega)$-norm) to the application of a collection of ∂_θ. Finally, for a function $h : \partial\Omega_t \to \mathbf{R}^2$, define

$$\|h\|_{H^s(\partial\Omega_t)}^2 = \|\langle\partial_\theta\rangle^s h\|_{L^2(\partial\Omega_t)}^2$$

for a real number s.

2.4   Cut-off functions.

Fix d_0 such that the normal N to $\partial\Omega_t$ can be extended into the image of the set $\{y \in \mathbf{R}^2 : 1 - d_0 < |y|\}$ under x. This fact is used in lemma 2.1 which is presented in section 2.5. Let η_i and ζ_i be radial functions which form a partition of unity subordinate to the sets $\{y \in \mathbf{R}^2 : |y| < 1 - \frac{d_0}{2i}\}$ and $\{y \in \mathbf{R}^2 : 1 - \frac{d_0}{i} < |y|\}$ respectively. This means that η_i takes the value 1 on the set $\{y \in \mathbf{R}^2 : |y| \leq 1 - \frac{d_0}{i}\}$ and ζ_i takes the value 1 on the set $\{y \in \mathbf{R}^2 : 1 - \frac{d_0}{2i} \leq |y|\}$. We will also let η_i and ζ_i denote the analogous functions in the Eulerian frame.

2.5   Hodge-decomposition inequalities.

In this section we present two divergence-curl estimates which are used throughout this text. The first allows point-wise control on all derivatives near the boundary of Ω_t by the divergence, the curl and tangential derivatives. Letting $\zeta = \zeta_i$ we have the following:

LEMMA 2.1 *Let α be a vector-field on Ω_t. Define $(curl\,\alpha)_{jk} = \partial_j \alpha_k - \partial_k \alpha_j$ and $div\,\alpha = \partial_j \alpha^j$. Then we have the following point-wise estimate on Ω_t:*

$$(2.1) \qquad\qquad |\zeta \nabla \alpha| \leq |\zeta\, curl\,\alpha| + |\zeta\, div\,\alpha| + |\zeta\, \partial_\theta \alpha|,$$

where $|\cdot|$ denotes the usual Euclidean distance.

Using lemma 2.1 and an induction argument we have the following lemma:

LEMMA 2.2 *For $1 \leq s \leq 5$,*

$$(2.2) \qquad \|\zeta\alpha\|_{H^s(\Omega_t)} \leq P\big[\|x\|_5\big]\left[\|\zeta\alpha\|_{L^2(\Omega_t)} + \|\zeta\, curl\,\alpha\|_{H^{s-1}(\Omega_t)} + \|\zeta\, div\,\alpha\|_{H^{s-1}(\Omega_t)} + \sum_{j=1}^{s}\|\zeta\partial_\theta^j\alpha\|_{L^2(\Omega_t)}\right].$$

We will also use the following estimates which allows $H^s(\Omega_t)$ control in terms of the divergence, the curl and boundary derivatives:

LEMMA 2.3 *Let $div\,\alpha$ and $curl\,\alpha$ be defined as in lemma B.1. Then, for $s \leq 5$,*

$$(2.3) \qquad \|\alpha\|_{H^s(\Omega_t)} \leq P\big[\|x\|_5\big]\left[\|\alpha\|_{L^2(\Omega_t)} + \|div\,\alpha\|_{H^{s-1}(\Omega_t)} + \|curl\,\alpha\|_{H^{s-1}(\Omega_t)} + \|(\langle\partial_\theta\rangle^{s-\frac{1}{2}}\alpha)\cdot N\|_{L^2(\partial\Omega_t)}\right],$$

where N is the outward unit normal to $\partial\Omega_t$. Also, for $s \leq 5$,

$$(2.4) \qquad \|\alpha\|_{H^s(\Omega_t)} \leq P\big[\|x\|_5\big]\left[\|\alpha\|_{L^2(\Omega_t)} + \|div\,\alpha\|_{H^{s-1}(\Omega_t)} + \|curl\,\alpha\|_{H^{s-1}(\Omega_t)} + \|(\langle\partial_\theta\rangle^{s-\frac{1}{2}}\alpha)\cdot Q\|_{L^2(\partial\Omega_t)}\right]$$

where Q is a unit vector which is tangent to $\partial\Omega_t$.

3   Elliptic estimates.

In section 4, we will prove energy estimates for (1.1) - (1.7) and to prepare for this, we need the elliptic estimates for p and ϕ contained in this section.

3.1   Estimates for ϕ.

In this section we prove the following theorem:

THEOREM 3.1 $\|\nabla\phi\|_{H^5(\Omega_t)} \leq P\big[\|x\|_5\big]$, *where P is a polynomial.*

Using the cut-off functions defined in section 2.4, we have $\|\nabla\phi\|_{H^5(\Omega_t)} \leq \|\eta_1\nabla\phi\|_{H^5(\Omega_t)} + \|\zeta_1\nabla\phi\|_{H^5(\Omega_t)}$. This allows us to consider interior and boundary regularity separately.

3.1.1   Interior regularity.

In this section we prove the following result:

THEOREM 3.2 *For any $1 \leq i$ and $1 \leq s \leq 5$*

$$(3.1) \qquad \|\nabla^s[\eta_i \nabla \phi]\|_{L^2(\Omega_t)} \leq P\left[\|x\|_5\right]$$

where P is a polynomial.

We prove that (3.1) holds by induction on s. Suppose that $s = 0$. We have $\|\eta_i \nabla \phi\|_{L^2(\Omega_t)} \leq \|\eta_i\|_{L^\infty(\Omega_t)}$ $\times \|\nabla \phi\|_{L^2(\Omega_t)}$ and

$$(3.2) \qquad \|\nabla \phi\|^2_{L^2(\Omega_t)} = \int_{\Omega_t} (\partial_j \phi)(\partial^j \phi) dx = \int_{\partial \Omega_t} N^j (\partial_j \phi)\, \phi\, dx - \int_{\Omega_t} \Delta \phi\, \phi\, dx.$$

Since we have $\|\phi\|_{L^\infty(\Omega_t)} \leq P\left[\|x_\kappa\|_2\right]$ and $\|\nabla \phi\|_{L^\infty(\Omega_t)} \leq P\left[\|x_\kappa\|_2\right]$, we control both terms in (3.2) appropriately. This proves (3.1) for $s = 1$. Now suppose that $s = 5$ and that we have the result for smaller s. Then

$$(3.3) \qquad \|\nabla^5[\eta_i \nabla \phi]\|^2_{L^2(\Omega_t)} = \int_{\Omega_t} (\partial_{j_1} \ldots \partial_{j_5}[\eta_i \partial_{j_6} \phi])(\partial^{j_1} \ldots \partial^{j_5}[\eta_i \partial^{j_6} \phi]) dx.$$

Now,

$$(3.4) \qquad \partial^{j_1} \ldots \partial^{j_5}[\eta_i \partial^{j_6} \phi] = \eta_i(\partial^{j_1} \ldots \partial^{j_5} \partial^{j_6} \phi) + \sum (\nabla^{k_1} \eta_i)(\nabla^{k_2+1} \phi)$$

where the sum is over $k_1 + k_2 = 5$ such that $k_2 \leq 4$. To control the second term in (3.4) we use the following procedure: Let $i_1 = i$. Suppose that we have found $i_1, \ldots, i_l$. The support of $\nabla^k \eta_{i_l}$ is contained in the image under x of the set $\{y \in \mathbf{R}^2 : 1 - \frac{d_0}{i_l} < |y| < 1 - \frac{d_0}{2i_l}\}$. Pick i_{l+1} such that $1 - \frac{d_0}{2i_l} \leq 1 - \frac{d_0}{i_{l+1}}$. Then $\eta_{i_{l+1}}$ takes the value 1 on the set $\{y \in \mathbf{R}^2 : |y| \leq 1 - \frac{d_0}{i_{l+1}}\}$ and $\{y \in \mathbf{R}^2 : 1 - \frac{d_0}{i_l} < |y| < 1 - \frac{d_0}{2i_l}\} \subseteq \{y \in \mathbf{R}^2 : |y| \leq 1 - \frac{d_0}{i_{l+1}}\}$.

LEMMA 3.3 *For $k_2 \geq 1$ we have*

$$(3.5) \qquad (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2} \phi) = \sum (\nabla^{k_1} \eta_{i_1})(\nabla^{l_2} \eta_{i_2}) \ldots (\nabla^{l_{n-1}} \eta_{i_{n-1}})(\nabla^{l_n}[\eta_{i_n} \nabla \phi])$$

where the sum is over all $l_2 + \ldots + l_n = k_2 - 1$; where for instance if $l_2 = 0$ the term $\nabla^{l_2} \eta_{i_2}$ is taken to not be present in the sum; and where if $l_n = 0$ the term $\nabla^{l_n}[\eta_{i_n} \nabla \phi]$ is taken to be $\eta_{i_n} \nabla \phi$.

PROOF: We prove this by induction on k_2. For $k_2 = 1$ we have $(\nabla^{k_1} \eta_{i_1})(\nabla \phi) = \eta_{i_2}(\nabla^{k_1} \eta_{i_1})(\nabla \phi)$, which is of the correct form. Suppose that $k_2 \geq 2$ and that we have the result for smaller k_2. Then

$$(3.6) \qquad (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2} \phi) = \eta_{i_2} (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2} \phi)$$

$$(3.7) \qquad\qquad = (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2-1}[\eta_{i_2} \nabla \phi]) - (\nabla^{k_1} \eta_{i_1}) \sum_{l_1+l_2=k_2-1,\, l_2 \leq k_2-2} (\nabla^{l_1} \eta_{i_2})(\nabla^{l_2+1} \phi)$$

$$(3.8) \qquad\qquad = (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2-1}[\eta_{i_2} \nabla \phi])$$

$$(3.9) \qquad\qquad - (\nabla^{k_1} \eta_{i_1}) \sum_{l_1+l_2=k_2-1,\, l_2 \leq k_2-2} \sum (\nabla^{l_1} \eta_{i_2})(\nabla^{m_2} \eta_{i_3}) \ldots (\nabla^{m_{n-1}} \eta_{i_{n-1}})(\nabla^{m_n}[\eta_{i_n} \nabla \phi])$$

which again is of the correct form.   ∎

Integrating the first term in (3.4) by parts twice we have

$$(3.10) \quad -\int_{\Omega_t}(\partial_{j_1}\dots\partial_{j_5}\partial^{j_6}[\eta_i\partial_{j_6}\phi])\eta_i(\partial^{j_1}\dots\partial^{j_5}\phi)dx - \int_{\Omega_t}(\partial_{j_1}\dots\partial_{j_5}[\eta_i\partial_{j_6}f])(\partial^{j_6}\eta_i)(\partial^{j_1}\dots\partial^{j_5}\phi)dx$$

$$(3.11) \quad = \int_{\Omega_t}(\partial_{j_1}\dots\partial_{j_4}\partial^{j_6}[\eta_i\partial_{j_6}\phi])\eta_i(\partial^{j_1}\dots\partial^{j_5}\partial_{j_5}\phi)dx + \int_{\Omega_t}(\partial_{j_1}\dots\partial_{j_4}\partial^{j_6}[\eta_i\partial_{j_6}\phi])(\partial_{j_5}\eta_i)(\partial^{j_1}\dots\partial^{j_5}\phi)dx$$

$$(3.12) \quad -\int_{\Omega_t}(\partial_{j_1}\dots\partial_{j_5}[\eta_i\partial_{j_6}\phi])(\partial^{j_6}\eta_i)(\partial^{j_1}\dots\partial^{j_5}\phi)dx$$

where we can control the second and third term in (3.11) using lemma 3.3. Also, $\partial^{j_6}[\eta_i\partial_{j_6}\phi] = (\partial^{j_6}\eta_i)(\partial_{j_6}\phi) + \eta_i$ and therefore the first term in (3.11) is equal to

$$(3.13) \quad \int_{\Omega_t}\partial_{j_1}\dots\partial_{j_4}[(\partial^{j_6}\eta_i)(\partial_{j_6}\phi)]\eta_i(\partial^{j_1}\dots\partial^{j_5}\partial_{j_5}\phi)dx + \int_{\Omega_t}\partial_{j_1}\dots\partial_{j_4}[\eta_i]\eta_i(\partial^{j_1}\dots\partial^{j_5}\partial_{j_5}\phi)dx$$

The above terms in (3.13) can be controlled using lemma 3.3 and the inductive hypothesis. This concludes the proof of proposition 3.2.

3.1.2 Boundary regularity revised

Let ζ denote ζ_1. We have $\nabla^2[\zeta\phi] = (\nabla^2\zeta)\phi + (\nabla\zeta)(\nabla\phi) + \zeta(\nabla^2\phi)$ and $\nabla\zeta$ is supported in $\{y \in \mathbf{R}^2 : 1 - d_0 < |y| < 1 - \frac{d_0}{2}\}$. We can find i such that $1 - \frac{d_0}{2} \leq 1 - \frac{d_0}{i}$ and therefore $\{y \in \mathbf{R}^2 : 1 - d_0 < |y| < 1 - \frac{d_0}{2}\} \subseteq \{y \in \mathbf{R}^2 : |y| < 1 - \frac{d_0}{i}\}$ where η_i takes the value 1. Thus $(\nabla\zeta)(\nabla\phi) = \eta_i(\nabla\zeta)(\nabla\phi) = (\nabla\zeta)(\nabla[\eta_i\phi]) - (\nabla\zeta)(\nabla\eta_i)\phi$, which we control by theorem 3.2. Thus we need only be concerned with $\zeta\nabla^s\phi$. In this section we prove the following result:

THEOREM 3.4 *For* $1 \leq s \leq 6$

$$(3.14) \quad \|\zeta\nabla^s\phi\|_{L^2(\Omega_t)} \leq P\big[\|x\|_{5.5}\big]$$

where P is a polynomial.

Since integration by parts on Ω_t will yield a boundary term which is difficult to deal with because $\partial\Omega_t$ is the complement of the singular support of ϕ, we begin by expanding the region of integration: Suppose that $x(0, y) = y$ and define $\tilde{x} = E(x - y) + y$ where E is the extension operator on Ω — see, for instance, [10] — and define $\tilde{V} = \partial_t\tilde{x}$. Then both $\tilde{V}$ and $\tilde{x}$ are defined in all of $\mathbf{R}^2$ and such that $\|\tilde{x}\|_{H^s(\mathbf{R}^2)} \leq c\|x\|_{H^s(\Omega)}$ and similarly for $\tilde{V}$. Define $\tilde{B}_a^i = \frac{\partial\tilde{x}^i}{\partial y^a}$. Then since $\det(B) = 1$ on Ω, we can choose d_0 (possibly smaller than the one used before) so that $\tilde{x}$ is a change of variables on $\tilde{\Omega} = \{y \in \mathbf{R}^2 : |y| < 1 + d_0\}$ and such that d_0 is small enough that the normal N to $\partial\Omega_t$ can be extended into the region between $\partial\Omega_t$ and the boundary of $\tilde{\Omega}_t = \tilde{x}(t, \tilde{\Omega})$. This means that for every i, N can be extended into the support of ζ on both sides of $\partial\Omega_t$. Let $\tilde{A}$ be the inverse of $\tilde{B}$. We now define $\tilde{\phi}$ as follows:

$$(3.15) \quad \tilde{\phi}(t, x) = -\chi_{\Omega_t} * \Phi(x) \text{ for } x \text{ in } \tilde{\Omega}_t$$

where again Φ is the fundamental solution for the Laplacean. This means that on Ω_t, we have $\tilde{\phi} = \phi$ and therefore that $\tilde{\phi}$ and ϕ have the same regularity on Ω_t. It also means that $\tilde{\phi}$ is smooth on $\partial\tilde{\Omega}_t$. Finally, let the norms on the extended domains $\tilde{\Omega}$ and $\tilde{\Omega}_t$ be defined analogously to the norms on Ω and Ω_t.

LEMMA 3.5 *If* $\triangle f = g$ *where* $g(x) = 0$ *when* $dist(x, \partial\tilde{\Omega}_t) \geq d_0 > 0$, *then* $|\nabla^s f(x)| \leq C_s\|g\|_{L^1(\tilde{\Omega}_t)}$, *for* $x \in \partial\tilde{\Omega}_t$ *and* $|\partial_y^\alpha x| \leq C_\alpha$, *for* $x \in \partial\tilde{\Omega}_t$.

PROOF: We have

$$|\nabla^s f(x)| \le \int_{\tilde{\Omega}_t} |g(z)| \, |\Phi^{(s)}(|x-z|)| dz \le C\delta^{-s}\|g\|_{L^1(\tilde{\Omega}_t)}.$$

∎

LEMMA 3.6 *We have* $\|\nabla f\|_{L^2(\tilde{\Omega}_t)} \le c\|g\|_{L^1(\tilde{\Omega}_t)}$.

PROOF: We have

$$(3.16) \qquad \|\nabla f\|_{L^2(\tilde{\Omega}_t)}^2 = \int_{\tilde{\Omega}_t} (\partial_j \tilde{f})(\partial_j f) dx = \int_{\partial\tilde{\Omega}_t} N^j (\partial_j f) f \, dS(x) - \int_{\tilde{\Omega}_t} g f \, dx.$$

∎

PROPOSITION 3.7 *For $0 \le j \le 4$, we have*

$$(3.17) \qquad \|\zeta\nabla\partial_\theta^j\nabla f\|_{L^2(\Omega_t)} \le P\big[\|x\|_5\big]\Big(\sum_{k=0}^{j}\|\partial_\theta^k g\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j-3}\|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)} + \delta_{j4}\|g\|_{L^\infty(\tilde{\Omega}_t)}\Big).$$

where the sum of L^4 norms is only there if $j > 2$ and the L^∞ norm is only there if $j = 4$.

We begin by proving a lemma which says that we need only be concerned with tangential derivatives:

LEMMA 3.8 *Let f satisfy $\Delta f = g$ in $\tilde{\Omega}_t$. Then for $0 \le j \le 4$,*

$$(3.18) \quad \|\zeta\nabla\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} \le P\big[\|x\|_5\big]\Big(\sum_{k=0}^{j+1}\|\zeta\partial_\theta^k\nabla f\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j}\|\partial_\theta^k g\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j-3}\|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)} + \delta_{j4}\|g\|_{L^\infty(\tilde{\Omega}_t)}\Big),$$

where the sum of L^4 norms is only there if $j > 2$ and the L^∞ norm is only there if $j = 4$. Moreover, for $0 \le j \le 2$ we have

$$(3.19) \qquad \|\zeta\nabla\partial_\theta^j\nabla f\|_{L^4(\tilde{\Omega}_t)} \le P\big[\|x\|_5\big]\Big(\sum_{k=0}^{j+2}\|\zeta\partial_\theta^k\nabla f\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j}\|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)}\Big).$$

Furthermore

$$(3.20) \qquad \|\zeta\nabla\nabla f\|_{L^\infty(\tilde{\Omega}_t)} \lesssim \|g\|_{L^\infty(\tilde{\Omega}_t)} + \sum_{j=0}^{1}\|\zeta\nabla^j\partial_\theta\nabla f\|_{L^4(\tilde{\Omega}_t)}.$$

PROOF: We prove this result by induction. For $j = 0$ we have $\|\zeta\nabla^2 f\|_{L^2(\tilde{\Omega}_t)} \lesssim \|\zeta\Delta f\|_{L^2(\tilde{\Omega}_t)} + \|\zeta\partial_\theta\nabla f\|_{L^2(\tilde{\Omega}_t)}$, by Lemma 2.1, which is of the right form. Now suppose that $1 \le j \le 2$ and that we have (3.18) for smaller j. Then $\|\zeta\nabla\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} \le \|\zeta\text{div}\,\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|\zeta\text{curl}\,\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|\zeta\partial_\theta^{j+1}\nabla f\|_{L^2(\tilde{\Omega}_t)}$. Now $\zeta\text{div}\,\partial_\theta^j\nabla f = \zeta\partial_\theta^j\Delta f + \sum(\partial_\theta^k \tilde{A})(\zeta\nabla\partial_\theta^l\nabla f)$ where the sum is over $k+l = j$ such that $l \le j-1 \le 1$. We have $\|\zeta(\text{div}\,\partial_\theta^j\nabla f - \partial_\theta^j\Delta f)\|_{L^2(\tilde{\Omega}_t)} \le P\big[\|x\|_5\big]\|\zeta\nabla\partial_\theta^l\nabla f\|_{L^2(\tilde{\Omega}_t)}$ which we control by induction. Similarly, we also control $\|\zeta\text{curl}\,\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)}$. This proves (3.18) for $j \le 2$.

For $j = 0, 1$ we have by Lemma 2.1

$$(3.21) \quad \|\zeta\nabla\partial_\theta^j\nabla f\|_{L^4(\tilde{\Omega}_t)} \lesssim \sum_{k=0}^{j}\|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)} + \sum_{k=0}^{j+1}\|\zeta\partial_\theta^k\nabla f\|_{L^4(\tilde{\Omega}_t)} \lesssim \sum_{k=0}^{j}\|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)} + \sum_{k=0}^{j+1}\|\zeta\partial_\theta^k\nabla f\|_{H^1(\tilde{\Omega}_t)}$$

$$\lesssim \sum_{k=0}^{j}P\big[\|x\|_5\big]\left[\sum_{k=0}^{j}\|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)} + \sum_{k=0}^{j+2}\|\zeta\partial_\theta^k\nabla f\|_{L^2(\tilde{\Omega}_t)}\right]$$

using Sobolev's inequality, (3.18) for $j = 1, 2$ and Theorem 3.2. Thus we have (3.19) for $j = 0, 1$.

Now suppose that $j = 3$. Then for $0 \le k \le 2$ we have $\|(\partial_\theta^k \tilde{A})(\zeta \nabla \partial_\theta^l \nabla f)\|_{L^2(\tilde{\Omega}_t)} \le \|x\|_5 \|\zeta \nabla \partial_\theta^l \nabla f\|_{L^2(\tilde{\Omega}_t)}$ and for $k = 3$ we have $l = 0$ and $\|(\partial_\theta^k \tilde{A})(\zeta \nabla \partial_\theta^l \nabla f)\|_{L^2(\tilde{\Omega}_t)} \le \|\partial_\theta^3 \tilde{A}\|_{L^4(\tilde{\Omega}_t)} \|\zeta \nabla^2 f\|_{L^4(\tilde{\Omega}_t)}$, both of which we control appropriately. This proves (3.18) for $j = 3$ and using this we get (3.19) for $j = 2$ as well.

It remains to prove (3.18) for $j = 4$. We would need to control $\|(\partial_\theta^k \tilde{A})(\zeta \nabla \partial_\theta^l \nabla f)\|_{L^2(\tilde{\Omega}_t)}$ for $k + l = j = 4$. If $k \le 3$ this is controlled as above. It hence reain to control $\|(\partial_\theta^4 \tilde{A})(\zeta \nabla \nabla f)\|_{L^2(\tilde{\Omega}_t)}$, but by Lemma 2.1 and Sobolev's lemma

$$(3.22) \qquad \|\zeta \nabla \nabla f\|_{L^\infty(\tilde{\Omega}_t)} \lesssim \|\zeta \Delta f\|_{L^\infty(\tilde{\Omega}_t)} + \|\zeta \partial_\theta \nabla f\|_{L^\infty(\tilde{\Omega}_t)} \lesssim \|g\|_{L^\infty(\tilde{\Omega}_t)} + \sum_{j=0}^{1} \|\zeta \nabla^j \partial_\theta \nabla f\|_{L^4(\tilde{\Omega}_t)}.$$

∎

Using lemma 3.8, we see that it is enough to control $\|\zeta \partial_\theta^j \nabla f\|_{L^2(\tilde{\Omega}_t)}$ appropriately for $0 \le j \le 5$ which is the content of the following proposition:

PROPOSITION 3.9 *For $0 \le j \le 5$ we have*

$$(3.23) \qquad \|\zeta \partial_\theta^j \nabla f\|_{L^2(\tilde{\Omega}_t)} \le P\big[\|x\|_{5.5}\big]\Big(\|\langle \partial_\theta \rangle^{5.5} g\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j-1} \|\partial_\theta^k g\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j-4} \|\partial_\theta^k g\|_{L^4(\tilde{\Omega}_t)} + \delta_{j5} \|g\|_{L^\infty(\tilde{\Omega}_t)}\Big).$$

PROOF: We prove that (3.23) holds by induction on the order. To start the induction we have the analogue of Lemma 3.6. Now we suppose that we have $j = 5$ and suppose that we have already have appropriate control of the lower order cases of (3.23). We have

$$(3.24) \quad \|\zeta \partial_\theta^5 \nabla f\|_{L^2(\tilde{\Omega}_t)}^2 = \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^5 \partial_i f)(\zeta \partial_\theta^5 \partial^i f) dx = \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^5 \partial_i f)(\zeta \partial^i \partial_\theta^5 f) dx - \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^5 \partial_i f)(\partial^i \partial_\theta^5 x^l)(\zeta \partial_l f) dx + \text{L.O.}$$

where the lower order terms are easily controlled by induction and the previous estimates

$$(3.25) \qquad \text{L.O.} = \sum_{l_1 + \ldots + l_s = 5, \ l_1, \ldots, l_s \le 4} \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^5 \nabla f)(\nabla \partial_\theta^{l_1} x) \ldots (\nabla \partial_\theta^{l_{s-1}} x)(\zeta \partial_\theta^{l_s} \nabla f) dx$$

In fact, for $l_1, \ldots, l_{s-1} \le 2$ we control (3.25) by $P\big[\|x\|_5\big]\|\zeta \partial_\theta^5 \nabla f\|_{L^2(\tilde{\Omega}_t)} \|\zeta \partial_\theta^{l_s} \nabla f\|_{L^2(\tilde{\Omega}_t)}$, which we control by induction. Suppose that $3 \le l_1 \le 4$. Then $\|\nabla \partial_\theta^{l_1} x\|_{L^4(\tilde{\Omega}_t)} \le \|x\|_5$; and $l_2, \ldots, l_{s-1} \le 2$, so we control the other terms containing x. We also have $0 \le l_s \le 2$ and therefore $\|\zeta \partial_\theta^{l_s} \nabla f\|_{L^4(\tilde{\Omega}_t)} \le \|\zeta \nabla \partial_\theta^{l_s} \nabla f\|_{L^2(\tilde{\Omega}_t)}$, which we control appropriately by lemma 3.8. Integrating the first two terms in (3.24) by parts ignoring boundary terms and interior terms when the derivative fall on the cutoff ζ, that are easy to control with Lemma 3.5 respectively Theorem 3.2, gives

$$(3.26) \qquad -\int_{\tilde{\Omega}_t} (\zeta \partial^i \partial_\theta^5 \partial_i f)(\zeta \partial_\theta^5 f) dx + \int_{\tilde{\Omega}_t} (\zeta \partial^i \partial_\theta^5 \partial_i f)(\partial_\theta^5 x^l)(\zeta \partial_l f) dx + \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^5 \partial_i f)(\partial_\theta^5 x^l)(\zeta \partial^i \partial_l f) dx$$

We control the last term in (3.29). Also,

$$(3.27) \qquad \partial^i \partial_\theta^5 \partial_i f = \partial_\theta^5 \Delta f + \partial_\theta\big((\nabla_i \partial_\theta^4 x^l)(\nabla_l \nabla_i f)\big) + \sum (\nabla \partial_\theta^{l_1} x) \ldots (\nabla \partial_\theta^{l_{s-1}} x)(\nabla \partial_\theta^{l_s} \nabla f)$$

where $l_1 + \ldots + l_s = 5$ and $l_1, \ldots, l_s \le 4$. For $l_1, \ldots, l_{s-1} \le 2$ we control the last term in (3.27) appropriately by lemma 3.8. If $3 \le l_1 \le 4$ then we control $\nabla \partial_\theta^{l_1} x$ as before and $0 \le l_2, \ldots, l_{s-1} \le 2$ so we can control the other terms containing x in $L^\infty(\tilde{\Omega}_t)$. We also have $0 \le l_s \le 2$ so we can control $\|\zeta \nabla \partial_\theta^{l_s} \nabla f\|_{L^4(\tilde{\Omega}_t)}$ by lemma 3.8. We

therefore control the last term in (3.27). Let us now consider the first two term in (3.27) substituted into the first term in (3.29) and integrate by parts in θ to get modulo lower order terms

$$(3.28) \qquad + \int_{\tilde{\Omega}_t} \zeta \partial_\theta^4 g(\zeta \partial_\theta^6 f) dx - \int_{\tilde{\Omega}_t} (\nabla_i \partial_\theta^4 x^l)(\zeta \nabla_l \nabla_i f)(\zeta \partial_\theta^6 f) dx,$$

which is of the form we control. Let us now finally consider the first two term in (3.27) substituted into the second term in (3.29) we get modulo lower order terms

$$(3.29) \qquad - \int_{\tilde{\Omega}_t} \zeta \partial_\theta^5 g(\partial_\theta^5 x^l)(\zeta \partial_l f) dx + \int_{\tilde{\Omega}_t} (\partial_\theta \nabla_i \partial_\theta^4 x^k)(\zeta \nabla_k \nabla_i f)(\partial_\theta^5 x^l)(\zeta \partial_l f) dx,$$

This can now be estimated as in appendix A. ∎

From now on we will restrict the result to $\Omega_t \subset \tilde{\Omega}_t$. By Propositions above

COROLLARY 3.10 *If* $\triangle f = g$ *where* $g(x) = 0$ *when* $x \notin \Omega_t$ *then, for* $0 \leq j \leq 4$ *we have*

$$(3.30) \quad \|\zeta \nabla \partial_\theta^j \nabla f\|_{L^2(\Omega_t)} \lesssim P\big[\|x\|_{5.5}\big]\Big(\|\langle \partial_\theta \rangle^{5.5} g\|_{L^2(\tilde{\Omega}_t)} + \sum_{k=0}^{j} \|\partial_\theta^k g\|_{L^2(\Omega_t)} + \sum_{k=0}^{j-3} \|\partial_\theta^k g\|_{L^4(\Omega_t)} + \delta_{j4}\|g\|_{L^\infty(\Omega_t)}\Big)$$

LEMMA 3.11 *Let* f *satisfy* $\Delta f = g$ *in* Ω_t. *Writing* ∂ *to denote both* ∂_θ *and* ∂_r *we have*

$$(3.31) \qquad \|\zeta \nabla \partial^j \nabla f\|_{L^2(\Omega_t)} \leq P\big[\|x\|_5\big] \sum_{k=0}^{j} \|\zeta \partial_\theta \partial^k \nabla f\|_{L^2(\Omega_t)} + \sum_{k=0}^{j} \|\partial^k g\|_{L^2(\Omega_t)}$$

for $0 \leq j \leq 4$; *and for* $0 \leq j \leq 2$ *we have*

$$(3.32) \qquad \|\zeta \nabla \partial^j \nabla f\|_{L^\infty(\Omega_t)} \leq P\big[\|x\|_5\big] \sum_{k=0}^{j+1} \|\zeta \partial_\theta^2 \partial^k \nabla f\|_{L^2(\Omega_t)} + \sum_{k=0}^{j} \|\partial^k g\|_{L^\infty(\Omega_t)}.$$

PROOF: We prove this result by induction. For $j = 0$ we have $\|\zeta \nabla^2 f\|_{L^2(\Omega_t)} \leq \|\zeta \Delta f\|_{L^2(\Omega_t)} + \|\zeta \partial_\theta \nabla f\|_{L^2(\Omega_t)}$, by lemma 2.1, which is of the right form. Now suppose that $1 \leq j \leq 2$ and that we have (3.31) for smaller j. Then $|\zeta \nabla \partial^j \nabla f| \leq |\zeta \operatorname{div} \partial^j \nabla f| + |\zeta \operatorname{curl} \partial^j \nabla f| + |\zeta \partial_\theta \partial^j \nabla f|$. Now $\zeta \operatorname{div} \partial^j \nabla f = \zeta \partial^j \Delta f + \sum (\partial^k A)(\zeta \nabla \partial^l \nabla f)$ where the sum is over $k + l = j$ such that $l \leq j - 1 \leq 1$. We have $\|\zeta(\operatorname{div} \partial^j \nabla f - \partial^j \Delta f)\|_{L^2(\Omega_t)} \leq P\big[\|x\|_5\big]\|\zeta \nabla \partial^l \nabla f\|_{L^2(\Omega_t)}$ which we control by induction. Similarly, we also control $\|\zeta \operatorname{curl} \partial^j \nabla f\|_{L^2(\Omega_t)}$. For $j = 0$ we have

$$(3.33) \qquad \|\zeta \nabla \partial^j \nabla f\|_{L^\infty(\Omega_t)} \leq \|g\|_{L^\infty(\Omega_t)} + \|\zeta \partial_\theta \nabla f\|_{L^\infty(\Omega_t)}$$
$$(3.34) \qquad \leq \|g\|_{L^\infty(\Omega_t)} + \|\zeta \nabla \partial \partial_\theta \nabla f\|_{L^2(\Omega_t)}$$
$$(3.35) \qquad \leq P\big[\|x\|_5\big] \big[\|g\|_{L^\infty(\Omega_t)} + \|\zeta \partial \partial_\theta^2 \nabla f\|_{L^2(\Omega_t)}\big]$$

using Sobolev's inequality and lemma 3.8. Thus we have (3.32) for $j = 0$. Now suppose that $j = 3$. Then for $0 \leq k \leq 2$ we have $\|(\partial^k A)(\zeta \nabla \partial^l \nabla f)\|_{L^2(\Omega_t)} \leq \|x\|_5 \|\zeta \nabla \partial^l \nabla f\|_{L^2(\Omega_t)}$ and for $k = 3$ we have $l = 0$ and $\|(\partial^k A)(\zeta \nabla \partial^l \nabla f)\|_{L^2(\Omega_t)} \leq \|\partial^3 A\|_{L^2(\Omega_t)}\|\zeta \nabla^2 f\|_{L^\infty(\Omega_t)}$, both of which we control appropriately. Now we can prove (3.32) for $j = 1$ and using that result we can prove (3.31) for $j = 4$. And using that result we prove (3.32) for $j = 2$. ∎

Using lemma 3.11 and an induction argument we control $\|\zeta \nabla^s \phi\|_{L^2(\Omega_t)}$ for $0 \leq s \leq 5$ and hence we obtain theorem 3.1. ∎

3.2 Estimates for p.

Taking the divergence of (1.1) we see that p satisfies $\Delta p = -(\partial_i v^j)(\partial_j v^i) + 1$ on Ω_t and $p = 0$ on $\partial\Omega_t$. Thus we have the following:

PROPOSITION 3.12 *For all $i \geq 1$ we have*

$$(3.36) \qquad \|\eta_i \nabla p\|_{H^5(\Omega_t)} \leq P\big[\|x\|_5, \|V\|_5\big]$$

and

$$(3.37) \qquad \|\nabla p\|_{H^4(\Omega_t)} \leq P\big[\|x\|_5, \|V\|_4\big]$$

for $0 \leq j \leq 4$. Moreover, we have

$$(3.38) \qquad \|\nabla p\|_{H^{4.5}(\Omega_t)} \leq P\big[\|x\|_{5.5}, \|V\|_{4.5}\big]$$

and

$$(3.39) \qquad \|\nabla \dot{p}\|_{H^3(\Omega_t)} \leq P\big[\|x\|_{5.5}, \|V\|_5\big]$$

where $\dot{p} = \partial_t p$.

PROOF: The estimate (3.36) follows similarly to theorem 3.2. The estimates (3.37) and (3.38) follow similarly to theorem 3.1. The estimate (3.39) follows similarly to theorem 3.1 using theorem 3.1 and (3.37) above. For a detailed explanation of these proofs, see [12]. ∎

4 Energy estimates.

Finally, we are ready to prove the energy estimate in theorem 1.1. We control $\|V\|_5$ using lemma 2.2 and lemma 2.3, together with E_1 and E_2 below: Let

$$(4.1) \qquad E_1(t) = \big\|\zeta \partial_\theta^5 v\big\|_{L^2(\Omega_t)}^2 + \|\eta v\|_{H^5(\Omega_t)}^2$$

and $E_2(t) = \|\operatorname{curl}(v)\|_{H^{4.5}(\Omega_t)}$, where $\zeta = \zeta_1$ is the cut-off function supported near the boundary of Ω_t and $\eta = \eta_1$ is the cut-off function supported in the interior of Ω_t, as defined in section 2.4. Note that we will ignore terms which arise from the derivative falling on the cut-off function because these terms will be of lower order. To build regularity for $\|x\|_{5.5}$ we use lemma 2.3 together with E_3 and E_4 below: Let

$$(4.2) \qquad E_3(t) = \int_{\partial\Omega_t} (-\nabla p \cdot N)\big[(\partial_\theta^5 x) \cdot N\big]^2 \, dS(x),$$

where N is the external unit normal to $\partial\Omega_t$. Note that the term in (4.2) should read $\int_{\partial\Omega_t}(-\nabla p \cdot N)\big[(\partial_\theta^5 x) \circ x^{-1} \cdot N\big]^2 dS(x)$, but in the interest of creating tidy computations we will hereinafter omit some terms − such as the '$\circ x^{-1}$' above − which are not crucial to understanding the argument. And let

$$E_4(t) = \|\operatorname{div}[\partial x]\|_{H^{3.5}(\Omega_t)}^2 + \|\operatorname{curl}[\partial x]\|_{H^{3.5}(\Omega_t)}^2$$

where $\operatorname{div}[\partial x] = \partial_i[(\partial x^i) \circ x^{-1}]$ and ∂ is an arbitrary derivative in the Lagrangian frame. We now explain how E_3 and E_4 will be used to bound $\|x\|_{5.5}$. Let $\sum x_k e^{ik\theta}$ be the tangential Fourier expansion of x. Then $(\langle\partial_\theta\rangle^5 x) \cdot N = \sum \langle k\rangle^5 x_k \cdot N e^{ik\theta}$ and $(\partial_\theta^j x) \cdot N = \sum (ik)^j x_k \cdot N e^{ik\theta}$. Therefore

$$(4.3) \qquad \|(\langle\partial_\theta\rangle^5 x) \cdot N\|_{L^2(\partial\Omega_t)}^2 \leq \sum \langle k\rangle^{10} |x_k \cdot N|^2 \leq \sum |x_k \cdot N|^2 + \sum k^2 |x_k \cdot N|^2 \ldots + \sum k^{10} |x_k \cdot N|^2$$

and

$$(4.4) \qquad \sum k^{2j} |x_k \cdot N|^2 \leq \|(\partial_\theta^j x) \cdot N\|^2_{L^2(\partial\Omega_t)}.$$

Using the trace theorem and the fact that $x(t,y) = y + \int_{[0,t]} V(s,y)ds$ we control the terms with $0 \leq j \leq 4$. For the highest order term we have

$$(4.5) \qquad \|(\partial_\theta^5 x) \cdot N\|^2_{L^2(\partial\Omega_t)} = \int_{\partial\Omega_t} \left[(\partial_\theta^5 x) \cdot N \right]^2 dS(x) \leq \frac{1}{c_0} \int_{\partial\Omega_t} (-\nabla p \cdot N) \left[(\partial_\theta^5 x) \cdot N \right]^2 dS(x) \leq \frac{E_3}{c_0}.$$

Using E_4 we control $\|\mathrm{div}\,[\partial x]\|^2_{H^{3.5}(\Omega_t)}$ and $\|\mathrm{curl}\,[\partial x]\|^2_{H^{3.5}(\Omega_t)}$, and therefore, from lemma 2.3, we have, by (4.5),

$$(4.6) \qquad \|\partial\partial_\theta x\|^2_{H^{4.5}(\Omega_t)} \leq P\big[\|x\|_5\big]\left[E_4 + \frac{E_3}{c_0} \right].$$

This means that we similarly control $\|\partial_\theta \partial x\|^2_{H^{3.5}(\Omega_t)}$ and therefore $\|\partial x\|^2_{H^4(\partial\Omega_t)}$. Hence we control $\|\partial x\|^2_{4.5}$ and therefore $\|x\|_{5.5}$.

4.1   Almost E_1.

The time derivative of E_1 is equal to

$$(4.7) \quad -2\int_{\Omega_t} (\zeta\partial_\theta^5 v^i)(\zeta\partial_\theta^5 \partial_i p)dx - 2\int_{\Omega_t} (\zeta\partial_\theta^5 v^i)(\zeta\partial_\theta^5 \partial_i \phi)dx - 2\int_{\Omega_t} (\eta\partial_\theta^5 v^i)(\zeta\partial_\theta^5 \partial_i p)dx - 2\int_{\Omega_t} (\eta\partial_\theta^5 v^i)(\zeta\partial_\theta^5 \partial_i \phi)dx$$

using (1.1). Using proposition 3.2 and (3.36) from proposition 3.12 we control the third and fourth term in (4.7). The second term in (4.7) can be controlled using theorem 3.4. It now remains to control the first term in (4.7). We will deal with this term in section 4.3.1.

4.2 E_2.

We have $[\partial_t, \partial_i]x^j = -(\partial_i v^j)$ and therefore

$$(4.8) \qquad \partial_t \mathrm{curl}\,(v) = \partial_1\partial_t v_2 - \partial_2\partial_t v_1 - (\partial_1 v^j)(\partial_j v_2) + (\partial_2 v^k)(\partial_k v_1)$$
$$(4.9) \qquad \qquad = -(\partial_1 v_1)(\partial_1 v_2)(\partial_1 v_2)(\partial_2 v_2) + (\partial_2 v_1)(\partial_1 v_1) + (\partial_2 v_2)(\partial_2 v_1)$$
$$(4.10) \qquad \qquad = -(\partial_1 v_2)[(\partial_1 v_1) + (\partial_2 v_2)] + (\partial_2 v_1)[(\partial_1 v_1) + (\partial_2 v_2)]$$
$$(4.11) \qquad \qquad = -\mathrm{curl}\,(v)\mathrm{div}\,(v)$$
$$(4.12) \qquad \qquad = 0.$$

Thus $\mathrm{curl}\,(v)(t) = \mathrm{curl}\,(v)(0)$ and therefore $\|\mathrm{curl}\,(v)(t)\|_{H^{4.5}(\Omega_t)} = \|\mathrm{curl}\,(v)(0)\|_{H^{4.5}(\Omega_t)}$.

4.3 E_3.

The time derivative of E_3 is equal to

$$(4.13) \qquad \int_{\partial\Omega_t} \partial_t |\nabla p| \left[(\partial_\theta^5 x) \cdot N \right]^2 dS(x) + 2\int_{\partial\Omega_t} |\nabla p| \left[(\partial_\theta^5 x) \cdot (\partial_t N) \right] \left[(\partial_\theta^5 x) \cdot N \right] dS(x)$$

$$(4.14) \qquad + 2\int_{\partial\Omega_t} |\nabla p| \left[(\partial_t \partial_\theta^5 x) \cdot N \right] \left[(\partial_\theta^5 x) \cdot N \right] dS(x).$$

In (4.13) we control the first and second term. Using the fact that $\frac{-\partial_i p}{|\nabla p|} = N_i$ on the third term in (4.13) we have

$$(4.15) \qquad -2\int_{\partial\Omega_t}(\partial_\theta^5 v^i)N_i(\partial_\theta^5 x^j)(\partial_j p)dS(x) = -2\int_{\Omega_t}(\partial_i\partial_\theta^5 v^i)(\partial_\theta^5 x^j)(\partial_j p)dx - 2\int_{\Omega_t}(\partial_\theta^5 v^i)(\partial_i\partial_\theta^5 x^j)(\partial_j p)dx$$

$$(4.16) \qquad\qquad\qquad\qquad -2\int_{\Omega_t}(\partial_\theta^5 v^i)(\partial_\theta^5 x^j)(\partial_i\partial_j p)dx$$

using the divergence theorem. We control the third term in (4.15). In the first term in (4.15) we commute a ∂_θ outside the ∂_i this generates a lower order term and also

$$(4.17) \qquad\qquad -2\int_{\Omega_t}(\partial_\theta\partial_i\partial_\theta^4 v^i)(\partial_\theta^5 x^j)(\partial_j p)dx \le \|\langle\partial_\theta\rangle^{\frac{1}{2}}\partial_i\partial_\theta^4 v^i\|\|\langle\partial_\theta\rangle^{\frac{1}{2}}[(\partial_\theta^5 x)(\nabla p)]\|$$

using lemma A.4. By lemma A.1 and the fact that $\operatorname{div}[v] = 0$, we can control the above. The second term in (4.15) remains. We deal with this term in the next section.

4.3.1 The rest.

Combining the first term from (4.7) and the second term from (4.15) gives

$$(4.18) \qquad\qquad = -2\int_{\Omega_t}(\zeta\partial_\theta^5 v^i)(\zeta\partial_i\partial_\theta^5 p)dx + 2\sum\int_{\Omega_t}\zeta^2(\partial_\theta^5 v)(\nabla\partial_\theta^{k_1}x)\ldots(\nabla\partial_\theta^{k_{s-1}}x)(\partial_\theta^{k_s}\nabla p)$$

where $k_1 + \ldots + k_s = 5$ and $k_1,\ldots,k_s \le 4$. In the first term in (4.18) we integrate by parts to obtain

$$(4.19) \qquad\qquad\qquad\qquad \int_{\Omega_t}(\zeta\partial_i\partial_\theta^5 v^i)(\zeta\partial_\theta^5 p)dx.$$

Again, we integrate half of one of the ∂_θ from $\operatorname{div}\partial_\theta^5 v$ to the other side. The result can be controlled by (3.38) from proposition 3.12 and an argument from above. We can control the sum in (4.18) using (3.37) from proposition 3.12.

4.4 E_4.

First we deal with the divergence term. We have

$$(4.20) \qquad\qquad \partial_t\operatorname{div}[\partial x] = (\nabla v)(\partial^2 x) + \operatorname{div}[\partial_t\partial x] = (\nabla v)(\partial^2 x) + \nabla\operatorname{div} v.$$

Therefore we have an equation of the form $\partial_t f = g$. Since $H^{3.5}(\Omega_t)$ is an algebra, we control the first term in (4.20) by $\|V\|_{4.5}\|x\|_{5.5}$. We now consider two time derivatives on $\operatorname{curl}[\partial x]$:

$$(4.21) \qquad \partial_t^2\operatorname{curl}[\partial x] = \partial_t\left[(\nabla v)(\partial^2 x) + \operatorname{curl}\partial_t[\partial x]\right]$$

$$(4.22) \qquad\qquad\qquad = \partial_t\left[(\nabla v)(\partial^2 x)\right] + (\nabla v)(\nabla^2 v) + \operatorname{curl}\partial_t^2[\partial x]$$

$$(4.23) \qquad\qquad\qquad = \partial_t\left[(\nabla v)(\partial^2 x)\right] - (\nabla\partial_t v)(\partial^2 x) + (\partial x)(\partial\partial_t^2 x)$$

since $\operatorname{curl}\partial_t^2 x = 0$. Equation (4.23) is of the form $\partial_t[(\partial_t f) - g] = h$. Integrating with respect to time once yields $(\partial_t f)(t) = (\partial_t f)(0) + g(t) - g(0) + \int_{[0,t]}h(u)du$. Another integration with respect to time again gives

$$(4.24) \qquad\qquad f(t) = f(0) + t(\partial_t f)(0) + \int_{[0,t]}g(u)du - tg(0) + \int_{[0,t]}\int_{[0,u_2]}h(u_1)du_1 du_2.$$

Here $f = \operatorname{curl}[\partial x]$ so we control $f(0)$ and $(\partial_t f)(0)$. We have already seen that we can control the first term in (4.23). The second term in (4.23) can be controlled using the fact that $H^{3.5}(\Omega_t)$ is an algebra, (3.38) from proposition 3.12 and theorem 3.1.

A   Properties of $\langle \partial_\theta \rangle$.

In this section we prove a result concerning how $\langle \partial_\theta \rangle^{\frac{1}{2}}$ acts on a product and also an integration by parts type result.

LEMMA A.1 *Let f and g be functions on Ω. Then $\|\langle \partial_\theta \rangle^{\frac{1}{2}}[fg] - \langle \partial_\theta \rangle^{\frac{1}{2}}[f]g\|^2 \le c\|f\|^2 \|\langle \partial_\theta \rangle^{\frac{1}{2}+a}g\|^2$ for $a > \frac{1}{2}$.*

PROOF: Let $\sum f_k(\rho)e^{ik\theta}$ and $\sum g_l(\rho)e^{il\theta}$ be tangential Fourier expansions of $f(\rho, \theta)$ and $g(\rho, \theta)$ respectively. Then

$$fg = \sum_k \sum_l f_k g_l e^{i[k+l]\theta} = \sum_m \left[\sum_{k+l=m} f_k g_l \right] e^{im\theta}$$

and therefore

(A.1)
$$\langle \partial_\theta \rangle^{\frac{1}{2}}[fg] = \sum_m \langle m \rangle^{\frac{1}{2}} \left[\sum_{k+l=m} f_k g_l \right] e^{im\theta}$$

where $\langle m \rangle = [1 + |m|^2]^{\frac{1}{2}}$. Also $\langle \partial_\theta \rangle^{\frac{1}{2}}[f] = \sum_k \langle k \rangle^{\frac{1}{2}} f_k e^{ik\theta}$ and therefore

(A.2)
$$\langle \partial_\theta \rangle^{\frac{1}{2}}[f]g = \sum_m \left[\sum_{k+l=m} \langle k \rangle^{\frac{1}{2}} f_k g_l \right] e^{im\theta}.$$

The difference between (A.1) and (A.2) is

(A.3)
$$\sum_m \sum_{k+l=m} \left[\langle k+l \rangle^{\frac{1}{2}} - \langle k \rangle^{\frac{1}{2}} \right] f_k g_l e^{im\theta}.$$

We can control this using the following lemma.

LEMMA A.2 *Let k and l be points in $\mathbf{Z}$. Then*

$$\left| \langle k+l \rangle^{\frac{1}{2}} - \langle k \rangle^{\frac{1}{2}} \right| \le c\langle l \rangle^{\frac{1}{2}},$$

where c is a constant.

PROOF: Suppose that k and l are such that $0 \le |k| \le |l|$. Then

$$\left| \langle k+l \rangle^{\frac{1}{2}} - \langle k \rangle^{\frac{1}{2}} \right| \le c\langle l \rangle^{\frac{1}{2}}.$$

Now suppose that k and l are such that $0 \le |l| < |k|$. Then

$$
\begin{aligned}
\langle k \rangle^{\frac{1}{2}} \left| \frac{(1+(k+l)(k+l))^{\frac{1}{4}}}{(1+k^2)^{\frac{1}{4}}} - 1 \right| &= \langle k \rangle^{\frac{1}{2}} \left| \left(\frac{1+k^2+2kl+l^2}{1+k^2} \right)^{\frac{1}{4}} - 1 \right| \\
&= \langle k \rangle^{\frac{1}{2}} \left| \left(1 + \frac{2kl+l^2}{1+k^2} \right)^{\frac{1}{4}} - 1 \right|.
\end{aligned}
$$

Define $f(x) = (1+x)^{\frac{1}{4}} - 1$. Then there is a constant c which bounds $\frac{f(x)}{|x|}$, for all x in $(-4, 4)$. Therefore,

(A.4)
$$\langle k \rangle^{\frac{1}{2}} \left| \left(1 + \frac{2kl+l^2}{1+k^2} \right)^{\frac{1}{4}} - 1 \right| \le c\langle k \rangle^{\frac{1}{2}} \frac{2kl+l^2}{\langle k \rangle^2} \le c\langle k \rangle^{\frac{1}{2}} \frac{\langle k \rangle \langle l \rangle + \langle l \rangle^2}{\langle k \rangle^2} \le c\frac{\langle l \rangle}{\langle k \rangle^{\frac{1}{2}}}.$$

Since $|l| < |k|$, $\frac{\langle k \rangle^{\frac{1}{2}}}{\langle l \rangle^{\frac{1}{2}}} > 1$, from where the result follows.   ■

And from lemma A.2 we see that (A.3) can be estimated in $L^2(\Omega)$ by

$$\text{(A.5)} \qquad \sum_m \left[\sum_{k+l=m} \langle l \rangle^{\frac{1}{2}} \frac{\langle l \rangle^a}{\langle l \rangle^a} |f_k||g_l| \right]^2 \leq \sum_m \left[\sum_{k+l=m} \langle l \rangle^{2(\frac{1}{2}+a)} |g_l|^2 \right] \left[\sum_{k+l=m} \langle l \rangle^{-2a} |f_k|^2 \right].$$

Since $\sum \langle l \rangle^{-2a}$ is convergent for $2a > 1$ we must have $a > \frac{1}{2}$. This proves lemma A.1.   ■

¿From this proof we also have the following result:

COROLLARY A.3 *Let f and g be functions on $\partial\Omega$. Then*

$$\text{(A.6)} \qquad \|\langle \partial_\theta \rangle^{\frac{1}{2}} [fg] - \langle \partial_\theta \rangle^{\frac{1}{2}} [f] g \|^2_{L^2(\partial\Omega)} \leq c\|f\|^2_{L^2(\partial\Omega)} \|\langle \partial_\theta \rangle^{\frac{1}{2}+a} g\|^2_{L^2(\partial\Omega)}$$

for $a > \frac{1}{2}$.

LEMMA A.4 *Let f and g be functions on Ω. and let $(\ ,\ )$ be the $L^2(\Omega)$-inner product, then*
$$|(f, \partial_\theta g)| \leq c\|\langle \partial_\theta \rangle^{\frac{1}{2}} f\| \|\langle \partial_\theta \rangle^{\frac{1}{2}} g\|.$$

PROOF: Let $\sum f_k(\rho) e^{ik\theta}$ and $\sum g_l(\rho) e^{il\theta}$ be tangential Fourier expansions of $f(\rho, \theta)$ and $g(\rho, \theta)$ respectively. Then

$$\text{(A.7)} \qquad |(f, \partial_\theta g)| = \left| \int_0^1 \int_0^{2\pi} f(\rho, \theta) g(\rho, \theta) \rho d\rho d\theta \right|$$

$$\text{(A.8)} \qquad \leq \left| \int_0^1 \sum_l \langle l \rangle^{\frac{1}{2}} f_l(\rho) \langle l \rangle^{\frac{1}{2}} g_l(\rho) \rho d\rho \right|$$

$$\text{(A.9)} \qquad \leq \|\langle \partial_\theta \rangle^{\frac{1}{2}} f\| \|\langle \partial_\theta \rangle^{\frac{1}{2}} g\|.$$

                                                                    ■

B Hodge-decomposition inequalities.

In this section we prove the results whose proofs were omitted in the body of the text. We begin with a lemma which says that in the support of ζ we can control all derivatives by the curl the divergence and a tangential derivative.

LEMMA B.1 *Let α be a vector-field on $\tilde{\Omega}_t$. Define $(curl\,\alpha)_{jk} = \partial_j \alpha_k - \partial_k \alpha_j$ and $div\,\alpha = \partial_j \alpha^j$. Then we have the following pointwise estimate on Ω_t:*

$$\text{(B.1)} \qquad |\zeta \nabla \alpha| \leq |\zeta\,curl\,\alpha| + |\zeta\,div\,\alpha| + |\zeta\,\partial_\theta \alpha|,$$

where $|\cdot|$ denotes the usual Euclidean distance.

PROOF: Here we will suppress the index on ζ, letting it be denoted simply by ζ. Define $(def\,\alpha)_{jk} = \partial_j \alpha_k + \partial_k \alpha_j$. Thus $2\nabla\alpha = curl\,\alpha + def\,\alpha$. Let $\beta = diag(\partial_1 \alpha_1, \dots, \partial_n \alpha_n)$ and define $\gamma = \zeta def\,\alpha - \zeta\beta$. Then $|\zeta \nabla \alpha| \leq |\zeta curl\,\alpha| + |\zeta div\,\alpha| + |\gamma|$. It remains to control γ. Also define

$$\text{(B.2)} \qquad Q^{jk} = \delta^{jk} - N^j N^k,$$

the projection onto tangential vector-fields. Hence

$$
\begin{aligned}
|\gamma|^2 &= \delta^{ij}\delta^{kl}\gamma_{ik}\gamma_{jl}\\
&= \left(Q^{ij}+N^iN^j\right)\left(Q^{kl}+N^kN^l\right)\gamma_{ik}\gamma_{jl}\\
&= Q^{ij}Q^{kl}\gamma_{ik}\gamma_{jl}+Q^{ij}N^kN^l\gamma_{ik}\gamma_{jl}+N^iN^jQ^{kl}\gamma_{ik}\gamma_{jl}\\
&+ N^iN^jN^kN^l\gamma_{ik}\gamma_{jl}.
\end{aligned}
$$

Since γ is symmetric, $N^iN^jQ^{kl}\gamma_{ik}\gamma_{jl}=Q^{ij}N^kN^l\gamma_{ik}\gamma_{jl}$. Also,

$$
(B.3)\qquad N^iN^jN^kN^l\gamma_{ik}\gamma_{jl}=[N^iN^k\gamma_{ik}]^2=[\delta^{ik}\gamma_{ik}-Q^{ik}\gamma_{ik}]^2=[Q^{ik}\gamma_{ik}]^2\le Q^{ij}Q^{kl}\gamma_{ik}\gamma_{jl},
$$

since for a symmetric matrix M we have $[\mathrm{Tr}(M)]^2\le c\mathrm{Tr}(M^2)$. ¿From (B.3),

$$
\begin{aligned}
Q^{ij}Q^{kl}\gamma_{ik}\gamma_{jl}+Q^{ij}N^kN^l\gamma_{ik}\gamma_{jl} &+ N^iN^jQ^{kl}\gamma_{ik}\gamma_{jl}+N^iN^jN^kN^l\gamma_{ik}\gamma_{jl}\\
&\le Q^{ij}Q^{kl}\gamma_{ik}\gamma_{jl}+2Q^{ij}N^kN^l\gamma_{ik}\gamma_{jl}+cQ^{ij}Q^{kl}\gamma_{ik}\gamma_{jl}\\
&\le 2cQ^{ij}(Q^{kl}+N^kN^l)\gamma_{ik}\gamma_{jl}\\
&= 2cQ^{ij}\delta^{kl}\gamma_{ik}\gamma_{jl}.
\end{aligned}
$$

Using the fact that $\gamma=\zeta\mathrm{def}\,\alpha-\zeta\beta$ we have

$$
(B.4)\quad Q^{ij}\delta^{kl}\gamma_{ik}\gamma_{jl}=Q^{ij}\delta^{kl}(\zeta\mathrm{def}\,\alpha)_{ik}(\zeta\mathrm{def}\,\alpha)_{jl}+Q^{ij}\delta^{kl}(\zeta\mathrm{def}\,\alpha)_{ik}\zeta\beta_{jl}+Q^{ij}\delta^{kl}\zeta\beta_{ik}(\zeta\mathrm{def}\,\alpha)_{jl}+Q^{ij}\delta^{kl}\zeta\beta_{ik}\zeta\beta_{jl}
$$

where the second and third term can be controlled by $\varepsilon|\zeta\nabla\alpha|^2+\frac{1}{\varepsilon}|\zeta\mathrm{div}\,\alpha|^2$ and the fourth term can be controlled by $|\zeta\mathrm{div}\,\alpha|^2$. The first term in (B.4) can be controlled as follows:

$$
\begin{aligned}
(B.5)\qquad Q^{ij}\delta^{kl}(\zeta\mathrm{def}\,\alpha)_{ik}(\zeta\mathrm{def}\,\alpha)_{jl} &= Q^{ij}\delta^{kl}(\zeta\partial_i\alpha_k+\zeta\partial_k\alpha_i)(\zeta\partial_j\alpha_l+\zeta\partial_l\alpha_j)\\
(B.6)\qquad &= Q^{ij}\delta^{kl}(\zeta\partial_i\alpha_k)(\zeta\partial_j\alpha_l)+Q^{ij}\delta^{kl}(\zeta\partial_i\alpha_k)(\zeta\partial_l\alpha_j)\\
(B.7)\qquad &+ Q^{ij}\delta^{kl}(\zeta\partial_k\alpha_i)(\zeta\partial_j\alpha_l)+Q^{ij}\delta^{kl}(\zeta\partial_k\alpha_i)(\zeta\partial_l\alpha_j).
\end{aligned}
$$

Let $\nabla_Q^i=Q^{ij}\partial_j$. Since $Q^{ij}=\delta_{mn}Q^{im}Q^{jn}$, the first term in (B.6) can be bounded by $|\zeta\nabla_Q[\alpha]|^2$. The second and third term in (B.6) can be bounded by $\varepsilon|\zeta\nabla\alpha|^2+\frac{1}{\varepsilon}|\zeta\nabla_Q[\alpha]|^2$. The fourth term we manipulate as follows: $Q^{ij}\delta^{kl}(\zeta\partial_k\alpha_i)(\zeta\partial_l\alpha_j)=\delta_{mn}Q^{mi}(\zeta\partial_k\alpha_i)Q^{nj}(\zeta\partial^k\alpha_j)$ and

$$
\begin{aligned}
(B.8)\qquad Q^{mi}(\zeta\partial_k\alpha_i) &= Q^{mi}(\zeta\partial_i\alpha_k)+Q^{mi}[\zeta\partial_k\alpha_i-\zeta\partial_i\alpha_k]\\
(B.9)\qquad &= \zeta\nabla_Q^m[\alpha_k]+Q^{mi}(\zeta\mathrm{curl}\,\alpha)_{ki}.
\end{aligned}
$$

Thus the fourth term in (B.6) can be controlled by $(1+\frac{1}{\varepsilon})|\zeta\nabla_Q[\alpha]|^2+|\zeta\mathrm{curl}\,\alpha|^2+\varepsilon|\zeta\nabla\alpha|^2$. This concludes the proof. ∎

¿From lemma B.1 we have the following result:

Lemma B.2 *For* $1\le s\le 5$,

$$
(B.10)\qquad \|\zeta\alpha\|_{H^s(\Omega_t)}\le P[\|x\|_5]\left[\|\zeta\alpha\|_{L^2(\Omega_t)}+\|\zeta\,curl\,\alpha\|_{H^{s-1}(\Omega_t)}+\|\zeta\,div\,\alpha\|_{H^{s-1}(\Omega_t)}+\sum_{j=1}^s\|\zeta\partial_\theta^j\alpha\|_{L^2(\Omega_t)}\right].
$$

Proof: The base case is when $s=1$ we have on Ω_t, according to lemma B.1, $\|\zeta\nabla\alpha\|\le\|\zeta\mathrm{curl}\,\alpha\|_{L^2(\Omega_t)}+\|\zeta\mathrm{div}\,\alpha\|_{L^2(\Omega_t)}+\|\zeta\partial_\theta\alpha\|_{L^2(\Omega_t)}$, which means that (B.10) holds. Now suppose that $s=5$ and that we have the result for smaller s. Then, by lemma B.1 we see that

$$
(B.11)\qquad \|\zeta\nabla^5\alpha\|_{L^2(\Omega_t)}\le\|\zeta\mathrm{curl}\,\alpha\|_{H^4(\Omega_t)}+\|\zeta\mathrm{div}\,\alpha\|_{H^4(\Omega_t)}+\|\zeta\partial_\theta\nabla^4\alpha\|_{L^2(\Omega_t)}.
$$

To manipulate the second to last term in (B.11) we write

$$\text{(B.12)} \qquad \zeta \nabla^4 \partial_\theta \alpha - \zeta \partial_\theta \nabla^4 \alpha = \sum (\nabla^j \partial_\theta x)(\zeta \nabla^{k+1} \alpha)$$

summing over $j + k = 4$ such that $k \leq 3$. For $0 \leq j \leq 2$ we have $\|\nabla^j \partial_\theta x\|_{L^\infty(\Omega_t)} \leq \|x\|_5$ and we control $\|\zeta \nabla^{k+1} \alpha\|_{L^2(\Omega_t)}$ by induction. For $3 \leq j \leq 4$ we have $\|\nabla^j \partial_\theta x\|_{L^2(\Omega_t)} \leq \|x\|_5$ and $\|\zeta \nabla^2 \alpha\|_{L^\infty(\Omega_t)} \leq P[\|x\|_5]\|\zeta\alpha\|_{H^4(\Omega_t)}$ by Sobolev's inequality. This we control by induction. Now

$$\text{(B.13)} \qquad \|\zeta \nabla^4 \partial_\theta \alpha\|_{L^2(\Omega_t)} \leq \|\zeta \operatorname{curl} \nabla^3 \partial_\theta \alpha\|_{L^2(\Omega_t)} + \|\zeta \operatorname{div} \nabla^3 \partial_\theta \alpha\|_{L^2(\Omega_t)} + \|\zeta \partial_\theta \nabla^3 \partial_\theta \alpha\|_{L^2(\Omega_t)}$$

$$\text{(B.14)} \qquad \leq \|\zeta \nabla^3 [(\nabla \partial_\theta x)(\nabla \alpha)]\|_{L^2(\Omega_t)} + \|\zeta \nabla^3 \partial_\theta \operatorname{curl} \alpha\|_{L^2(\Omega_t)} + \|\zeta \nabla^3 \partial_\theta \operatorname{div} \alpha\|_{L^2(\Omega_t)}$$

$$\text{(B.15)} \qquad + \|\zeta \partial_\theta \nabla^3 \partial_\theta \alpha\|_{L^2(\Omega_t)}.$$

The first term in (B.14) is controlled by

$$\text{(B.16)} \qquad \sum \|(\nabla^{j+1} \partial_\theta x)(\zeta \nabla^{k+1} \alpha)\|_{L^2(\Omega_t)}$$

where the sums is over $j + k = 3$. This term can be controlled by $\|x\|_5 \|\zeta\alpha\|_{H^4(\Omega_t)}$. We control the second term in (B.14) by

$$\text{(B.17)} \qquad \sum \|(\nabla^j \partial_\theta x)(\zeta \nabla^{k+1} \operatorname{curl} \alpha)\|_{L^2(\Omega_t)}$$

where the sum is over $j + k = 3$. We control this term by $\|x\|_5 \|\zeta \operatorname{curl} \alpha\|_{H^4(\Omega_t)}$. Similarly for the third term in (B.14). $\blacksquare$

In this section we prove the following lemma.

LEMMA B.3 *Let $\operatorname{div}\alpha$ and $\operatorname{curl}\alpha$ be defined as in lemma B.1. Then, for $s \leq 5$,*

$$\text{(B.18)} \qquad \|\alpha\|_{H^s(\Omega_t)} \leq P\big[\|x\|_5\big] \left[\|\alpha\|_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{curl}\alpha\|_{H^{s-1}(\Omega_t)} + \|(\langle \partial_\theta \rangle^{s-\frac{1}{2}} \alpha) \cdot N\|_{L^2(\partial\Omega_t)} \right],$$

where N is the outward unit normal to $\partial\Omega_t$ and where $p(s)$ is a polynomial which depends on s. Also, for $s \leq 5$,

$$\text{(B.19)} \qquad \|\alpha\|_{H^s(\Omega_t)} \leq P\big[\|x\|_5\big] \left[\|\alpha\|_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{curl}\alpha\|_{H^{s-1}(\Omega_t)} + \|(\langle \partial_\theta \rangle^{s-\frac{1}{2}} \alpha) \cdot Q\|_{L^2(\partial\Omega_t)} \right]$$

where Q is a unit vector which is tangent to $\partial\Omega_t$.

PROOF: First we prove (B.18) and (B.19) for $s = 1$, then we will use lemma B.2 to obtain the higher order results. Finally, we will use interpolation to obtain the result for real s. Now

$$\text{(B.20)} \qquad \|\nabla\alpha\|_{L^2(\Omega_t)} = \int_{\Omega_t} \partial_j \alpha_i \partial^j \alpha^i \, dx = \int_{\partial\Omega_t} \alpha_i N_j \partial^j \alpha^i \, dS(x) - (\alpha, \Delta\alpha)_{\Omega_t}$$

where we define $(\alpha, \Delta\alpha)_{\Omega_t} = \int_{\Omega_t} \alpha_i \partial_j \partial^j \alpha^i \, dx$. And

$$\text{(B.21)}$$

$$-(\alpha, \Delta\alpha)_{\Omega_t} = - \int_{\Omega_t} \alpha_i \left[\partial^i \partial^j \alpha_j + \partial_j \partial^j \alpha^i - \partial^i \partial^j \alpha_j \right] dx = \int_{\Omega_t} \alpha_i \left[-\partial^i \operatorname{div}\alpha + \partial^j (\operatorname{curl}\alpha)^i_j \right] dx$$

$$\text{(B.22)} \qquad = - \int_{\partial\Omega_t} N^i \alpha_i \operatorname{div}\alpha \, dS(x) + \int_{\Omega_t} [\operatorname{div}\alpha]^2 \, dx + \int_{\partial\Omega_t} \alpha_i N^j (\operatorname{curl}\alpha)^i_j \, dS(x) - \int_{\Omega_t} \partial^j \alpha_i (\operatorname{curl}\alpha)^i_j \, dx.$$

Also,

(B.23)
$$-\int_{\Omega_t} \partial^j \alpha_i \, (\mathrm{curl}\,\alpha)_j^i \, dx = -\int_{\Omega_t} (\mathrm{curl}\,\alpha)_i^j \, (\mathrm{curl}\,\alpha)_j^i \, dx - \int_{\Omega_t} \partial_i \alpha^j \, (\mathrm{curl}\,\alpha)_j^i \, dx$$

and

(B.24)
$$-\int_{\Omega_t} \partial_i \alpha^j \, (\mathrm{curl}\,\alpha)_j^i \, dx = -\int_{\partial\Omega_t} N_i \alpha^j \, (\mathrm{curl}\,\alpha)_j^i \, dS(x) + \int_{\Omega_t} \alpha^j \partial_i \, (\mathrm{curl}\,\alpha)_j^i \, dx.$$

Moreover,

(B.25)
$$\int_{\Omega_t} \alpha^j \partial_i \, (\mathrm{curl}\,\alpha)_j^i \, dx = \int_{\Omega_t} \alpha^j \partial_i \left[\partial^i \alpha_j - \partial_j \alpha^i \right] dx$$

(B.26)
$$= (\alpha, \Delta\alpha)_{\Omega_t} - \int_{\Omega_t} \alpha^j \partial_i \partial_j \alpha^i \, dx$$

(B.27)
$$= (\alpha, \Delta\alpha)_{\Omega_t} - \int_{\partial\Omega_t} \alpha^j N_j \partial_i \alpha^i \, dS(x) + \int_{\Omega_t} [\mathrm{div}\,\alpha]^2 \, dx.$$

From (B.27) we see that

(B.28)
$$-2 \, (\alpha, \Delta\alpha)_{\Omega_t} = 2 \int_{\Omega_t} [\mathrm{div}\,\alpha]^2 \, dx - \int_{\Omega_t} (\mathrm{curl}\,\alpha)_i^j \, (\mathrm{curl}\,\alpha)_j^i \, dx$$

(B.29)
$$- 2 \int_{\partial\Omega_t} \alpha \cdot N \mathrm{div}\,\alpha \, dS(x) + \int_{\partial\Omega_t} \left(\alpha_i N^j - N_i \alpha^j \right) (\mathrm{curl}\,\alpha)_j^i \, dS(x).$$

The boundary terms from (B.20) and (B.28) are

(B.30)
$$\int_{\partial\Omega_t} \alpha_i N_j \partial^j \alpha^i \, dS(x) - \int_{\partial\Omega_t} \alpha \cdot N \mathrm{div}\,\alpha \, dS(x) + \frac{1}{2} \int_{\partial\Omega_t} \left(\alpha_i N^j - N_i \alpha^j \right) (\mathrm{curl}\,\alpha)_j^i \, dS(x).$$

The second term in (B.30) can be manipulated using Q: On $\partial\Omega_t$, $\alpha = \alpha \cdot NN + Q\alpha$ and therefore

(B.31)
$$-\alpha \cdot N \mathrm{div}\,\alpha = -(\partial_i N^i)[\alpha \cdot N]^2 - \alpha \cdot N \nabla_N [\alpha \cdot N] - \alpha \cdot NN_i \nabla_N [Q^i \alpha] - \alpha \cdot N \nabla_{Qi} [Q^i \alpha]$$

where $\nabla_N = N^i \partial_i$. In the above, $-\alpha \cdot NN_i \nabla_N [Q^i \alpha] = -[\alpha \cdot N]^2 N_i \nabla_N [Q^{ij}] N_j - \alpha \cdot NN_i \nabla_N [Q^{ij}] Q_j \alpha$. And the third term from (B.30) we manipulate as follows:

(B.32)
$$\frac{1}{2} \left(\alpha_i N^j - N_i \alpha^j \right) (\mathrm{curl}\,\alpha)_j^i \;=\; \frac{1}{2} \left(\alpha_i N^j - N_i \alpha^j \left(\partial^i \alpha_j - \partial_j \alpha^i \right) \right)$$

(B.33)
$$=\; \frac{1}{2} \left[\alpha_i N^j \partial^i \alpha_j - \alpha_i N^j \partial_j \alpha^i - N_i \alpha^j \partial^i \alpha_j + N_i \alpha^j \partial_j \alpha^i \right]$$

(B.34)
$$=\; \alpha_i N^j \partial^i \alpha_j - \alpha_i N^j \partial_j \alpha^i.$$

The second term in (B.34) cancels the first term in (B.30). The first term in (B.34) we deal with as follows:

(B.35)
$$\alpha_i N^j \partial^i \alpha_j = \alpha_i \partial^i [\alpha \cdot N] - \alpha_i \alpha_j (\partial^i N^j) = \alpha \cdot N \nabla_N [\alpha \cdot N] + Q_i \alpha \nabla_Q^i [\alpha \cdot N] - \alpha_i \alpha_j (\partial^i N^j).$$

The first term in (B.35) cancels the second term in (B.31). The remaining terms therefore are

(B.36)
$$\int_{\partial\Omega_t} \left[-(\partial_i N^i)[\alpha \cdot N]^2 - [\alpha \cdot N]^2 N_i \nabla_N [Q^{ij}] N_j - \alpha \cdot NN_i \nabla_N [Q^{ij}] Q_j \alpha \right] dS(x)$$

(B.37)
$$+ \int_{\partial\Omega_t} \left[-\alpha \cdot N \nabla_{Qi} [Q^i \alpha] + Q_i \alpha \nabla_Q^i [\alpha \cdot N] - \alpha_i \alpha_j (\partial^i N^j) \right] dS(x).$$

To get the lower order terms into the form we want, we use the fact that we can trade normal and tangential boundary components: Define $\tau_{ij} = 2\alpha_i\alpha_j - \delta_{ij}(\alpha^k)(\alpha_k)$. Then

$$
\begin{aligned}
\left| \|\alpha \cdot N\|^2_{L^2(\partial\Omega_t)} - \|\alpha \cdot Q\|^2_{L^2(\partial\Omega_t)} \right| &= \left| \int_{\partial\Omega_t} \left[N^i N^j \alpha_i \alpha_j - Q^{ij}\alpha_i\alpha_j \right] dS(x) \right| \\
&= \left| \int_{\partial\Omega_t} \left[2N^i N^j - \delta^{ij} \right] [\alpha_i\alpha_j] dS(x) \right| \\
&\leq \sum_{k=1}^{\mu} \left| \int_{\partial\Omega_t} \zeta_k N^i N^j \tau_{ij} dS(x) \right| \\
&\leq \sum_{k=1}^{\mu} \left| \int_{U_k \cap \Omega_t} \zeta_k (\partial^i N^j) \tau_{ij} dx \right| + \sum_{k=1}^{\mu} \left| \int_{U_k \cap \Omega_t} \zeta_k N^j \partial^i \tau_{ij} dx \right|.
\end{aligned}
$$

Now

$$
\begin{aligned}
\partial^i \tau_{ij} &= 2\operatorname{div}\alpha\alpha_j + 2\alpha_i(\partial_i\alpha_j) - (\partial_j\alpha_k)(\alpha^k) - (\alpha_k)(\partial_j\alpha^k) \\
&= 2\operatorname{div}\alpha\alpha_j + 2\alpha_i(\partial_i\alpha_j) + \left(-(\partial_j\alpha_k)(\alpha^k) + (\partial_k\alpha_j)(\alpha^k) \right) - (\partial_k\alpha_j)(\alpha^k) \\
&\quad + \left(-(\alpha^k)(\partial_j\alpha^k) + (\alpha^k)(\partial_k\alpha^j) \right) - (\alpha^k)(\partial_k\alpha^j).
\end{aligned}
$$

Thus,

(B.38) $\qquad \left| \|\alpha \cdot N\|^2_{L^2(\partial\Omega_t)} - \|\alpha \cdot Q\|^2_{L^2(\partial\Omega_t)} \right| \leq \|x\|_5 \left[\|\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|_{L^2(\Omega_t)} + \|\operatorname{curl}\alpha\|_{L^2(\Omega_t)} \right].$

Hence all the lower order terms in (B.36) can be controlled by

$$
\|\alpha \cdot N\|^2_{L^2(\partial\Omega_t)} + \|\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{curl}\alpha\|^2_{L^2(\Omega_t)}
$$

or

(B.39) $\qquad \|\alpha \cdot Q\|^2_{L^2(\partial\Omega_t)} + \|\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{curl}\alpha\|^2_{L^2(\Omega_t)}.$

To control the fourth term in (B.36) we use lemma A.1:

(B.40) $\qquad\qquad \int_{\partial\Omega_t} \alpha \cdot N \nabla_{Q^i}[Q^i\alpha] dS(x) = \int_{\partial\Omega_t} \alpha \cdot N c \partial_\theta [Q^i\alpha] dS(x)$

(B.41) $\qquad\qquad\qquad\qquad\qquad\qquad \leq \|\alpha \cdot N\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \|\alpha \cdot Q\|_{H^{\frac{1}{2}}(\partial\Omega_t)}.$

By using the fact that $ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ and the trace theorem we see that the fourth term in (B.36) can be controlled by both

$$
\frac{1}{\varepsilon}\|x\|^2_4 \left[\|\alpha \cdot N\|_{H^{\frac{1}{2}}(\partial\Omega_t)} + \|\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{curl}\alpha\|^2_{L^2(\Omega_t)} \right] + \varepsilon\|x\|^2_4\|\alpha\|^2_{H^1(\Omega_t)}
$$

and

$$
\frac{1}{\varepsilon}\|x\|^2_4 \left[\|\alpha \cdot Q\|_{H^{\frac{1}{2}}(\partial\Omega_t)} + \|\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|^2_{L^2(\Omega_t)} + \|\operatorname{curl}\alpha\|^2_{L^2(\Omega_t)} \right] + \varepsilon\|x\|^2_4\|\alpha\|^2_{H^1(\Omega_t)}.
$$

The fifth term in (B.36) can be controlled similarly. This proves (B.18) and (B.19) for $s = 1$. We now prove the estimate in terms of $\alpha \cdot N$, with the estimate in terms of $\alpha \cdot Q$ following similarly.

Suppose that $2 \leq s \leq 3$ and that we have the result for smaller s. Using lemma B.2, we have

(B.42)

$$
\|\alpha\|^2_{H^s(\Omega_t)} = \|\alpha\|^2_{L^2(\Omega_t)} + \|\nabla\alpha\|^2_{H^{s-1}(\Omega_t)}
$$

(B.43) $\qquad \leq P\big[\|x\|_5\big] \left[\|\alpha\|_{L^2(\Omega_t)} + \|\nabla\alpha\|_{L^2(\Omega_t)} + \|\operatorname{curl}\alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{div}\alpha\|_{H^{s-1}(\Omega_t)} + \sum_{i=1}^{s-1} \|\partial^i\nabla\alpha\|_{L^2(\Omega_t)} \right]$

where ∂ denotes both ∂_θ and $\eta\nabla$. Then

$$(B.44) \qquad \partial^{s-1}\nabla\alpha - \nabla\partial^{s-1}\alpha = \sum(\nabla^{i+1}x)(\nabla^{j+1}\alpha)$$

where the sum is over $i + j = s - 1$ such that $j \leq s - 2$. The commutator in (B.44) can be controlled by $\|x\|_5\|\alpha\|_{H^{s-1}(\Omega_t)}$. Using the above computation for the case $s = 1$ we have

$$(B.45) \quad \|\nabla\partial^{s-1}\alpha\|_{L^2(\Omega_t)} \leq P\big[\|x\|_5\big]\big[\|\partial^{s-1}\alpha\|_{L^2(\Omega_t)} + \|\mathrm{div}\,\partial^{s-1}\alpha\|_{L^2(\Omega_t)} + \|\mathrm{curl}\,\partial^{s-1}\alpha\|_{L^2(\Omega_t)} + \|(\partial^{s-1}\alpha)\cdot N\|_{H^{\frac12}(\partial\Omega_t)}\big]$$

$$(B.46) \qquad \leq P\big[\|x\|_5\big]\big[\|\alpha\|_{H^{s-1}(\Omega_t)} + \|\mathrm{div}\,\alpha\|_{H^{s-1}(\Omega_t)} + \|\mathrm{curl}\,\alpha\|_{H^{s-1}(\Omega_t)} + \|(\partial_\theta^{s-1}\alpha)\cdot N\|_{H^{\frac12}(\partial\Omega_t)}\big].$$

Now suppose that $s = 4$. Then we control the commutator in (B.44) as follows: For $0 \leq i \leq 2$ we estimate this term as above. For $i = 3$ we have $j = 0$ and we can control the commutator by $\|x\|_4\|\alpha\|_{H^3(\Omega_t)}$. The case for $s = 5$ follows similarly. Using corollary A.3 we see that

$$\|(\partial_\theta^{s-1}\alpha)\cdot N\|_{H^{\frac12}(\partial\Omega_t)} \leq \|(\langle\partial_\theta\rangle^{\frac12}\partial_\theta^{s-1}\alpha)\cdot N\|_{L^2(\partial\Omega_t)}\|N\|_{L^\infty(\partial\Omega_t)} + \|\alpha\|_{H^{s-1}(\partial\Omega_t)}\|\langle\partial_\theta\rangle^2 N\|_{L^2(\partial\Omega_t)}.$$

Let α have tangential Fourier expansion $\sum\alpha_k e^{ik\theta}$. Then $\partial_\theta^{s-1}\alpha^j = \sum(ik)^{s-1}\alpha_k^j e^{ik\theta}$ and $\langle\partial_\theta\rangle^{\frac12}\partial_\theta^{s-1}\alpha^j = \sum\langle k\rangle^{\frac12}(ik)^{s-1}\alpha_k^j e^{ik\theta}$. Therefore $(\langle\partial_\theta\rangle^{\frac12}\partial_\theta^{s-1}\alpha)\cdot N = \sum\langle k\rangle^{\frac12}(ik)^{s-1}\alpha_k\cdot N e^{ik\theta}$. Thus

$$(B.47) \qquad \|(\langle\partial_\theta\rangle^{\frac12}\partial_\theta^{s-1}\alpha)\cdot N\|_{L^2(\partial\Omega_t)}^2 = \sum\langle k\rangle k^{2(s-1)}|\alpha_k\cdot N|^2.$$

Also, $(\langle\partial_\theta\rangle^{s-\frac12}\alpha)\cdot N = \sum\langle k\rangle^{s-\frac12}\alpha_k\cdot N e^{ik\theta}$. Thus

$$(B.48) \qquad \|(\langle\partial_\theta\rangle^{s-\frac12}\alpha)\cdot N\|_{L^2(\partial\Omega_t)}^2 = \sum\langle k\rangle^{2(s-\frac12)}|\alpha_k\cdot N|^2.$$

Thus $\|(\langle\partial_\theta\rangle^{\frac12}\partial_\theta^{s-1}\alpha)\cdot N\|_{L^2(\partial\Omega_t)} \leq \|(\langle\partial_\theta\rangle^{s-\frac12}\alpha)\cdot N\|_{L^2(\partial\Omega_t)}$. By interpolation we now obtain the result for non-integer s. This concludes the proof. ∎

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