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Well-posedness for the equations of motion of an inviscid, incompressible, self-gravitating fluid with free boundary

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**Abstract:**

We prove that the equations of motion of an incompressible, inviscid, self-gravitating fluid with free boundary are well-posed in Sobolev space. The methodology consists of a fixed-point argument using a tangential smoothing operator, followed by energy estimates

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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Well-posedness for the equations of motion of an inviscid,  
incompressible, self-gravitating fluid with free boundary.**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Mathematics

by

Karl Håkan Nordgren

Committee in charge:

Professor Hans Lindblad, Chair  
Professor Thomas Bewley  
Professor William Helton  
Professor Michael Holst  
Professor Miroslav Krstic

2008

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The dissertation of Karl Håkan Nordgren is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2008

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## VITA

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ABSTRACT OF THE DISSERTATION

**Well-posedness for the equations of motion of an inviscid,  
incompressible, self-gravitating fluid with free boundary.**

by

Karl Håkan Nordgren

Doctor of Philosophy in Mathematics

University of California San Diego, 2008

Professor Hans Lindblad, Chair

We prove that the equations of motion of an incompressible, inviscid, self-gravitating fluid with free boundary are well-posed in Sobolev space. The methodology consists of a fixed-point argument using a tangential smoothing operator, followed by energy estimates.

# Chapter 1

## Introduction.

Let  $\Omega_t \subseteq \mathbf{R}^3$  be the domain occupied by a fluid at time  $t \in [0, T]$  and suppose that the fluid has velocity  $v(t, x)$  and pressure  $p(t, x)$  at a point  $x$  in  $\Omega_t$ . For an inviscid, self-gravitating fluid these two quantities are related by Euler's equation

$$(\partial_t + v^i \partial_i)v_j = -\partial_j p - \partial_j \phi \quad (1.0.1)$$

in  $\Omega_t$ , where  $\partial_i = \frac{\partial}{\partial x^i}$  and  $v^i = \delta^{ij}v_j$  and where  $\phi$  is the Newtonian gravity-potential defined by

$$\phi(t, x) = -\chi_{\Omega_t} * \Phi(x) \quad (1.0.2)$$

on  $\Omega_t$ , where  $\chi_{\Omega_t}$  is a function which takes the value 1 on  $\Omega_t$  and the value 0 on the complement of  $\Omega_t$  and where  $\Phi$  is the fundamental solution to the Laplacean. We can impose the condition that the fluid be incompressible by requiring that the fluid-velocity be divergence-free:

$$\operatorname{div} v = \partial_i v^i = 0 \text{ in } \Omega_t. \quad (1.0.3)$$

The absence of surface-tension is imposed with the following boundary condition:

$$p = 0 \text{ on } \partial\Omega_t \quad (1.0.4)$$

and to make the free-boundary move with the fluid-velocity, we have

$$\partial_t + v^i \partial_i \text{ is in the tangent-space of } \cup_{t \in [0, T]} [\Omega_t \times \{t\}]. \quad (1.0.5)$$

The problem is, then, is to prove the existence of a triple  $(v, p, \Omega_t)$  satisfying (1.0.1) - (1.0.5) in some interval  $[0, T]$ , given the initial-conditions

$$v = v_0 \text{ on } \Omega_0, \quad (1.0.6)$$

where  $v_0$  and  $\Omega = \Omega_0$  are known. We will also assume that initially there is a constant  $c_0$  such that

$$\nabla p \cdot N \leq -c_0 < 0 \text{ on } \partial\Omega \quad (1.0.7)$$

where  $N$  is the exterior unit normal to  $\partial\Omega$ .

The main theorem we will prove in this paper is the following:

**THEOREM 1.0.1** *Let the initial domain  $\Omega$  be in  $H^8$  and let  $v_0$  be in  $H^{7.5}(\Omega)$ . Then there is a unique solution  $(v, p, \Omega_t)$  satisfying (1.0.1), (1.0.3), (1.0.4) and (1.0.6), in some interval  $[0, T]$ , such that  $\Omega_t$  is in  $H^8$  and  $v$  is in  $L^2[0, T], H^{7.5}(\Omega_t)$ .*

## 1.1 Background.

Past progress has been made in three situations: The first progress was made on the water-wave problem under the assumption that the fluid be irrotational – that is, the curl of the fluid-velocity is zero –, incompressible and that the free-boundary not be subject to surface-tension. Notable results in this area are Wu’s papers [11] and [12] where she uses Clifford analysis to show well-posedness in two and then three dimensions in an infinitely deep fluid; and also Lannes’ paper [4] where the Nash-Moser technique is used to prove well-posedness in arbitrary space-dimensions for a fluid of finite depth.

In [2], Christodoulou and Lindblad proved *a priori* estimates for the incompressible Euler’s equation, without the assumption of irrotationality. They were not sufficient to obtain the existence result, however, because no approximation-scheme was discovered which did not destroy the structure in the equations on which the estimates relied. In [5] Lindblad proved that the equations obtained by linearising Euler’s equation around a solution are well-posed. Using the fact that the linearised operator was invertible, Lindblad then used the Nash-Moser approximation scheme to obtain the full well-posedness in [6]. Well-posedness was also proved by Coutand and Shkoller in [3], using a fixed-point argument which relies on smoothing the fluid-velocity only – crucially – in the direction tangential to the boundary. This is followed by energy estimates which we will discuss in detail in chapter 5. Also, in [10], Shatah and Zeng prove *a priori* estimates under these conditions by considering Euler’s equation as the geodesic equation on the group of volume-preserving diffeomorphisms. The latter two papers also consider the case of positive surface-tension.

## 1.2 A conserved quantity.

Let us begin by defining Lagrangian coordinates and then noting a conserved quantity for the flow: Suppose that  $v$  satisfies (1.0.1) and (1.0.3), and that  $x$  satisfies

$$\frac{dx}{dt}(t, y) = v(t, x(t, y)) \text{ and } x(0, y) = y \quad (1.2.1)$$

for  $y$  in  $\Omega$  and for  $t$  in some time interval  $[0, T]$ . This means that  $x(t, \cdot) : \Omega \rightarrow \Omega_t$  is such that

$$\partial_t \det \left( \frac{\partial x}{\partial y} \right) (t, y) = \operatorname{div} v \circ x(t, y) = 0. \quad (1.2.2)$$

And since  $\det \left( \frac{\partial x}{\partial y} \right) (0, y) = 1$  we therefore have  $\det \left( \frac{\partial x}{\partial y} \right) = 1$  in  $\Omega$ . Let

$$E(t) = \int_{\Omega_t} v(t, x) \cdot v(t, x) dx + \int_{\Omega_t} \phi(t, x) dx \quad (1.2.3)$$

$$= \int_{\Omega} v(t, x(t, y)) \cdot v(t, x(t, y)) dy + \int_{\Omega} \phi(t, x(t, y)) dy. \quad (1.2.4)$$

Then

$$E(0) = \int_{\Omega} v_0(x) \cdot v_0(x) dx - \int_{\Omega} \phi(0, x) dx \quad (1.2.5)$$

where the second integral in (1.2.5) converges. Using (1.0.1), the time derivative of the first term in  $E$  is equal to

$$-2 \int_{\Omega} (\partial_i p)(t, x(t, y)) v^i(t, x(t, y)) dy - 2 \int_{\Omega} (\partial_i \phi)(t, x(t, y)) v^i(t, x(t, y)) dy. \quad (1.2.6)$$

The first integral in (1.2.6) can be shown to be zero using integration by parts. Now  $\phi(t, x) = \int_{\Omega_t} -\Phi(|x - z|) dz$ . Thus

$$(\partial_i \phi)(t, x(t, y)) = \int_{\Omega} [x(t, y) - x(t, z)]_i \Phi'(|x(t, y) - x(t, z)|) dz.$$

And therefore the second integral in (1.2.6) is equal to

$$-2 \int_{\Omega} \int_{\Omega} [x(t, y) - x(t, z)] \cdot v(t, x(t, y)) \Phi'(|x(t, y) - x(t, z)|) dy dz. \quad (1.2.7)$$

The time derivative of the second term in  $E$  is equal to

$$\int_{\Omega} \int_{\Omega} [x(t, y) - x(t, z)] \cdot [v(t, x(t, y)) - v(t, x(t, z))] \Phi'(|x(t, y) - x(t, z)|) dy dz \quad (1.2.8)$$

$$= \int_{\Omega} \int_{\Omega} [x(t, y) - x(t, z)] \cdot v(t, x(t, y)) \Phi'(|x(t, y) - x(t, z)|) dy dz \quad (1.2.9)$$

$$+ \int_{\Omega} \int_{\Omega} [x(t, z) - x(t, y)] \cdot v(t, x(t, z)) \Phi'(|x(t, z) - x(t, y)|) dz dy \quad (1.2.10)$$

$$= 2 \int_{\Omega} \int_{\Omega} [x(t, y) - x(t, z)] \cdot v(t, x(t, y)) \Phi'(|x(t, y) - x(t, z)|) dy dz. \quad (1.2.11)$$

Thus (1.2.7) cancels (1.2.11), which means that  $\dot{E} = 0$ . Since  $\int_{\Omega_t} \phi(t, x) dx < \infty$ , this means that  $\|v\|_{L^2(\Omega_t)}$  is always finite. We will prove higher order versions of this in chapter 5.

### 1.3 Summary of the argument.

We show well-posedness for (1.0.1) - (1.0.5) under the initial conditions (1.0.6) and (1.0.7), using the methodology developed by Coutand and Shkoller in [3]: We suppose to begin with that we have been given  $\Omega$  in  $H^9$  and  $v_0$  in  $H^8(\Omega)$ . We prove elliptic estimates for  $p$  and  $\phi$  in chapter 3 and chapter 4. These are used subsequently, both to prove *a priori* estimates in chapter 5 and to prove the existence of a fixed point in chapter 6. In chapter 5, we smooth  $V$  in

the directions tangential to the boundary using a convolution-type operator, with the smoothing controlled by the parameter  $\kappa$ . Using this smoothing we can write down a version of Euler's equation where the transportation velocity is smoothed. Let the flow of  $V_\kappa$  be defined by

$$x_\kappa(t, y) = y + \int_{[0,t]} V_\kappa(s, y) ds.$$

We now define the energy

$$E_\kappa(t) = \sup_{[0,t]} [\|V\|_{7.5} + \|x_\kappa\|_8 + \kappa\|V\|_8] \quad (1.3.1)$$

and prove that there is  $T_1 > 0$ , which does not depend on  $\kappa$ , such that  $E_\kappa(T_1) \leq E_0$ , where  $E_0$  depends on the fact that  $\Omega$  is in  $H^8$ , on  $\|v_0\|_8$  and on  $E_\kappa(0)$ . Now let  $V$  be a point in the space

$$C_\kappa(T) = \left\{ f \in L^2([0, T], H^8(\Omega)) : \sup_{[0, T]} \|f\|_8 \leq \kappa^{-1} E_0 + 1 \right\}.$$

Thus we also control the flow  $x$  of  $V$  in  $H^8(\Omega)$ . To find a solution to Euler's equation with smoothed transport-velocity, one seeks a fixed-point of the operator

$$\Lambda^i(V) = v_0^i - \int_{[0,t]} (\partial_\kappa^i p_\kappa) ds - \int_{[0,t]} (\partial_\kappa^i \phi_\kappa) ds.$$

Here  $\partial_\kappa^i = \delta^{ij} \frac{\partial}{\partial x_\kappa^j}$  are derivatives with respect to the coordinates  $x_\kappa$ . We define  $p_\kappa$  as follows:

$$\Delta p_\kappa = - \left( \frac{\partial v_\kappa^j}{\partial x_\kappa^i} \right) \left( \frac{\partial v^i}{\partial x_\kappa^j} \right) + 1 \text{ on } \Omega_t \text{ with boundary condition } p_\kappa = 0 \text{ on } \partial\Omega_t.$$

And we define  $\phi_\kappa$  by  $\phi_\kappa(t, x) = -\chi_{\Omega_t} * \Phi(x)$ . To show that  $\Lambda : C_\kappa(T) \rightarrow C_\kappa(T)$  we use the following elliptic-regularity theorem from chapter 3 and chapter 4:

**THEOREM 1.3.1** *We have*

$$\|(\nabla_\kappa p_\kappa) \circ x_\kappa\|_8 \leq P [\|x_\kappa\|_8, \|\partial_\theta x_\kappa\|_8, \|V\|_8] \quad (1.3.2)$$

and

$$\|(\nabla_\kappa \phi_\kappa) \circ x_\kappa\|_8 \leq P \left[ \|x_\kappa\|_8, \|\partial_\theta^{\frac{1}{2}} x_\kappa\|_8 \right]. \quad (1.3.3)$$

where  $\partial_\theta$  is a derivative which is tangential to the boundary.

Theorem 1.3.1 makes clear the main difficulty with this problem; namely, that the geometry of the domain contributes terms of the highest order. The smoothing along the boundary allows the order to be reduced, at the cost of powers of  $\frac{1}{\kappa}$ :  $\|\partial_\theta x_\kappa\|_8 = \kappa^{-1} \|x_\kappa\|_8$  which we can control.

**REMARK:** The elliptic estimates for  $\nabla_\kappa \phi_\kappa$  are new. Their proof uses a Hodge-decomposition inequality introduced by Lindblad in [5] and also an extension of the coordinate system  $x_\kappa$ , to

avoid the fact that *a priori*  $\phi_\kappa$  may be ill-behaved along  $\partial\Omega_t$ . It should also be noted that the estimates are in terms of  $\nabla_\kappa p_\kappa$  and  $\nabla_\kappa \phi_\kappa$  and not  $p_\kappa$  and  $\phi_\kappa$ , which saves commutators.

An application of theorem 1.3.1 proves that for  $T_2$  small enough,  $\Lambda$  is invariant on  $C_\kappa(T_2)$ . Similar estimates show that  $\Lambda$  is a contraction which provides a unique fixed point  $V$ , which depends on  $\kappa$ , defined on a time interval  $[0, T_2]$ . Using the *a priori* estimates we now extend this solution to the whole interval  $[0, T_1]$  which does not depend on  $\kappa$ . Thus the fixed-point solutions converge to a solution of Euler's equation.

Finally, in chapter 7, we suppose that the initial domain  $\Omega$  is in  $H^8$  and that the initial data  $v_0$  is given in  $H^{7.5}(\Omega)$ . We can smooth the initial data using a standard convolution to obtain  $\Omega_\varepsilon$  in  $H^9$  and  $v_{0,\varepsilon}$  in  $H^8(\Omega)$ . The previous argument then provides us with solutions  $v_\varepsilon$  to Euler's equation with initial data  $v_{0,\varepsilon}$ . We prove that these solutions converge in  $H^{7.5}(\Omega)$  using an energy-type argument.

# Chapter 2

## Preliminaries.

In this chapter we define the coordinates, the cut-off functions, the derivatives and the norms which we will be using in this paper.

### 2.1 Coordinates and derivatives.

Let  $U_1, \dots, U_\mu \subseteq \mathbf{R}^3$  be an open cover for  $\partial\Omega$  such that for each  $U_i$  with  $i = 1, \dots, \mu$  there is a change of variables  $\Psi_i : \{z \in \mathbf{R}^3 : |z_j| \leq 1 \text{ for } j = 1, 2, 3\} \rightarrow U_i$  with

$$\Psi_i : \{z \in \mathbf{R}^3 : |z_j| \leq 1 \text{ for } j = 1, 2 \text{ and } -1 \leq z_3 \leq 0\} \rightarrow U_i \cap \overline{\Omega}$$

and,

$$\Psi_i : \{z \in \mathbf{R}^3 : |z_j| \leq 1 \text{ for } j = 1, 2 \text{ and } z_3 = d_0\} \rightarrow U_i \cap \partial\Omega^{d_0}.$$

Here  $\Omega^{d_0} = \{y_1 + y_2 \in \mathbf{R}^3 : y_1 \in \Omega \text{ and } d(y_2) < d_0\}$  where  $d(y) = \text{dist}(y, \partial\Omega)$ . Let  $U_{\mu+1}, \dots, U_\nu$  be an open cover for the rest of  $\Omega$  such that for each  $U_i$  with  $i = \mu + 1, \dots, \nu$  there is a change of variables  $\Psi_i : \{z \in \mathbf{R}^3 : |z_j| \leq 1 \text{ for } j = 1, 2, 3\} \rightarrow U_i$ . Let  $\xi_1, \dots, \xi_\nu$  be a partition of unity subordinate to  $U_1, \dots, U_\nu$ . We will let  $z' = (z^1, z^2)$  denote the tangential directions and we will let  $z^3$  denote the final, normal, direction.

We will let  $x$  be coordinates on  $\Omega_t$  and  $\partial_i, \partial_j, \partial_k, \dots$  be derivatives on  $\Omega_t$ ; we will let  $y$  be coordinates in a patch on  $\Omega$  and  $\partial_a, \partial_b, \partial_c, \dots$  will denote derivatives in such a patch; and we will let  $z$  and  $\frac{\partial}{\partial z^i}$  to denote the coordinates and derivatives on  $(-1, 1)^3$ . Moreover,  $\nabla$  will denote an arbitrary derivative on  $\Omega_t$  and  $\partial$  will denote an arbitrary derivative in a patch on  $\Omega$ .

#### 2.1.1 Tangential derivatives.

In this paper we will use two types of tangential derivatives: The first class contains the derivatives  $1 + \xi_k \left( \frac{\partial \Psi_k^\alpha}{\partial z^i} \right) \circ \Psi_k^{-1} \frac{\partial}{\partial y^\alpha}$  where  $k = 1, \dots, \mu$  and  $i = 1, 2$  which are tangential to

the boundary of  $\Omega$ . We will abuse notation and let  $\partial_\theta$  denote this type of derivative both on  $\Omega$  and on  $\Omega_t$ . We also define a fractional tangential derivative  $\langle \partial_\theta \rangle^s$  for a function  $f$  on  $\Omega$  to be the operator which sends

$$\int_{\mathbf{R}^2} \hat{f}_j(\alpha', z^3) e^{i\alpha' \cdot z'} d\alpha' \text{ to } \int_{\mathbf{R}^2} \langle \alpha' \rangle^s \hat{f}_j(\alpha', z^3) e^{i\alpha' \cdot z'} d\alpha',$$

where  $\alpha' = (\alpha_1, \alpha_2)$ ;

$$\hat{f}_j(\alpha', z^3) = \int_{\mathbf{R}^2} f_j(z^1, z^2, z^3) e^{i\alpha' \cdot z'} dz';$$

where  $\langle \alpha' \rangle = [1 + |\alpha_1|^2 + |\alpha_2|^2]^{\frac{1}{2}}$ ; and where  $f_j = (\xi_j f) \circ \Psi_j$ .

## 2.2 Norms.

In this section we define the norms on  $(-1, 1)^2 \times (-1, 0)$ ,  $\Omega$  and  $\Omega_t$ .

### 2.2.1 $\|\cdot\|_{H^s((-1,1)^2 \times (-1,0))}$ .

Let  $f$  be a function compactly supported on  $(-1, 1)^2 \times (-1, 0)$  then for an integer  $k \geq 0$  we define

$$\|f\|_{H^k((-1,1)^2 \times (-1,0))}^2 = \sum_{j=0}^k \|\nabla^j f\|_{L^2((-1,1)^2 \times (-1,0))}^2$$

where  $\nabla$  denotes  $(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3})$ . Now because  $f$  is compactly supported,

$$\|\nabla^j f\|_{L^2((-1,1)^2 \times (-1,0))}^2 = \|\nabla^j f\|_{L^2(\mathbf{R}^3)}^2 = \int_{\mathbf{R}^3} |\alpha|^{2j} |\hat{f}|^2(\alpha) d\alpha$$

where  $\hat{f}(\alpha) = \int_{\mathbf{R}^3} f(z) e^{i\alpha \cdot z} dz$ . Thus we define

$$\|f\|_{H^s((-1,1)^2 \times (-1,0))}^2 = \int_{\mathbf{R}^3} \langle \alpha \rangle^{2s} |\hat{f}|^2(\alpha) d\alpha$$

### 2.2.2 $\|\cdot\|_{H^s(\Omega)}$ , $\|\cdot\|_s$ and $\|\cdot\|_{H^s(\partial\Omega)}$ .

Let  $f$  be a function on  $\Omega$ . Then  $\xi_j f$  is a function supported in a coordinate neighbourhood. For integer  $k \geq 0$  we define

$$\|f\|_{H^k(\Omega)}^2 = \sum_{j=1}^{\nu} \sum_{i=1}^k \|\nabla^i [\xi_j f]\|^2$$

where  $\nabla = (\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$  and  $\|\cdot\|$  denotes the  $L^2(\Omega)$ -norm. We define the intermediate spaces by interpolation, see for instance [8] and [1]. Define  $f_j = (\xi_j f) \circ \Psi_j$ , which is compactly supported in  $\mathbf{R}^3$  and

$$\|f\|_s^2 = \sum_{j=1}^{\nu} \int_{\mathbf{R}^3} \langle \alpha \rangle^{2s} |\hat{f}_j|^2(\alpha) d\alpha.$$



**LEMMA 2.2.1**  $\|\cdot\|_{H^s(\Omega)}$  and  $\|\cdot\|_s$  are equivalent.

**PROOF:** Let  $f$  be defined on  $\Omega$ . Then  $\|f\|_{H^k(\Omega)}^2 = \sum_{j=1}^{\nu} \sum_{i=1}^k \|\nabla^i[\xi_j f]\|_{L^2(\Omega)}^2$ . We have  $\nabla^i[\xi_j f] = \xi_j(\nabla^i f) + (\nabla \xi_j)(\nabla^{i-1} f) + \dots + (\nabla^i \xi) f$  and

$$\begin{aligned} \|\xi_j(\nabla^i f)\|_{L^2(\Omega)}^2 &= \int_{\Omega \cap V_j} \xi_j^2 [(\partial/\partial y)^i f]^2 dy \\ &= \int_{\Omega \cap V_j} \xi_j (\partial/\partial y)^{i-1} [(\partial \Psi_j^{-1}/\partial y) \partial/\partial z [f \circ \Psi_j]] \xi_j (\partial/\partial y)^i f dy \\ &\leq c \|(\xi_j f) \circ \Psi_j\|_{H^i((-1,1)^2 \times (-1,0))} \|\xi_j(\nabla^i f)\|_{L^2(\Omega)}. \end{aligned}$$

■

Finally, for a function  $g$  on  $\partial\Omega$  we define  $\|g\|_{H^s(\partial\Omega)} = \|\langle \partial_\theta \rangle^s [g]\|$ .

### 2.2.3 $\|\cdot\|_{H^s(\Omega_t)}$ and $\|\cdot\|_{H^s(\partial\Omega_t)}$ .

Let  $f$  be a function on  $\Omega_t$ . For integer  $k \geq 0$  we define  $\|f\|_{H^k(\Omega_t)}^2 = \|f\|_{L^2(\Omega_t)}^2 + \dots + \|\nabla^k f\|_{L^2(\Omega_t)}^2$ , where  $\nabla = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$ . We define the intermediate Sobolev spaces by interpolation. We also define  $\|\langle \partial_\theta \rangle^s [f]\|_{L^2(\Omega_t)} = \|\langle \partial_\theta \rangle^s [f \circ x]\|_{L^2(\Omega)}$  and for a function  $g$  on  $\partial\Omega_t$  we define  $\|g\|_{H^s(\partial\Omega_t)} = \|g \circ x\|_{H^s(\partial\Omega)}$ . For integral  $s$  the operator  $\langle \partial_\theta \rangle^s$  is equivalent (in the  $L^2(\Omega)$ - and  $L^2(\partial\Omega)$ -norm) to the application of multiples  $\partial_\theta$ .

### 2.2.4 Regularity of the domain.

We will use the following norm to quantify the regularity of the domain:  $\|\Omega\|_s = \sum_{i=1}^{\mu} \|\Psi_i\|_{H^s([-1,1]^3)}$ .

## 2.3 Smoothing.

Now we define the smoothing operator which we will be using. It is the main idea from [3]: Let  $\vartheta : \mathbf{R}^2 \rightarrow \mathbf{R}$  be a smooth function which is compactly supported on  $\{z' \in \mathbf{R}^2 : |z'| \leq 1\}$  and such that  $\int_{\mathbf{R}^2} \vartheta(z') dz' = 1$ . For  $\kappa > 0$ , define

$$\vartheta_\kappa(z') = \frac{1}{\kappa^2} \vartheta\left(\frac{z'}{\kappa}\right)$$

then  $\vartheta_\kappa$  is compactly supported on  $\{z' \in \mathbf{R}^2 : |z'| \leq \kappa\}$ . Let  $f$  be compactly supported in  $\{z' \in \mathbf{R}^2 : |z'| \leq 1\}$  and let  $\kappa$  be smaller than the distance from the support of  $f$  to the boundary of  $\{z' \in \mathbf{R}^2 : |z'| \leq 1\}$ . Define the tangential convolution of  $\vartheta_\kappa$  and  $f$  by

$$\vartheta_\kappa * f(z) = \int_{\mathbf{R}^2} \vartheta_\kappa(z' - z'') f(z'', z^3) dz''.$$

It will be obvious from the context whether by  $*$  we mean the usual or the tangential convolution. As can be seen, tangential convolution smooths in the tangential direction:

$$\partial_\theta [\vartheta_\kappa * f \circ \Psi_i^{-1}] = \vartheta_\kappa * f \circ \Psi_i^{-1} + \sum_{j=1,2} \int_{\mathbf{R}^2} \frac{\partial}{\partial z^j} [\vartheta_\kappa](z' - z'') f(z', z'') dz'' \quad (2.3.1)$$

$$= [1 + \kappa^{-1}] \vartheta_\kappa * f \circ \Psi_i^{-1}. \quad (2.3.2)$$

Now suppose that  $f$  is a function defined on  $\Omega$  and define the smoothed version of  $f$  to be

$$f_\kappa = \sum_{i=1}^{\mu} \xi_i^{\frac{1}{2}} \vartheta_\kappa * \vartheta_\kappa * \left[ \left( \xi_i^{\frac{1}{2}} f \right) \circ \Psi_i \right] \circ \Psi_i^{-1} + \sum_{i=\mu+1}^{\nu} \xi_i f. \quad (2.3.3)$$

Sometimes we will let  $f_b$  denote the first term in (2.3.3) – the part which is supported near the boundary of  $\Omega$  – and we will let  $f_m$  denote the second term in (2.3.3) – the part which is supported in the interior of  $\Omega$ . Finally, Lagrangian-flow associated with  $f$  is given by

$$y + \int_{[0,t]} f(y, s) ds.$$

## 2.4 Cut off functions.

Fix  $d_0$  such that the normal  $N$  to  $\partial\Omega_t$  can be extended into the image of the set  $\{y \in \Omega : d(y) < d_0\}$  under  $x$ . This fact is used in lemma 2.5.1 below. Let  $\eta_i$  and  $\zeta_i$  be radial functions which form a partition of unity subordinate to the sets  $\{y \in \mathbf{R}^3 : \frac{d_0}{2i} < d(y)\}$  and  $\{y \in \mathbf{R}^3 : d(y) < \frac{d_0}{i}\}$  respectively. This means that  $\eta_i$  takes the value 1 on the set  $\{y \in \mathbf{R}^3 : \frac{d_0}{i} \leq d(y)\}$  and  $\zeta_i$  takes the value 1 on the set  $\{y \in \mathbf{R}^3 : d(y) \leq \frac{d_0}{2i}\}$ . We will also let  $\eta_i$  and  $\zeta_i$  denote the analogous functions in the Eulerian frame.

## 2.5 Hodge-decomposition inequalities.

In this section we present three divergence-curl estimates which are used throughout this text. Their proofs can be found in appendix B. The first allows pointwise control on all derivatives near the boundary of  $\Omega_t$  by the divergence, the curl and tangential derivatives. Letting  $\zeta = \zeta_i$  we have the following:

**LEMMA 2.5.1** *Let  $\alpha$  be a vector-field on  $\Omega_t$ . Define  $(curl \alpha)_{jk} = \partial_j \alpha_k - \partial_k \alpha_j$  and  $div \alpha = \partial_j \alpha^j$ . Then we have the following pointwise estimate on  $\Omega_t$ :*

$$|\zeta \nabla \alpha| \leq |\zeta curl \alpha| + |\zeta div \alpha| + |\zeta \partial_\theta \alpha|, \quad (2.5.1)$$

where  $|\cdot|$  denotes the usual Euclidean distance.

Using lemma 2.5.1 and an induction argument we have the following lemma:

**LEMMA 2.5.2** For  $0 \leq s \leq 8$ ,

$$\|\zeta\alpha\|_{H^s(\Omega_t)} \leq P[\|x\|_8] [\|\zeta\alpha\|_{L^2(\Omega_t)} + \|\zeta\operatorname{curl}\alpha\|_{H^{s-1}(\Omega_t)} + \|\zeta\operatorname{div}\alpha\|_{H^{s-1}(\Omega_t)}] \quad (2.5.2)$$

$$+ P[\|x\|_8] \|\zeta\langle\partial_\theta\rangle^s\alpha\|_{L^2(\Omega_t)}. \quad (2.5.3)$$

We will also use the following estimates which allows  $H^s(\Omega_t)$  control in terms of the divergence, the curl and boundary derivatives:

**LEMMA 2.5.3** Let  $\operatorname{div}\alpha$  and  $\operatorname{curl}\alpha$  be defined as in lemma 2.5.1. Then, for  $0 \leq s \leq 8$ ,

$$\|\alpha\|_{H^s(\Omega_t)} \leq P[\|x\|_s] [\|\alpha\|_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{curl}\alpha\|_{H^{s-1}(\Omega_t)}] \quad (2.5.4)$$

$$+ P[\|x\|_s] \|(\langle\partial_\theta\rangle^{s-\frac{1}{2}}\alpha) \cdot N\|_{L^2(\partial\Omega_t)}, \quad (2.5.5)$$

where  $N$  is the outward unit normal to  $\partial\Omega_t$ . Also,

$$\|\alpha\|_{H^s(\Omega_t)} \leq P[\|x\|_s] [\|\alpha\|_{L^2(\Omega_t)} + \|\operatorname{div}\alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{curl}\alpha\|_{H^{s-1}(\Omega_t)}] \quad (2.5.6)$$

$$+ P[\|x\|_s] \|(\langle\partial_\theta\rangle^{s-\frac{1}{2}}\alpha) \cdot Q\|_{L^2(\partial\Omega_t)} \quad (2.5.7)$$

where  $Q$  is a unit vector which is tangent to  $\partial\Omega_t$ .

# Chapter 3

## Elliptic estimates for $p$ .

In this chapter  $\kappa > 0$  is fixed. As was mentioned in the introduction, we will prove *a priori* estimates for a smoothed Euler's equation. Existence will follow a fixed-point argument applied to a map  $\Lambda$  (which we will define in chapter 6) defined on the space

$$C_\kappa(T) = \left\{ f \in L^2\left([0, T], H^8(\Omega)\right) : \sup_{t \in [0, T]} \|f\|_8(t) \leq \kappa^{-1} E_0 + 1 \right\} \quad (3.0.1)$$

where  $E_0 = E_0(\|\Omega\|_9, \|v_0\|_8)$ , under the assumption that we control  $\|\Omega\|_9$  and  $\|v_0\|_8$ . To prove the *a priori* estimates and to apply the fixed point argument, elliptic estimates for  $p$  and  $\phi$  are required and in this chapter we prove the estimates for  $p$ . The result in section 3.1 will be used to show that  $\Lambda$  is invariant on  $C(T)$ . The result in section 3.2 will be used to show that  $\Lambda$  is also a contraction. In section 3.3, we prove the estimates for  $p$  which will be used for the energy estimates in chapter 5 and estimates for establishing optimal regularity in chapter 7. The estimates for  $\phi$  are in chapter 4.

### 3.1 Estimates to show that $\Lambda$ is invariant.

Let  $U$  be a point in  $C(T)$  and let  $x_\kappa$  be the flow of  $U_\kappa$ , where  $U_\kappa$  is the smoothed version of  $U$ . We have

$$\sup_{[0, T]} \|\partial x_\kappa - \text{Id}\|_7 \leq \sup_{[0, T]} \int_{[0, t]} \|\partial U_\kappa\|_7 ds \leq T \sup_{[0, T]} \|\partial U_\kappa\|_7 \leq T \sup_{[0, T]} \|U\|_8 \quad (3.1.1)$$

and therefore for small enough  $T$ ,  $B_a^i = \frac{\partial x_\kappa^i}{\partial y^a}$  is invertible and its inverse  $A_i^a = \frac{\partial y^a}{\partial x_\kappa^i}$  is well-defined. By choosing  $T$  to be smaller if necessary, we can assume that  $\frac{1}{2} \leq c_1 \leq \det(B) \leq c_2 \leq \frac{3}{2}$ .

**LEMMA 3.1.1** *For  $0 \leq i \leq 7$ , we have  $\|A\|_i \leq P[\|x_\kappa\|_8]$ .*

**PROOF:** This follows an induction argument by applying derivatives to the relation  $\delta_i^j = A_i^a B_a^j$  and using interpolation. ■

Let  $V$  and  $W$  be points in  $C(T)$  and define  $v_\kappa = V_\kappa \circ x_\kappa^{-1}$  and  $w = W \circ x_\kappa^{-1}$ . Define a function  $f$  on  $\Omega_t = x_\kappa(\Omega, t)$  by

$$\Delta f = -(\nabla v_\kappa)(\nabla w) + 1 \text{ on } \Omega_t, \quad (3.1.2)$$

where  $\nabla$  denotes derivatives with respect to the coordinates  $x_\kappa$ , with boundary condition

$$f = 0 \text{ on } \partial\Omega_t. \quad (3.1.3)$$

In this section we prove the following theorem:

**THEOREM 3.1.2**  $\|\nabla f\|_{H^s(\Omega_t)} \leq P[\|x_\kappa\|_8, \kappa^{-1}\|x_\kappa\|_8, \|v_\kappa\|_{H^s(\Omega_t)}, \|w\|_{H^s(\Omega_t)}]$  where  $P$  is a polynomial which is linear in  $\kappa^{-1}\|x_\kappa\|_8$ .

Using the cut off functions defined in chapter 2 we have  $\|\nabla f\|_{H^s(\Omega_t)} \leq \|\eta_1 \nabla f\|_{H^s(\Omega_t)} + \|\zeta_1 \nabla f\|_{H^s(\Omega_t)}$ . We begin by proving interior estimates for  $f$ .

### 3.1.1 Interior estimates.

In this section we prove the following estimate:

**PROPOSITION 3.1.3** For all  $1 \leq i$  and all  $0 \leq s \leq 8$  we have

$$\|\nabla^s[\eta_i \nabla f]\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8, \|v_\kappa\|_{H^s(\Omega_t)}, \|w\|_{H^s(\Omega_t)}]. \quad (3.1.4)$$

We prove proposition 3.1.3 by induction on  $s$ . For  $s = 0$ , we have  $\|\eta_i \nabla f\|_{L^2(\Omega_t)} \leq \|\eta_i\|_{L^\infty(\Omega_t)} \|\nabla f\|_{L^2(\Omega_t)}$  and

$$\|\nabla f\|_{L^2(\Omega_t)}^2 = \int_{\Omega_t} (\partial_j f)(\partial^j f) dx = \int_{\Omega_t} f(\nabla v_\kappa)(\nabla w) dx - \int_{\Omega_t} f dx. \quad (3.1.5)$$

By Poincaré's inequality, the terms in (3.1.5) can be controlled by

$$P[\|x_\kappa\|_8, \|v_\kappa\|_{H^4(\Omega_t)}, \|w\|_{H^4(\Omega_t)}] \|\nabla f\|_{L^2(\Omega_t)}. \quad (3.1.6)$$

This proves the case for  $s = 0$ . Now suppose that  $s = 8$  and that we have proposition 3.1.3 for smaller  $s$ . We have

$$\|\nabla^8[\eta_i \nabla f]\|_{L^2(\Omega_t)}^2 = \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8}[\eta_i \partial_{j_9} f])(\partial^{j_1} \dots \partial^{j_8}[\eta_i \partial^{j_9} f]) dx. \quad (3.1.7)$$

Now

$$\partial^{j_1} \dots \partial^{j_8}[\eta_i \partial^{j_9} f] = \eta_i(\partial^{j_1} \dots \partial^{j_8} \partial^{j_9} f) + \sum (\nabla^{k_1} \eta_i)(\nabla^{k_2+1} f) \quad (3.1.8)$$

where the sum is over  $k_1 + k_2 = 8$  such that  $k_2 \leq 7$ . To control the second term in (3.1.8) we use the following procedure: Let  $i_1 = i$ . Suppose that we have found  $i_1, \dots, i_l$ . The support of  $\nabla^k \eta_{i_l}$  is contained in the image under  $x_\kappa$  of the set  $\{y \in \mathbf{R}^3 : \frac{d_0}{2i_l} < d(y) < \frac{d_0}{i_l}\}$ . Pick  $i_{l+1}$  such that  $\frac{d_0}{i_{l+1}} \leq \frac{d_0}{2i_l}$ . Then  $\eta_{i_{l+1}}$  takes the value 1 on the set  $\{y \in \mathbf{R}^3 : \frac{d_0}{i_{l+1}} \leq d(y)\}$  and  $\{y \in \mathbf{R}^3 : \frac{d_0}{2i_l} < d(y) < \frac{d_0}{i_l}\} \subseteq \{y \in \mathbf{R}^3 : \frac{d_0}{i_{l+1}} \leq d(y)\}$ . Thus we have lemma 3.1.4:

**LEMMA 3.1.4** For  $k_2 \geq 1$  we have

$$(\nabla^{k_1} \eta_{i_1})(\nabla^{k_2} f) = \sum (\nabla^{k_1} \eta_{i_1})(\nabla^{l_2} \eta_{i_2}) \dots (\nabla^{l_{n-1}} \eta_{i_{n-1}})(\nabla^{l_n} [\eta_{i_n} \nabla f]) \quad (3.1.9)$$

where the sum is over all  $l_2 + \dots + l_n = k_2 - 1$ ; where for instance if  $l_2 = 0$  the term  $\nabla^{l_2} \eta_{i_2}$  is taken to not be present in the sum; and where if  $l_n = 0$  the term  $\nabla^{l_n} [\eta_{i_n} \nabla f]$  is taken to be  $\eta_{i_n} \nabla f$ .

**PROOF:** We prove this by induction on  $k_2$ . For  $k_2 = 1$  we have  $(\nabla^{k_1} \eta_{i_1})(\nabla f) = \eta_{i_2}(\nabla^{k_1} \eta_{i_1})(\nabla f)$ , which is of the correct form. Suppose that  $k_2 \geq 2$  and that we have the result for smaller  $k_2$ . Then

$$(\nabla^{k_1} \eta_{i_1})(\nabla^{k_2} f) = \eta_{i_2} (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2} f) \quad (3.1.10)$$

$$= (\nabla^{k_1} \eta_{i_1})(\nabla^{k_2-1} [\eta_{i_2} \nabla f]) \quad (3.1.11)$$

$$- (\nabla^{k_1} \eta_{i_1}) \sum_{l_1+l_2=k_2-1, l_2 \leq k_2-2} (\nabla^{l_1} \eta_{i_2})(\nabla^{l_2+1} f) \quad (3.1.12)$$

applying the inductive hypothesis to the second term in (3.1.10) gives the result.  $\blacksquare$

On  $\Omega_t$ , we are considering  $\eta_i \circ x_\kappa^{-1}$  and therefore  $\partial_j [\eta_i \circ x_\kappa^{-1}] = (\partial_a \eta_i) A_j^a$ . Thus  $\nabla \eta_i$  has the same regularity as  $A$  and by lemma 3.1.4, we can therefore control the second term in (3.1.8). Integrating the first term in (3.1.8) by parts twice we have

$$- \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} \partial^{j_9} [\eta_i \partial_{j_9} f]) \eta_i (\partial^{j_1} \dots \partial^{j_8} f) dx \quad (3.1.13)$$

$$- \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} [\eta_i \partial_{j_9} f]) (\partial^{j_9} \eta_i) (\partial^{j_1} \dots \partial^{j_8} f) dx \quad (3.1.14)$$

$$= \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_7} \partial^{j_9} [\eta_i \partial_{j_9} f]) \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} f) dx \quad (3.1.15)$$

$$+ \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_7} \partial^{j_9} [\eta_i \partial_{j_9} f]) (\partial_{j_8} \eta_i) (\partial^{j_1} \dots \partial^{j_8} f) dx \quad (3.1.16)$$

$$- \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} [\eta_i \partial_{j_9} f]) (\partial^{j_9} \eta_i) (\partial^{j_1} \dots \partial^{j_8} f) dx \quad (3.1.17)$$

where we can control the second and third term in (3.1.15) using lemma 3.1.4. Also,  $\partial^{j_9} [\eta_i \partial_{j_9} f] = (\partial^{j_9} \eta_i) (\partial_{j_9} f) - \eta_i (\nabla v_\kappa) (\nabla w) + \eta_i$  and therefore the first term in (3.1.15) is equal to

$$\int_{\Omega_t} \partial_{j_1} \dots \partial_{j_7} [(\partial^{j_9} \eta_i) (\partial_{j_9} f)] \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} f) dx \quad (3.1.18)$$

$$- \int_{\Omega_t} \partial_{j_1} \dots \partial_{j_7} [\eta_i (\nabla v_\kappa) (\nabla w)] \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} f) dx \quad (3.1.19)$$

$$+ \int_{\Omega_t} \partial_{j_1} \dots \partial_{j_7} [\eta_i] \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} f) dx \quad (3.1.20)$$

The above terms in (3.1.18) can be controlled using lemma 3.1.4 and the inductive hypothesis. This concludes the proof of proposition 3.1.3.

### 3.1.2 Boundary estimates.

In this section we let  $\zeta = \zeta_1$ . For  $0 \leq s \leq 8$ , we have

$$\nabla^s[\zeta \nabla f] = \zeta(\nabla^{s+1} f) + \sum_{k_1+k_2=s, k_2 \leq s-1} (\nabla^{k_1} \zeta)(\nabla^{k_2+1} f). \quad (3.1.21)$$

Since  $\nabla \zeta$  is supported in the interior of  $\Omega_t$  and has the same regularity as  $A$ , we can control the sum by proposition 3.1.3, using lemma 3.1.4. Therefore, in this section we prove the following proposition:

**PROPOSITION 3.1.5** *For all  $1 \leq s \leq 8$  we have*

$$\|\zeta \nabla^s f\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8, \|v_\kappa\|_{H^s(\Omega_t)}, \|w\|_{H^s(\Omega_t)}] \quad (3.1.22)$$

and

$$\|\zeta \nabla^9 f\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8, \|\partial_\theta x_\kappa\|_8, \|v_\kappa\|_{H^8(\Omega_t)}, \|w\|_{H^8(\Omega_t)}] \quad (3.1.23)$$

where  $P$  is a polynomial which is linear in  $\|\partial_\theta x_\kappa\|_8$ .

We will build regularity using the following lemma which is a corollary of lemma 2.5.1.

**LEMMA 3.1.6** *For  $1 \leq s \leq 8$ ,*

$$\|\zeta \nabla^{s+1} f\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8] \left[ 1 + \|v_\kappa\|_{H^s(\Omega_t)} \|w\|_{H^s(\Omega_t)} + \sum_{0 \leq j \leq s} \|\zeta \partial_\theta^j \nabla f\|_{L^2(\Omega_t)} \right]. \quad (3.1.24)$$

**PROOF:** We prove this by induction on  $s$ . For  $s = 1$  we have, according to lemma 2.5.1,  $|\zeta \nabla^2 f| \leq |\nabla v_\kappa| |\nabla w| + 1 + |\zeta \partial_\theta \nabla f|$  which is of the correct form for (3.1.24). Now suppose that  $s = 8$  and that we have (3.1.24) for  $1 \leq s \leq 7$ . Again by lemma 2.5.1, we have  $|\zeta \nabla^{s+1} f| \leq |\zeta \nabla^{s-1} \text{curl} \nabla f| + |\zeta \nabla^{s-1} \text{div} \nabla f| + |\zeta \partial_\theta \nabla^s f|$ . Here we have  $\text{div} \nabla f = \Delta f = -(\nabla v_\kappa)(\nabla w) + 1$  and therefore pointwise on  $\Omega_t$  we have  $|\zeta \nabla^{s-1} \text{div} \nabla f| = |\zeta \nabla^{s-1} [(\nabla v_\kappa)(\nabla w)]|$ . We also have

$$\zeta \nabla^{s-1} \partial_\theta \nabla f = \zeta \nabla^{s-1} [(\partial_\theta x_\kappa)(\nabla^2 f)] = \zeta \partial_\theta \nabla^s f + \sum_{k_1+k_2=s-1, k_2 \leq s-2} (\nabla^{k_1} \partial_\theta x_\kappa) \zeta (\nabla^{k_2+2} f). \quad (3.1.25)$$

For  $0 \leq k_1 \leq 4$ , we have  $\|\nabla^{k_1} \partial_\theta x_\kappa\|_{L^\infty(\Omega_t)} \leq \|x_\kappa\|_8$  and  $\|\zeta \nabla^{k_2+2} f\|_{L^2(\Omega_t)}$  is controlled by the inductive hypothesis since  $k_2 + 2 \leq s = 8$ . For  $5 \leq k_1 \leq 7$ , we have  $\|\nabla^{k_1} \partial_\theta x_\kappa\|_{L^2(\Omega_t)} \leq \|x_\kappa\|_8$  and in this case  $0 \leq k_2 \leq 2$ , and therefore  $\|\zeta \nabla^{k_2+2} f\|_{L^\infty(\Omega_t)} \leq \|\zeta \nabla^{k_2+2+3} f\|_{L^2(\Omega_t)}$ , in addition to terms which we control by proposition 3.1.3. We control  $\|\zeta \nabla^{k_2+2+3} f\|_{L^2(\Omega_t)}$  by the inductive hypothesis because  $k_2 + 2 + 3 \leq 7$ . Now  $|\zeta \nabla^{s-1} \partial_\theta \nabla f| \leq |\zeta \nabla^{s-2} \text{curl} \partial_\theta \nabla f| + |\zeta \nabla^{s-2} \text{div} \partial_\theta \nabla f| + |\zeta \partial_\theta \nabla^{s-2} \partial_\theta \nabla f|$  and  $\zeta \nabla^{s-2} \text{div} \partial_\theta \nabla f = \zeta \nabla^{s-2} [(\nabla \partial_\theta x_\kappa)(\nabla^2 f) + \partial_\theta \Delta f]$ . We have

$$\zeta \nabla^{s-2} [(\nabla \partial_\theta x_\kappa)(\nabla^2 f)] = \sum_{k_1+k_2=s-2} (\nabla^{k_1+1} \partial_\theta x_\kappa) \zeta (\nabla^{k_2+2} f). \quad (3.1.26)$$

For  $0 \leq k_1 \leq 3$ , we have  $\|\nabla^{k_1+1}\partial_\theta x_\kappa\|_{L^\infty(\Omega_t)} \leq \|x_\kappa\|_8$  and  $\|\zeta(\nabla^{k_2+2}f)\|_{L^2(\Omega_t)}$  is controlled by the inductive hypothesis. For  $4 \leq k_1 \leq 6$  we have  $\|\nabla^{k_1+1}\partial_\theta x_\kappa\|_{L^2(\Omega_t)} \leq \|x_\kappa\|_8$  and in this case  $0 \leq k_2 \leq 2$  and we control  $\|\zeta(\nabla^{k_2+2}f)\|_{L^\infty(\Omega_t)} \leq \|\zeta(\nabla^{k_2+2+3}f)\|_{L^2(\Omega_t)}$  using the inductive hypothesis. Also,  $\zeta\nabla^{s-2}\partial_\theta\Delta f = \zeta\nabla^{s-2}\partial_\theta[(\nabla v_\kappa)(\nabla w)]$  which we control appropriately. This concludes the proof.  $\blacksquare$

By lemma 3.1.6, to prove proposition 3.1.5, it is enough to prove that we control  $\|\zeta\partial_\theta^j\nabla f\|_{L^2(\Omega_t)}$  for  $0 \leq j \leq 8$  which is the content of the following proposition:

**PROPOSITION 3.1.7** *For  $0 \leq j \leq 7$ , we have*

$$\|\zeta\partial_\theta^j\nabla f\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8, \|v_\kappa\|_{H^8(\Omega_t)}, \|w\|_{H^8(\Omega_t)}] \quad (3.1.27)$$

and

$$\|\zeta\partial_\theta^8\nabla f\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8, \|\partial_\theta x_\kappa\|_8, \|v_\kappa\|_{H^8(\Omega_t)}, \|w\|_{H^8(\Omega_t)}] \quad (3.1.28)$$

where  $P$  is a polynomial which is linear in  $\|\partial_\theta x_\kappa\|_8$ .

We prove this result by induction. We have already proved the base-case. Since the case for  $j = 7$  follows similarly to the case for  $j = 8$  we will now prove the case for  $j = 8$  and suppose that we have (3.1.27) for  $0 \leq j \leq 7$ . We have

$$\|\zeta\partial_\theta^8\nabla f\|_{L^2(\Omega_t)}^2 = \int_{\Omega_t} (\zeta\partial_\theta^8\partial_i f)(\zeta\partial_\theta^8\partial^i f) dx \quad (3.1.29)$$

$$= \int_{\Omega_t} (\zeta\partial_\theta^8\partial_i f)(\zeta\partial^i\partial_\theta^8 f) dx - \int_{\Omega_t} (\zeta\partial_\theta^8\nabla f)(\nabla\partial_\theta^8 x_\kappa)(\zeta\nabla f) dx \quad (3.1.30)$$

$$+ \sum \int_{\Omega_t} (\zeta\partial_\theta^8\nabla f)(\nabla\partial_\theta^{k_1} x_\kappa) \dots (\nabla\partial_\theta^{k_{l-1}} x_\kappa)(\zeta\partial_\theta^{k_l}\nabla f) \quad (3.1.31)$$

where the sum is over  $k_1 + \dots + k_l = 8$  such that  $k_1, \dots, k_l \leq 7$ , which means that we can control all the terms in the sum using the inductive hypothesis. The second term in (3.1.30) we control appropriately as well. The first term in (3.1.30) we integrate by parts to obtain

$$- \int_{\Omega_t} (\zeta\partial_\theta^8\nabla f)(\nabla\zeta)(\partial_\theta^8 f) dx - \int_{\Omega_t} (\zeta\partial^i\partial_\theta^8\partial_i f)(\zeta\partial_\theta^8 f) dx \quad (3.1.32)$$

where we control the first term by proposition 3.1.3 because  $\nabla\zeta$  is supported in the interior of  $\Omega_t$ . The second term in (3.1.32) is equal to

$$\int_{\Omega_t} (\nabla\partial_\theta^8 x_\kappa)(\zeta\nabla^2 f)(\zeta\partial_\theta^8 f) dx - \int_{\Omega_t} (\zeta\partial_\theta^8\Delta f)(\zeta\partial_\theta^8 f) dx \quad (3.1.33)$$

$$- \sum \int_{\Omega_t} (\nabla\partial_\theta^{k_1} x_\kappa) \dots (\nabla\partial_\theta^{k_{l-1}} x_\kappa)(\zeta\nabla\partial_\theta^{k_l}\nabla f)(\zeta\partial_\theta^8 f) dx \quad (3.1.34)$$

where the sum is over  $k_1 + \dots + k_l = 8$  such that  $k_1, \dots, k_l \leq 7$ . Using lemma 3.1.6 and the fact that for  $1 \leq k \leq 9$ ,

$$\|\partial_\theta^k f\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8] \sum_{0 \leq l \leq k-1} \|\partial_\theta^l \nabla f\|_{L^2(\Omega_t)} \quad (3.1.35)$$



we control the first term in (3.1.33) appropriately. The second term in (3.1.33) we integrate by parts to obtain

$$-\int_{\Omega_t} (\zeta \partial_\theta^7 [(\nabla v_\kappa)(\nabla w)])(\partial_\theta \zeta)(\partial_\theta^8 f) dx - \int_{\Omega_t} \zeta \partial_\theta^7 [(\nabla v_\kappa)(\nabla w)](\zeta \partial_\theta^9 f) dx \quad (3.1.36)$$

where no boundary terms arise because the components of  $\partial_\theta$  are orthogonal to  $N$ . The first term in (3.1.36) vanishes because  $\partial_\theta \zeta = 0$ . We control the second term in (3.1.36) appropriately by (3.1.35). This concludes the proof of proposition 3.1.7.

## 3.2 Estimates to show that $\Lambda$ is a contraction.

To show that  $\Lambda$  is a contraction we need the following estimates: First, let  $U, V$  and  $W$  be points in  $C(T)$ . Let  $x_\kappa$  be the flow of  $U_\kappa$ , where  $U_\kappa$  is the smoothed version of  $U$ . Define  $\Omega_t = x_\kappa(\Omega, t)$ ,  $v_\kappa = V_\kappa \circ x_\kappa^{-1}$  and  $w = W \circ x_\kappa^{-1}$ . Define a function  $f$  by

$$\Delta f = -(\nabla v_\kappa)(\nabla w) \text{ on } \Omega_t, \quad (3.2.1)$$

where  $\nabla$  denotes differentiation with respect to the coordinates  $x_\kappa$ , with boundary condition

$$f = 0 \text{ on } \partial\Omega_t. \quad (3.2.2)$$

Employing the same approach as in section 3.1, we can prove the following theorem:

**THEOREM 3.2.1** *For  $f$  defined by (3.2.1) and (3.2.2) we have*

$$\|\nabla f\|_{H^7(\Omega_t)} \leq P[\|x_\kappa\|_8] \|v_\kappa\|_{H^7(\Omega_t)} \|w\|_{H^7(\Omega_t)} \quad (3.2.3)$$

and

$$\|\nabla f\|_{H^8(\Omega_t)} \leq P[\|x_\kappa\|_8, \kappa^{-1}\|x_\kappa\|_8] \|v_\kappa\|_{H^8(\Omega_t)} \|w\|_{H^8(\Omega_t)} \quad (3.2.4)$$

where  $P$  is linear in  $\kappa^{-1}\|x_\kappa\|_8$ .

Second, let  $U_1, U_2, V$  and  $W$  be in  $C(T)$ . Smooth  $U_1, U_2$  and  $V$  to obtain  $(U_1)_\kappa, (U_2)_\kappa$  and  $V_\kappa$ . Let  $u_1$  and  $u_2$  be the flows of  $(U_1)_\kappa$  and  $(U_2)_\kappa$  respectively. Let  $x_1$  and  $x_2$  be the coordinates on  $\Omega_{t,1} = u_1(t, \Omega)$  and  $\Omega_{t,2} = u_2(t, \Omega)$  respectively. For  $k = 1, 2$ , define

$$\Delta f_k = - \left( \frac{\partial}{\partial x_k^i} [V_\kappa^j \circ u_k^{-1}] \right) \left( \frac{\partial}{\partial x_k^j} [W^i \circ u_k^{-1}] \right) + 1 \text{ on } \Omega_{t,k} \quad (3.2.5)$$

with boundary condition

$$p = 0 \text{ on } \partial\Omega_{t,k}. \quad (3.2.6)$$

The estimates to compare  $f_1$  and  $f_2$  must be performed in Lagrangian coordinates. We now prove the following theorem:

**THEOREM 3.2.2**

$$\|(\nabla_1 f_1) \circ u_1 - (\nabla_2 f_2) \circ u_2\|_8 \quad (3.2.7)$$

$$\leq [1 + \kappa^{-1}] \|U_1 - U_2\|_8 P [\|U_1\|_8, \|U_2\|_8, \kappa^{-1}\|U_1\|_8, \kappa^{-1}\|U_2\|_8, \|V\|_8, \|W\|_8] \quad (3.2.8)$$

where  $\nabla_k$  denotes derivatives with respect to the coordinates  $x_k$  on  $\Omega_{t,k}$ .

**3.2.1 Interior estimates.**

In this case, as in section 3.1, the interior estimates follow more readily than the boundary estimates. We therefore state the interior estimates and omit the proof:

**PROPOSITION 3.2.3** *For all  $1 \leq i$ , we have*

$$\|(\eta_i \nabla_1 f_1) \circ u_1 - (\eta_i \nabla_2 f_2) \circ u_2\|_8 \quad (3.2.9)$$

$$\leq \|U_1 - U_2\|_8 P [\|U_1\|_8, \|U_2\|_8, \kappa^{-1}\|U_1\|_8, \kappa^{-1}\|U_2\|_8, \|V\|_8, \|W\|_8]. \quad (3.2.10)$$

**3.2.2 Boundary estimates.**

To obtain the boundary estimates we now build regularity in much the same way that we did in section 3.1. Define

$$A(k)_i^a(y) = \left( \frac{\partial(u_k^{-1})^a}{\partial x_k^i} \right) (s, u_k(s, y)) \text{ and } B(k)_a^i(y) = \left( \frac{\partial u_k^i}{\partial y^a} \right) (s, y).$$

First we prove a lemma showing a relationship between  $A(1)$  and  $A(2)$ , and  $U_1$  and  $U_2$ .

**LEMMA 3.2.4** *For  $0 \leq j \leq 4$  we have*

$$\|\partial^j A(1) - \partial^j A(2)\|_\infty \leq P[\|U_1\|_8, \|U_2\|_8] \|U_1 - U_2\|_8 \quad (3.2.11)$$

and for  $I$  with  $0 \leq j \leq 7$  we have

$$\|\partial^j A(1) - \partial^j A(2)\| \leq P[\|U_1\|_8, \|U_2\|_8] \|U_1 - U_2\|_8. \quad (3.2.12)$$

Also,  $\|\partial^j B(1) - \partial^j B(2)\| \leq P[\|U_1\|_8, \|U_2\|_8] \|U_1 - U_2\|_8$ .

**PROOF:** Since  $A(1) - A(2) = B(1)^{-1} - B(2)^{-1}$  and

$$B(k)^{-1} = \frac{C(k)}{\det(B(k))},$$

where  $C(k)$  is the cofactor matrix of  $B(k)$  we have  $\|A(1) - A(2)\| \leq c\|U_1 - U_2\|_1$  and  $\|A(1) - A(2)\|_\infty \leq c\|U_1 - U_2\|_4$ . This proves the base-case of both (3.2.11) and (3.2.12). Now for  $j \geq 1$ ,

$$\partial^j A(k) = - \sum (\partial^{j_1} A(k)) (\partial^{j_2} B) (\partial^{j_3} A(k))$$

where the sum is over  $j_1 + j_2 + j_3 = j$  and such that  $j_2, j_3 \leq j - 1$ . Thus

$$\partial^j A(1) - \partial^j A(2) = \sum (\partial^{j_1} A(2))(\partial^{j_2} B(2))(\partial^{j_3} A(2)) - \sum (\partial^{j_1} A(1))(\partial^{j_2} B(1))(\partial^{j_3} A(1)) \quad (3.2.13)$$

$$= \sum (\partial^{j_1} A(2))(\partial^{j_2} B(2))(\partial^{j_3} A(2)) - \sum (\partial^{j_1} A(1))(\partial^{j_2} B(2))(\partial^{j_3} A(2)) \quad (3.2.14)$$

$$+ \sum (\partial^{j_1} A(1))(\partial^{j_2} B(2))(\partial^{j_3} A(2)) - \sum (\partial^{j_1} A(1))(\partial^{j_2} B(1))(\partial^{j_3} A(2)) \quad (3.2.15)$$

$$+ \sum (\partial^{j_1} A(1))(\partial^{j_2} B(1))(\partial^{j_3} A(2)) - \sum (\partial^{j_1} A(1))(\partial^{j_2} B(1))(\partial^{j_3} A(1)) \quad (3.2.16)$$

$$= \sum [(\partial^{j_1} A(2)) - (\partial^{j_1} A(1))] (\partial^{j_2} B(2))(\partial^{j_3} A(2)) \quad (3.2.17)$$

$$+ \sum (\partial^{j_1} A(1)) [(\partial^{j_2} B(2)) - (\partial^{j_2} B(1))] (\partial^{j_3} A(2)) \quad (3.2.18)$$

$$+ \sum (\partial^{j_1} A(1))(\partial^{j_2} B(1)) [(\partial^{j_3} A(2)) - (\partial^{j_3} A(1))] . \quad (3.2.19)$$

One infers (3.2.11) from (3.2.13). Now suppose that  $1 \leq j \leq 7$  and that we have (3.2.12) for smaller  $j$ . In the first sum, if  $j_1 \leq 4$  then we control  $\|\partial^{j_1} A(2) - \partial^{j_1} A(1)\|_\infty$  appropriately using (3.2.11). If  $j_2 \leq 4$  then we control  $\|\partial^{j_2} B(2)\|_\infty$  by  $\|U_2\|_8$ . In this case,  $0 \leq j_3 \leq 6$  and therefore we can control  $\|\partial^{j_3} A(2)\|$  by  $\|U_2\|_8$ . If  $5 \leq j_2 \leq 7$  then we can control  $\|\partial^{j_2} B(2)\|$  by  $\|U_2\|_8$ ; and now  $0 \leq j_1, j_3 \leq 2$  so we can control  $\|\partial^{j_1} A(2) - \partial^{j_1} A(1)\|_\infty$  using (3.2.11) and  $\|\partial^{j_3} A(2)\|_\infty$  by  $\|U_2\|_8$ . If  $5 \leq j_1 \leq 6$ , then we control  $\|\partial^{j_1} A(2) - \partial^{j_1} A(1)\|$  appropriately using the inductive hypothesis, since  $j_1 \leq j - 1$ . And  $1 \leq j_2, j_3 \leq 2$  which means that we control  $\|\partial^{j_2} B(1)\|_\infty$  and  $\|\partial^{j_3} A(2)\|_\infty$  appropriately. The second and third sum follow similarly.  $\blacksquare$

Just as lemma 2.5.1 was central to the proof of lemma 3.1.6, so lemma B.4.1 from appendix B is central to lemma 3.2.5 below. We let  $\zeta = \zeta_1$ , where the  $\zeta_i$  are cut off functions defined in chapter 2.

**LEMMA 3.2.5** For  $0 \leq j \leq 7$ ,

$$\|(\zeta \nabla_1 \partial^j \nabla_1 f_1) \circ u_1 - (\zeta \nabla_2 \partial^j \nabla_2 f_2) \circ u_2\| \quad (3.2.20)$$

$$\leq \|U_1 - U_2\|_8 P [\|U_1\|_8, \|U_2\|_8, \kappa^{-1}\|U_1\|_8, \kappa^{-1}\|U_2\|_8, \|V\|_8, \|W\|_8] \quad (3.2.21)$$

$$+ \|(\zeta \partial_\theta \partial^j \nabla_1 f_1) \circ u_1 - (\zeta \partial_\theta \partial^j \nabla_2 f_2) \circ u_2\|, \quad (3.2.22)$$

and for  $0 \leq j \leq 4$  we have

$$\|(\zeta \nabla_1 \partial^j \nabla_1 f_1) \circ u_1 - (\zeta \nabla_2 \partial^j \nabla_2 f_2) \circ u_2\|_\infty \quad (3.2.23)$$

$$\leq \|U_1 - U_2\|_8 P [\|U_1\|_8, \|U_2\|_8, \kappa^{-1}\|U_1\|_8, \kappa^{-1}\|U_2\|_8, \|V\|_8, \|W\|_8] \quad (3.2.24)$$

$$+ \|(\zeta \partial_\theta \partial^{j+3} \nabla_1 f_1) \circ u_1 - (\zeta \partial_\theta \partial^{j+3} \nabla_2 f_2) \circ u_2\|. \quad (3.2.25)$$

**PROOF:** Let  $j = 0$ . Then according to lemma B.4.1 we have,

$$|(\zeta \nabla_1^2 f_1) \circ u_1 - (\zeta \nabla_2^2 f_2) \circ u_2| \leq |(\zeta \Delta_1 f_1) \circ u_1 - (\zeta \Delta_2 f_2 \alpha_2) \circ u_2| \quad (3.2.26)$$

$$+ |(\zeta \partial_\theta \nabla_1 f_1) \circ u_1 - (\zeta \partial_\theta \nabla_2 f_2) \circ u_2| + |Q_1 - Q_2| |(\zeta \nabla_2^2 f_2) \circ u_2|. \quad (3.2.27)$$

Now

$$(\Delta_1 f_1) \circ u_1 - (\Delta_2 f_2) \circ u_2 = -A(1)(\partial V_\kappa)A(1)(\partial W) + A(2)(\partial V_\kappa)A(2)(\partial W) \quad (3.2.28)$$

$$= -A(1)(\partial V_\kappa)A(1)(\partial W) + A(2)(\partial V_\kappa)A(1)(\partial W) \quad (3.2.29)$$

$$- A(2)(\partial V_\kappa)A(1)(\partial W) + A(2)(\partial V_\kappa)A(2)(\partial W) \quad (3.2.30)$$

$$= [A(2) - A(1)](\partial V_\kappa)A(1)(\partial W) + A(2)(\partial V_\kappa)[A(2) - A(1)](\partial W) \quad (3.2.31)$$

and this can be controlled by  $\|U_1 - U_2\|_8 P[\|U_1\|_8, \|U_2\|_8] \|V\|_8 \|W\|_8$ . The second term in (3.2.26) can be controlled by  $\|(\partial_\theta \nabla_1 f_1) \circ u_1 - (\partial_\theta \nabla_2 f_2) \circ u_2\|$  and the third term can be controlled by

$$\|U_1 - U_2\|_8 P[\|U_2\|_8, \|V\|_8, \|W\|_8].$$

Now suppose that  $1 \leq j \leq 4$ , and that we have (3.2.20) for smaller  $j$ . For a function  $\theta$  defined on  $\Omega_{t,1}$ ,

$$\partial^j \frac{\partial}{\partial x_1^i} \theta - \frac{\partial}{\partial x_1^i} \partial^j \theta = \sum (\partial^{j_1} A(1)) B(1) (\nabla_1 \partial^{j_2} \theta)$$

where the sum is over  $j_1 + j_2 = j$  and  $j_2 \leq j - 1$  we see that

$$(\text{curl}_1 \partial^j \nabla_1 f_1) \circ u_1 - (\text{curl}_2 \partial^j \nabla_2 f_2) \circ u_2 \quad (3.2.32)$$

$$= \sum (\partial^{j_1} A(1)) B(1) (\nabla_1 \partial^{j_2} \nabla_1 f_1) \circ u_1 \quad (3.2.33)$$

$$- \sum (\partial^{j_1} A(2)) B(2) (\nabla_2 \partial^{j_2} \nabla_2 f_2) \circ u_2 \quad (3.2.34)$$

$$= \sum [(\partial^{j_1} A(1)) - (\partial^{j_1} A(2))] B(1) (\nabla_1 \partial^{j_2} \nabla_1 f_1) \circ u_1 \quad (3.2.35)$$

$$+ \sum (\partial^{j_1} A(2)) [B(1) - B(2)] (\nabla_1 \partial^{j_2} \nabla_1 f_1) \circ u_1 \quad (3.2.36)$$

$$+ \sum (\partial^{j_1} A(2)) B(2) [(\nabla_1 \partial^{j_2} \nabla_1 f_1) \circ u_1 - (\nabla_2 \partial^{j_2} \nabla_2 f_2) \circ u_2]. \quad (3.2.37)$$

We can control the above by the inductive hypothesis since  $j_2 \leq j - 1$ . In addition, to control  $(\text{div}_1 \partial^j \nabla_1 f_1) \circ u_1 - (\text{div}_2 \partial^j \nabla_2 f_2) \circ u_2$  we must also control

$$(\partial^j \Delta_1 f_1) \circ u_1 - (\partial^j \Delta_2 f_2) \circ u_2 = [(\partial^{j_1} A(2)) - (\partial^{j_1} A(1))] (\partial \partial^{j_2} V_\kappa) ((\partial^{j_3} A(1)) (\partial \partial^{j_4} W)) \quad (3.2.38)$$

$$+ (\partial^{j_1} A(2)) (\partial \partial^{j_2} V_\kappa^j) [((\partial^{j_3} A(2)) - (\partial^{j_3} A(1))) (\partial \partial^{j_4} W)]. \quad (3.2.39)$$

where the sum is over  $j_1 + j_2 + j_3 + j_4 = j$ . As we have seen, we can control the above terms by  $\|U_1 - U_2\|_8 P[\|U_1\|_8, \|U_2\|_8, \|V\|_8, \|W\|_8]$  for  $0 \leq j \leq 7$ . Now we prove (3.2.23) for  $0 \leq j \leq 1$ , and using this result we can prove (3.2.20) for  $j = 5$ . Using that result we can prove (3.2.23) for  $j = 2$ , etc.  $\blacksquare$

Suppose now that we control  $\|(\zeta\partial_\theta^j\nabla_1 f_1) \circ u_1 - (\zeta\partial_\theta^j\nabla_2 f_2) \circ u_2\|$  appropriately for  $0 \leq j \leq 8$ . Then (3.2.20) says that we can control  $\|(\zeta\nabla_1\partial_\theta^j\nabla_1 f_1) \circ u_1 - (\zeta\nabla_2\partial_\theta^j\nabla_2 f_2) \circ u_2\|$  for  $0 \leq j \leq 7$ . That is, we control  $\|(\zeta\partial_\theta^j\partial\nabla_1 f_1) \circ u_1 - (\zeta\partial_\theta^j\partial\nabla_2 f_2) \circ u_2\|$  for  $0 \leq j \leq 7$ . Using (3.2.20) again, this means that we can control  $\|(\zeta\nabla_1\partial\partial_\theta^j\nabla_1 f_1) \circ u_1 - (\zeta\nabla_2\partial\partial_\theta^j\nabla_2 f_2) \circ u_2\|$  for  $0 \leq j \leq 6$ . Inductively, then, we can control  $\|(\zeta\nabla_1 f_1) \circ u_1 - (\zeta\nabla_2 f_2) \circ u_2\|_8$ . It remains, therefore, to bound  $\|(\zeta\partial_\theta^j\nabla_1 f_1) \circ u_1 - (\zeta\partial_\theta^j\nabla_2 f_2) \circ u_2\|$  for  $0 \leq j \leq 8$ :

**PROPOSITION 3.2.6** *For  $0 \leq j \leq 8$ ,*

$$\|(\zeta\partial_\theta^j\nabla_1 f_1) \circ u_1 - (\zeta\partial_\theta^j\nabla_2 f_2) \circ u_2\| \quad (3.2.40)$$

$$\leq \kappa^{-1}\|U_1 - U_2\|_8, P[\|U_1\|_8, \|U_2\|_8, \kappa^{-1}\|U_1\|_8, \kappa^{-1}\|U_2\|_8, \|V\|_8, \|W\|_8]. \quad (3.2.41)$$

We prove this proposition by induction on  $j$ . We have

$$(\nabla_1 f_1) \circ u_1 - (\nabla_2 f_2) \circ u_2 = A(1)_i^a [(\partial_a F_1) - (\partial_a F_2)] + [A(1)_i^a - A(2)_i^a] (\partial_a F_2), \quad (3.2.42)$$

where  $F_k = f_k \circ u_k$ . We control the second term in (3.2.42) by lemma 3.2.4. Also,

$$\int_\Omega \delta^{ij} A(1)_i^a [(\partial_a F_1) - (\partial_a F_2)] A(1)_j^b [(\partial_b F_1) - (\partial_b F_2)] dy \quad (3.2.43)$$

$$= - \int_\Omega \partial_b [\delta^{ij} A(1)_i^a A(1)_j^b [(\partial_a F_1) - (\partial_a F_2)]] [F_1 - F_2] dy \quad (3.2.44)$$

$$\leq \|\partial_b [\delta^{ij} A(1)_i^a A(1)_j^b [(\partial_a F_1) - (\partial_a F_2)]]\| \|F_1 - F_2\|. \quad (3.2.45)$$

Using the Poincaré inequality we have  $\|F_1 - F_2\| \leq c\|\partial F_1 - \partial F_2\|$ .

$$\partial_b [\delta^{ij} A(1)_i^a A(1)_j^b [(\partial_a F_1) - (\partial_a F_2)]] = \partial_b [\delta^{ij} A(1)_i^a A(1)_j^b (\partial_a F_1)] - \partial_b [\delta^{ij} A(1)_i^a A(1)_j^b (\partial_a F_2)] \quad (3.2.46)$$

and

$$\partial_b [\delta^{ij} A(1)_i^a A(1)_j^b (\partial_a F_1)] = \delta^{ij} (\partial_b A(1)_i^a) A(1)_j^b (\partial_a F_1) + \delta^{ij} A(1)_i^a (\partial_b A(1)_j^b) (\partial_a F_1) \quad (3.2.47)$$

$$+ \delta^{ij} A(1)_i^a A(1)_j^b (\partial_a \partial_b F_1). \quad (3.2.48)$$

We have  $(\Delta f_1)(u_1(t, y)) = \delta^{ij} A(1)_j^b (\partial_b A(1)_i^a) (\partial_a F_1) + \delta^{ij} A(1)_j^b A(1)_i^a (\partial_a \partial_b F_1)$  and we therefore see that

$$\partial_b [\delta^{ij} A(1)_i^a A(1)_j^b (\partial_a F_1)] = -A(1)_i^a (\partial_a V_\kappa^j) A(1)_j^b (\partial_b W^i) + 1 + \delta^{ij} A(1)_i^a (\partial_b A(1)_j^b) (\partial_a F_1). \quad (3.2.49)$$

We also have,

$$\partial_b [\delta^{ij} A(1)_i^a A(1)_j^b (\partial_a F_2)] = \delta^{ij} (\partial_b A(1)_i^a) A(1)_j^b (\partial_a F_2) + \delta^{ij} A(1)_i^a (\partial_b A(1)_j^b) (\partial_a F_2) \quad (3.2.50)$$

$$+ \delta^{ij} A(1)_i^a A(1)_j^b (\partial_a \partial_b F_2) \quad (3.2.51)$$

$$= \delta^{ij} [(\partial_b A(1)_i^a) - (\partial_b A(2)_i^a)] A(1)_j^b (\partial_a F_2) \quad (3.2.52)$$

$$+ \delta^{ij} (\partial_b A(2)_i^a) [A(1)_j^b - A(2)_j^b] (\partial_a F_2) \quad (3.2.53)$$

$$+ \delta^{ij} [A(1)_i^a - A(2)_i^a] A(1)_j^b (\partial_a \partial_b F_2) \quad (3.2.54)$$

$$+ \delta^{ij} A(2)_i^a [A(1)_j^b - A(2)_j^b] (\partial_a \partial_b F_2) \quad (3.2.55)$$

$$+ \delta^{ij} (\partial_b A(2)_i^a) A(2)_j^b (\partial_a F_2) \quad (3.2.56)$$

$$+ \delta^{ij} A(2)_i^a A(2)_j^b (\partial_a \partial_b F_2) + \delta^{ij} A(1)_i^a (\partial_b A(1)_j^b) (\partial_a F_2). \quad (3.2.57)$$

We control the first four terms in (3.2.52) by  $P[\|U_1\|_8, \|U_2\|_8, \|V_\kappa\|_8, \|W\|_8] \|U_1 - U_2\|_8$ . Similarly to (3.2.49),  $\delta^{ij} (\partial_b A(2)_i^a) A(2)_j^b (\partial_a F_2) + \delta^{ij} A(2)_i^a A(2)_j^b (\partial_a \partial_b F_2) = -A(2)_i^a (\partial_a V_\kappa^j) A(2)_j^b (\partial_b W^i) + 1$ . Combining this with the first two terms from (3.2.49), we have

$$- A(1)_i^a (\partial_a V_\kappa^j) A(1)_j^b (\partial_b W^i) + A(2)_i^a (\partial_a V_\kappa^j) A(2)_j^b (\partial_b W^i) \quad (3.2.58)$$

$$= -A(1)_i^a (\partial_a V_\kappa^j) A(1)_j^b (\partial_b W^i) + A(2)_i^a (\partial_a V_\kappa^j) A(1)_j^b (\partial_b W^i) \quad (3.2.59)$$

$$- A(2)_i^a (\partial_a V_\kappa^j) A(1)_j^b (\partial_b W^i) + A(2)_i^a (\partial_a V_\kappa^j) A(2)_j^b (\partial_b W^i) \quad (3.2.60)$$

$$= [A(2)_i^a - A(1)_i^a] (\partial_a V_\kappa^j) A(1)_j^b (\partial_b W^i) \quad (3.2.61)$$

$$+ A(2)_i^a (\partial_a V_\kappa^j) [A(2)_j^b - A(1)_j^b] (\partial_b W^i) \quad (3.2.62)$$

which can be controlled by  $P[\|U_1\|_8, \|U_2\|_8, \|V_\kappa\|_8, \|W\|_8] \|U_1 - U_2\|_8$ . Finally, combining last term from (3.2.49) and the last term from (3.2.52) we have,  $\|\delta^{ij} A(1)_i^a (\partial_b A(1)_j^b) (\partial_a F_1 - \partial_a F_2)\| \leq \|U_1\|_8 \|\partial_b A(1)_j^b\|_\infty \|\partial_a F_1 - \partial_a F_2\|$ . The following lemma shows that  $\partial_b A(1)_j^b$  is small.

**LEMMA 3.2.7**

$$\|\partial_b A_j^b\|_\infty \leq T \|\det(A)\|_\infty \sup_{[0, T]} P[\|x_\kappa\|_8] \exp \left[ T \sup_{[0, T]} P[\|V_\kappa\|_1] \right].$$

**PROOF:** Using the formula

$$\nabla \det(M) = \det(M) \operatorname{tr} [M^{-1} \nabla M] \quad (3.2.63)$$

we have

$$\partial_j \det(B) = \det(B) A_i^a (\partial_j B_a^i) \quad (3.2.64)$$

$$= -\det(B) (\partial_j A_i^a) B_a^i \quad (3.2.65)$$

$$= -\det(B) (\partial_i A_j^a) B_a^i \quad (3.2.66)$$

$$= -\det(B) (\partial_a A_j^a). \quad (3.2.67)$$

Thus  $\|\partial_a A(1)_j^a\|_\infty \leq \|\det(A)\|_\infty \|\nabla \det(B)\|_\infty$ . Again, using (3.2.63) we have  $\partial_t \det(B) = \det(B) A_i^a (\partial_t \partial_a x_\kappa) = \det(B) A_i^a (\partial_a V_\kappa^i)$ . Since  $\det(B)(0, y) = 1$  we have

$$\det(B)(t, y) = \exp \left[ \int_{[0,t]} A_i^a (\partial_a V_\kappa^i) ds \right]$$

and

$$\nabla \det(B)(t, y) = \exp \left[ \int_{[0,t]} A_i^a (\partial_a V_\kappa^i) ds \right] \left[ \int_{[0,t]} [(\nabla A)(\partial_a V_\kappa) + A(\nabla^2 V_\kappa)] ds \right].$$

Thus

$$\|\nabla \det(B)\|_\infty \leq T \sup_{[0,T]} P[\|x_\kappa\|_8] \exp \left[ T \sup_{[0,T]} P[\|V_\kappa\|_1] \right].$$

■

This concludes the base case. Now suppose that  $j = 8$  and that we have proposition 3.2.6 for smaller  $j$ . Then

$$\|(\partial_\theta^8 \nabla_1 f_1) \circ u_1 - (\partial_\theta^8 \nabla_2 f_2) \circ u_2\|^2 \quad (3.2.68)$$

$$= \int_\Omega \delta^{ij} [(\partial_\theta^8 A(1)_i^a)(\partial_a F_1) - (\partial_\theta^8 A(2)_i^a)(\partial_a F_2)] [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.69)$$

$$+ \int_\Omega \delta^{ij} [A_i^a(1)(\partial_a \partial_\theta^8 F_1) - A_i^a(2)(\partial_a \partial_\theta^8 F_2)] [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.70)$$

$$+ \sum \int_\Omega \delta^{ij} [(\partial_\theta^{j_1} A_i^a(1))(\partial_a \partial_\theta^{j_2} F_1) - (\partial_\theta^{j_1} A_i^a(2))(\partial_a \partial_\theta^{j_2} F_2)] [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.71)$$

where the sum is over all  $j_1$  and  $j_2$  such that  $j_1 + j_2 = 8$  and  $j_1, j_2 \leq 7$ . The first factor in the third term in (3.2.69) is equal to

$$(\partial_\theta^{j_1} A_i^a(1))(\partial_a \partial_\theta^{j_2} F_1) - (\partial_\theta^{j_1} A_i^a(2))(\partial_a \partial_\theta^{j_2} F_2) = [(\partial_\theta^{j_1} A_i^a(1)) - (\partial_\theta^{j_1} A_i^a(2))](\partial_a \partial_\theta^{j_2} F_1) \quad (3.2.72)$$

$$+ (\partial_\theta^{j_1} A_i^a(2))[(\partial_a \partial_\theta^{j_2} F_1) - (\partial_a \partial_\theta^{j_2} F_2)] \quad (3.2.73)$$

thus the third term in (3.2.69) can be controlled appropriately. The first term in (3.2.69) gives

$$\int_\Omega \delta^{ij} [(\partial_\theta^8 A(1)_i^a)(\partial_a F_1) - (\partial_\theta^8 A(2)_i^a)(\partial_a F_2)] [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.74)$$

$$= - \int_\Omega \delta^{ij} [A_i^c(1)(\partial_c \partial_\theta^8 u_1^k)(\partial_{1k} f_1) \circ u_1 - A_i^c(2)(\partial_c \partial_\theta^8 u_2^k)(\partial_{2k} f_2) \circ u_1] \quad (3.2.75)$$

$$\times [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.76)$$

$$- \int_\Omega \delta^{ij} [(\partial_\theta^{j_1} A_i^c(1))(\partial_\theta^{j_2} B_c^k(1))(\partial_\theta^{j_3} A_k^a(1))(\partial_a F_1) - (\partial_\theta^{j_1} A_i^c(2))(\partial_\theta^{j_2} B_c^k(2))(\partial_\theta^{j_3} A_k^a(2))(\partial_a F_2)] \quad (3.2.77)$$

$$\times [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy. \quad (3.2.78)$$

The first factor in the the second integral in (3.2.75) is equal to

$$(\partial_\theta^{j_1} A_i^c(1))(\partial_\theta^{j_2} B_c^k(1))(\partial_\theta^{j_3} A_k^a(1))(\partial_a F_1) - (\partial_\theta^{j_1} A_i^c(2))(\partial_\theta^{j_2} B_c^k(2))(\partial_\theta^{j_3} A_k^a(2))(\partial_a F_2) \quad (3.2.79)$$

$$= [(\partial_\theta^{j_1} A_i^c(1)) - (\partial_\theta^{j_1} A_i^c(2))](\partial_\theta^{j_2} B_c^k(1))(\partial_\theta^{j_3} A_k^a(1))(\partial_a F_1) \quad (3.2.80)$$

$$+ (\partial_\theta^{j_1} A_i^c(2))[(\partial_\theta^{j_2} B_c^k(1)) - (\partial_\theta^{j_2} B_c^k(2))](\partial_\theta^{j_3} A_k^a(1))(\partial_a F_1) \quad (3.2.81)$$

$$+ (\partial_\theta^{j_1} A_i^c(2))(\partial_\theta^{j_2} B_c^k(2))[(\partial_\theta^{j_3} A_k^a(1)) - (\partial_\theta^{j_3} A_k^a(2))](\partial_a F_1) \quad (3.2.82)$$

$$+ (\partial_\theta^{j_1} A_i^c(2))(\partial_\theta^{j_2} B_c^k(2))(\partial_\theta^{j_3} A_k^a(2))[(\partial_a F_1) - (\partial_a F_2)] \quad (3.2.83)$$

and therefore we can control the second integral in (3.2.75) appropriately. Because of the tangential smoothing, the first integral in (3.2.75) is equal to

$$- \kappa^{-1} \int_\Omega \delta^{ij} [A_i^c(1)(\partial_c \partial_\theta^7 u_1^k)(\partial_{1k} f_1) \circ u_1 - A_i^c(2)(\partial_c \partial_\theta^7 u_2^k)(\partial_{2k} f_2) \circ u_1] \quad (3.2.84)$$

$$\times [(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.85)$$

which we control appropriately. Integrating the first half of the second term from (3.2.69) by parts gives

$$\int_\Omega \delta^{ij} A_i^a(1)(\partial_a \partial_\theta^8 F_1)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.86)$$

$$= - \int_\Omega \delta^{ij} (\partial_a A_i^a(1))(\partial_\theta^8 F_1)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.87)$$

$$- \int_\Omega \delta^{ij} (\partial_\theta^8 F_1)[(\partial_{1i} \partial_\theta^8 \partial_{1j} f_1) \circ u_1 - A_i^a(1) \partial_a (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.88)$$

Integrating the second half of the second term from (3.2.69) by parts gives

$$- \int_\Omega \delta^{ij} A_i^a(2)(\partial_a \partial_\theta^8 F_2)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.89)$$

$$= \int_\Omega \delta^{ij} (\partial_a A_i^a(2))(\partial_\theta^8 F_2)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy \quad (3.2.90)$$

$$+ \int_\Omega \delta^{ij} (\partial_\theta^8 F_2)[A_i^a(2) \partial_a (\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_{2i} \partial_\theta^8 \partial_{2j} f_2) \circ u_2] dy. \quad (3.2.91)$$

The first two factors in the first term in (3.2.87) combines with the first two factors in the first term in (3.2.90) to give

$$- \delta^{ij} (\partial_a A_i^a(1))(\partial_\theta^8 F_1)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] \quad (3.2.92)$$

$$+ \delta^{ij} (\partial_a A_i^a(2))(\partial_\theta^8 F_2)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] \quad (3.2.93)$$

$$= -\delta^{ij} [(\partial_a A_i^a(1)) - (\partial_a A_i^a(2))](\partial_\theta^8 F_1)[(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2] \quad (3.2.94)$$

$$- \delta^{ij} (\partial_a A_i^a(2))[(\partial_\theta^8 F_1) - (\partial_\theta^8 F_2)][(\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_\theta^8 \partial_{2j} f_2) \circ u_2]. \quad (3.2.95)$$

The first term in (3.2.94) we control. The second term in (3.2.94) will be controlled using the following lemma:



**LEMMA 3.2.8** For  $1 \leq j \leq 9$ ,

$$\|\partial_\theta^j F_1 - \partial_\theta^j F_2\| \leq \|U_1 - U_2\|_8 P[\|U_1\|_8, \|U_2\|_8, \|V\|_8, \|W\|_8] \quad (3.2.96)$$

$$+ P[\|U_1\|_8, \|U_2\|_8] \sum_{0 \leq k \leq j-1} \|(\partial_\theta^k \nabla_1 f) \circ u_1 - (\partial_\theta^k \nabla_2 f) \circ u_1\|. \quad (3.2.97)$$

**PROOF:** We have

$$\partial_\theta^j F_1 = \partial_\theta^{j-1} [\partial_\theta u_1^k (\partial_{1k} f) \circ u_1] \quad (3.2.98)$$

$$= \sum (\partial_\theta^{j_1+1} u_1^k) (\partial^{j_2} \partial_{1k} f_1) \circ u_1 + (\partial_\theta u_1^k) (\partial^{j-1} \partial_{1k} f_1) \circ u_1 \quad (3.2.99)$$

where the sum is over  $j_1 + j_2 = j - 1$  and  $j_2 \leq j - 2$ , and similarly for  $\partial_\theta^j F_2$ . Thus

$$\partial_\theta^j F_1 - \partial_\theta^j F_2 = \sum (\partial_\theta^{j_1+1} u_1^k) (\partial_\theta^{j_2} \partial_{1k} f_1) \circ u_1 - \sum (\partial_\theta^{j_1+1} u_2^k) (\partial_\theta^{j_2} \partial_{2k} f_2) \circ u_2 \quad (3.2.100)$$

$$+ (\partial_\theta u_1^k) (\partial_\theta^{j_1+1} \partial_{1k} f_1) \circ u_1 - (\partial_\theta u_2^k) (\partial_\theta^{j_1+1} \partial_{2k} f_2) \circ u_2 \quad (3.2.101)$$

$$= \sum [(\partial_\theta^{j_1+1} u_1^k) - (\partial_\theta^{j_1+1} u_2^k)] (\partial_\theta^{j_2} \partial_{1k} f_1) \circ u_1 \quad (3.2.102)$$

$$+ \sum (\partial_\theta^{j_1+1} u_2^k) [(\partial_\theta^{j_2} \partial_{1k} f_1) \circ u_1 - (\partial_\theta^{j_2} \partial_{2k} f_2) \circ u_2] \quad (3.2.103)$$

$$+ [(\partial_\theta u_1^k) - (\partial_\theta u_2^k)] (\partial_\theta^{j_1+1} \partial_{1k} f_1) \circ u_1 \quad (3.2.104)$$

$$+ (\partial_\theta u_2^k) [(\partial_\theta^{j_1+1} \partial_{1k} f_1) \circ u_1 - (\partial_\theta^{j_1+1} \partial_{1k} f_2) \circ u_1]. \quad (3.2.105)$$

■

The second term in (3.2.87) gives

$$- \delta^{ij} (\partial_\theta^8 F_1) (\partial_{1i} \partial_\theta^8 \partial_{1j} f_1) \circ u_1 + \delta^{ij} (\partial_\theta^8 F_1) A_i^\alpha(1) \partial_\alpha (\partial_\theta^8 \partial_{2j} f_2) \circ u_2 \quad (3.2.106)$$

$$= -\delta^{ij} (\partial_\theta^8 F_1) [(\partial_{1i} \partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_{2i} \partial_\theta^8 \partial_{2j} f_2) \circ u_2] \quad (3.2.107)$$

$$- \delta^{ij} (\partial_\theta^8 F_1) [A_i^\alpha(2) - A_i^\alpha(1)] \partial_\alpha (\partial_\theta^8 \partial_{2j} f_2) \circ u_2. \quad (3.2.108)$$

and the second term in (3.2.90) gives

$$\delta^{ij} (\partial_\theta^8 F_2) A_i^\alpha(2) \partial_\alpha (\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - \delta^{ij} (\partial_\theta^8 F_2) (\partial_{2i} \partial_\theta^8 \partial_{2j} f_2) \circ u_2 \quad (3.2.109)$$

$$= \delta^{ij} (\partial_\theta^8 F_2) [A_i^\alpha(2) - A_i^\alpha(1)] \partial_\alpha (\partial_\theta^8 \partial_{1j} f_1) \circ u_1 \quad (3.2.110)$$

$$+ \delta^{ij} (\partial_\theta^8 F_2) [(\partial_{1i} \partial_\theta^8 \partial_{1j} f_1) \circ u_1 - (\partial_{2i} \partial_\theta^8 \partial_{2j} f_2) \circ u_2]. \quad (3.2.111)$$

The second term from (3.2.107) combines with the first term from (3.2.110) to give

$$- \delta^{ij} (\partial_\theta^8 F_1) [A_i^\alpha(2) - A_i^\alpha(1)] \partial_\alpha (\partial_\theta^8 \partial_{2j} f_2) \circ u_2 + \delta^{ij} (\partial_\theta^8 F_2) [A_i^\alpha(2) - A_i^\alpha(1)] \partial_\alpha (\partial_\theta^8 \partial_{1j} f_1) \circ u_1 \quad (3.2.112)$$

$$= -\delta^{ij} [(\partial_\theta^8 F_1) - (\partial_\theta^8 F_2)] [A_i^\alpha(2) - A_i^\alpha(1)] \partial_\alpha (\partial_\theta^8 \partial_{2j} f_2) \circ u_2 \quad (3.2.113)$$

$$- \delta^{ij} (\partial_\theta^8 F_2) [A_i^\alpha(2) - A_i^\alpha(1)] [\partial_\alpha (\partial_\theta^8 \partial_{1j} f_1) \circ u_1 - \partial_\alpha (\partial_\theta^8 \partial_{2j} f_2) \circ u_2]. \quad (3.2.114)$$

Here we control the first of the above terms by lemma 3.2.8. The first term from (3.2.107) combines with the second term from (3.2.110) to give

$$- \int_{\Omega} \delta^{ij} [(\partial_{\theta}^8 F_1) - (\partial_{\theta}^8 F_2)] [(\partial_{1i} \partial_{\theta}^8 \partial_{1j} f_1) \circ u_1 - (\partial_{2i} \partial_{\theta}^8 \partial_{2j} f_2) \circ u_2] dy. \quad (3.2.115)$$

Here we commute  $\partial_{1i}$  through the  $\partial_{\theta}$ . This generates commutators which we can control because of the tangential smoothing. We also obtain the following term

$$- \int_{\Omega} \delta^{ij} [(\partial_{\theta}^8 F_1) - (\partial_{\theta}^8 F_2)] [(\partial_{\theta}^8 \Delta_1 f_1) \circ u_1 - (\partial_{\theta}^8 \Delta_2 f_2) \circ u_2] dy \quad (3.2.116)$$

$$= \int_{\Omega} \delta^{ij} [(\partial_{\theta}^9 F_1) - (\partial_{\theta}^9 F_2)] [(\partial_{\theta}^7 \Delta_1 f_1) \circ u_1 - (\partial_{\theta}^7 \Delta_2 f_2) \circ u_2] dy \quad (3.2.117)$$

where we control the first factor by lemma 3.2.8. Again, we use the fact that

$$\delta^{ij} A_i^a(1) \partial_a [A_j^b(1) (\partial_b F_1)] = -A_i^a(1) (\partial_a V^j) A_j^b(1) (\partial_b W^i) + 1 \quad (3.2.118)$$

which shows that we can control the second factor in (3.2.116) appropriately.  $\blacksquare$

### 3.3 Estimates for chapter 5 and chapter 7.

In this section we record the results which will be used to prove the energy estimates in chapter 5 and the optimal regularity result in chapter 7. Let  $U$  and  $V$  be points in  $C(T)$  and let  $x_{\kappa}$  be the flow of  $U_{\kappa}$ . Define  $v_{\kappa} = V_{\kappa} \circ x_{\kappa}^{-1}$ ,  $v = V \circ x_{\kappa}^{-1}$  and  $\Omega_t = x_{\kappa}(t, \Omega)$ . We define a function  $f$  on  $\Omega_t$  by

$$\Delta f = -(\nabla v_{\kappa})(\nabla v) + 1 \text{ on } \Omega_t \quad (3.3.1)$$

with boundary condition

$$f = 0 \text{ on } \partial\Omega_t. \quad (3.3.2)$$

Similarly to theorem 3.1.2 we have the following:

**THEOREM 3.3.1** *For  $f$  defined by (3.3.1) and (3.3.2) we have*

$$\|\nabla f\|_{H^{7.5}(\Omega_t)} \leq P[\|x_{\kappa}\|_8, \kappa^{-0.5}\|x_{\kappa}\|_8, \|v_{\kappa}\|_{H^{7.5}(\Omega_t)}, \|v\|_{H^{7.5}(\Omega_t)}] \quad (3.3.3)$$

where  $P$  is a polynomial which is linear in  $\kappa^{-0.5}\|x_{\kappa}\|_8$  and also

$$\|\nabla f\|_{H^7(\Omega_t)} \leq P[\|x_{\kappa}\|_8, \|v_{\kappa}\|_{H^7(\Omega_t)}, \|v\|_{H^7(\Omega_t)}]. \quad (3.3.4)$$

We will also need estimates for  $\partial_t \nabla f$  to establish the energy estimates. Thus we have the following theorem which follows similarly to theorem 3.1.2 and theorem 3.3.1. Note that to derive this estimate we need the estimates from chapter 4.

**THEOREM 3.3.2** *For  $f$  defined by (3.3.1) and (3.3.2) we have*

$$\|\partial_t \nabla f\|_{H^7(\Omega_t)} \leq P[\|x_\kappa\|_8, \|v_\kappa\|_{H^8(\Omega_t)}, \|v\|_{H^8(\Omega_t)}]. \quad (3.3.5)$$

To establish optimal regularity in chapter 7 we also need the following theorem which follows similarly to theorem 3.3.2

**THEOREM 3.3.3** *For  $f$  defined by (3.3.1) and (3.3.2) we have*

$$\|\eta \partial_t^2 \nabla f\|_{H^6(\Omega_t)} \leq P[\|x_\kappa\|_{7.5}, \|v_\kappa\|_{H^{6.5}(\Omega_t)}, \|v\|_{H^{6.5}(\Omega_t)}] \quad (3.3.6)$$

where  $\eta = \eta_1$  is the cut off function defined in chapter 2. We also have

$$\|\partial_t^2 \nabla f\|_{H^{5.5}(\Omega_t)} \leq P[\|x_\kappa\|_{7.5}, \|v_\kappa\|_{H^{6.5}(\Omega_t)}, \|v\|_{H^{6.5}(\Omega_t)}]. \quad (3.3.7)$$

# Chapter 4

## Elliptic estimates for $\phi$ .

In this chapter  $\kappa > 0$  is fixed. Now we prove the estimates for  $\phi$ : In section 4.1 we prove estimates to show that  $\Lambda$  is invariant on  $C(T)$  (defined in (3.0.1)); in section 4.2 we prove results needed to show that  $\Lambda$  is a contraction; and in section 4.3 we prove the results needed for the energy estimates in chapter 5 and the optimal regularity result in chapter 7.

Let  $U$  be a point in  $C(T)$ , which was defined in (3.0.1), and let  $x_\kappa$  be the smoothed flow associated with  $U$ . Define  $\Omega_t = x_\kappa(t, \Omega)$  and define the function  $\phi$  by

$$\phi(t, x) = -\chi_{\Omega_t} * \Phi(x), \tag{4.0.1}$$

where  $\chi_{\Omega_t}(x)$  takes the value 1 when  $x$  is a point in  $\Omega_t$  and the value 0 otherwise and where  $\Phi$  is the fundamental solution to the Laplacean.

### 4.1 Estimates to show that $\Lambda$ is invariant.

In this section we prove the following theorem:

**THEOREM 4.1.1** *We have  $\|\nabla\phi\|_{H^7(\Omega_t)} \leq P[\|x_\kappa\|_8]$  and  $\|\nabla\phi\|_{H^8(\Omega_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8]$  where  $P$  is a polynomial which is linear in  $\kappa^{-0.5}\|x_\kappa\|_8$ .*

Using the cut off functions defined in chapter 2, we have  $\|\nabla\phi\|_{H^8(\Omega_t)} \leq \|\eta_1\nabla\phi\|_{H^8(\Omega_t)} + \|\zeta_1\nabla\phi\|_{H^8(\Omega_t)}$ . We begin by proving the interior regularity.

#### 4.1.1 Interior regularity.

In this section we prove the following result:

**PROPOSITION 4.1.2** *For any  $1 \leq i$  and  $0 \leq s \leq 8$*

$$\|\nabla^s[\eta_i\nabla\phi]\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8] \tag{4.1.1}$$

where  $P$  is a polynomial.

We prove that (4.1.1) holds by induction on  $s$ . Suppose that  $s = 0$ . We have  $\|\eta_i \nabla \phi\|_{L^2(\Omega_t)} \leq \|\eta_i\|_{L^\infty(\Omega_t)} \|\nabla \phi\|_{L^2(\Omega_t)}$  and

$$\|\nabla \phi\|_{L^2(\Omega_t)}^2 = \int_{\Omega_t} (\partial_j \phi)(\partial^j \phi) dx = \int_{\partial\Omega_t} N^j (\partial_j \phi) \phi dx - \int_{\Omega_t} \Delta \phi \phi dx. \quad (4.1.2)$$

Since we have  $\|\phi\|_{L^\infty(\Omega_t)} \leq P[\|x_\kappa\|_3]$  and  $\|\nabla \phi\|_{L^\infty(\Omega_t)} \leq P[\|x_\kappa\|_3]$ , we control both terms in (4.1.2) appropriately. This proves (4.1.1) for  $s = 1$ . Now suppose that  $s = 8$  and that we have the result for smaller  $s$ . Then

$$\|\nabla^8[\eta_i \phi]\|_{L^2(\Omega_t)}^2 = \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} [\eta_i \partial_{j_9} \phi])(\partial^{j_1} \dots \partial^{j_8} [\eta_i \partial^{j_9} \phi]) dx. \quad (4.1.3)$$

Now as we saw in (3.1.8),

$$\partial^{j_1} \dots \partial^{j_8} [\eta_i \partial^{j_9} \phi] = \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial^{j_9} \phi) + \sum (\nabla^{k_1} \eta_i) (\nabla^{k_2+1} \phi) \quad (4.1.4)$$

where the sum is over  $k_1 + k_2 = 8$  such that  $k_2 \leq 7$ . We control the second term in (4.1.4) using lemma 3.1.4. Integrating the first term in (4.1.4) by parts twice we have

$$- \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} \partial^{j_9} [\eta_i \partial_{j_9} \phi]) \eta_i (\partial^{j_1} \dots \partial^{j_8} \phi) dx \quad (4.1.5)$$

$$- \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} [\eta_i \partial_{j_9} \phi]) (\partial^{j_9} \eta_i) (\partial^{j_1} \dots \partial^{j_8} \phi) dx \quad (4.1.6)$$

$$= \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_7} \partial^{j_9} [\eta_i \partial_{j_9} \phi]) \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} \phi) dx \quad (4.1.7)$$

$$+ \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_7} \partial^{j_9} [\eta_i \partial_{j_9} \phi]) (\partial_{j_8} \eta_i) (\partial^{j_1} \dots \partial^{j_8} \phi) dx \quad (4.1.8)$$

$$- \int_{\Omega_t} (\partial_{j_1} \dots \partial_{j_8} [\eta_i \partial_{j_9} \phi]) (\partial^{j_9} \eta_i) (\partial^{j_1} \dots \partial^{j_8} \phi) dx \quad (4.1.9)$$

where we can control the second and third term in (4.1.7) using lemma 3.1.4. Also,  $\partial^{j_9} [\eta_i \partial_{j_9} \phi] = (\partial^{j_9} \eta_i) (\partial_{j_9} \phi) + \eta_i$  and therefore the first term in (4.1.7) is equal to

$$\int_{\Omega_t} \partial_{j_1} \dots \partial_{j_7} [(\partial^{j_9} \eta_i) (\partial_{j_9} \phi)] \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} \phi) dx + \int_{\Omega_t} \partial_{j_1} \dots \partial_{j_7} [\eta_i] \eta_i (\partial^{j_1} \dots \partial^{j_8} \partial_{j_8} \phi) dx \quad (4.1.10)$$

The above terms in (4.1.10) can be controlled using lemma 3.1.4 and the inductive hypothesis. This concludes the proof of proposition 4.1.2.

### 4.1.2 Boundary regularity.

Let  $\zeta$  denote  $\zeta_1$ . We have  $\nabla[\zeta \nabla \phi] = (\nabla \zeta)(\nabla \phi) + \zeta (\nabla^2 \phi)$  and  $\nabla \zeta$  is supported in the interior of  $\Omega_t$ . Thus we control the first term by theorem 4.1.2 and we need only be concerned with terms of the form  $\zeta \nabla^s \phi$ , for  $1 \leq s \leq 9$ . In this section we prove the following result:

**THEOREM 4.1.3** For  $1 \leq s \leq 9$

$$\|\zeta \nabla^s \phi\|_{L^2(\Omega_t)} \leq P [\|x_\kappa\|_8, \kappa^{-0.5} \|x_\kappa\|_8] \quad (4.1.11)$$

where  $P$  is a polynomial which is linear in  $\kappa^{-0.5} \|x_\kappa\|_8$ .

Since integration by parts on  $\Omega_t$  will yield a boundary term which is difficult to deal with because  $\partial\Omega_t$  is the complement of the singular support of  $\phi$ , we now extend the region of integration: There is  $d_0 > 0$  such that  $\Omega^{d_0} \subseteq \cup_{i=1}^{\nu} U_i$ , where  $\Omega^{d_0} = \{y_1 + y_2 \in \mathbf{R}^3 : y_1 \in \Omega \text{ and } d(y_2) < d_0\}$  and therefore the norms  $\|\cdot\|_{H^s(\Omega^{d_0})}$  are well-defined. Now we define the extended flow  $\tilde{x}_\kappa = E(x_\kappa)$  where  $E$  is the extension operator on  $\Omega$  – see, for instance, [7] – and define  $\tilde{V}_\kappa = \partial_t \tilde{x}_\kappa$ . Then we have  $\|\tilde{x}_\kappa\|_{H^s(\Omega^{d_0})} \leq c \|x_\kappa\|_{H^s(\Omega)}$  and similarly for  $\tilde{V}_\kappa$ . Define  $\tilde{B}_a^i = \frac{\partial \tilde{x}_\kappa^i}{\partial y^a}$ . Then since  $\frac{1}{2} \leq \det(B) \leq \frac{3}{2}$  on  $\Omega$ , possibly by picking a smaller  $d_0$ ,  $\tilde{x}_\kappa$  is a change of variables on  $\tilde{\Omega} = \Omega^{d_0}$  and such that the normal  $N$  to  $\partial\Omega_t$  can be extended into the region between  $\partial\Omega_t$  and the boundary of  $\tilde{\Omega}_t = \tilde{x}_\kappa(t, \tilde{\Omega})$ . Let  $\tilde{A}$  be the inverse of  $\tilde{B}$ . We now define  $\tilde{\phi}$  as follows:

$$\tilde{\phi}(t, x) = -\chi_{\Omega_t} * \Phi(x) \text{ for } x \text{ in } \tilde{\Omega}_t \quad (4.1.12)$$

where again  $\Phi$  is the fundamental solution for the Laplacean. This means that on  $\Omega_t$ , we have  $\tilde{\phi} = \phi$  and therefore that  $\tilde{\phi}$  and  $\phi$  have the same regularity on  $\Omega_t$ . It also means that  $\tilde{\phi}$  is smooth on  $\partial\tilde{\Omega}_t$ . Finally, let the norms on the extended domains  $\tilde{\Omega}$  and  $\tilde{\Omega}_t$  be defined analogously to the norms on  $\Omega$  and  $\Omega_t$ .

Having extended the domain we now approximate  $\tilde{\phi}$ : Let  $\chi_m$  denote a smooth *radial* function compactly supported in  $\{y \in \tilde{\Omega} \setminus \Omega : d(y) < \frac{1}{m}\}$ , which takes the value 1 on the set  $\{y \in \Omega : \frac{1}{m} \leq d(y)\}$ . This means that  $\partial_\theta \chi_m = 0$ . By an abuse of notation we will also let  $\chi_m$  denote the analogous function in the Eulerian frame. Define  $\tilde{\phi}_m(t, x) = -\chi_m * \Phi(x)$  for  $x$  in  $\tilde{\Omega}_t$ . We now show that the approximations converge to  $\tilde{\phi}$ .

**LEMMA 4.1.4**  $\|\nabla \tilde{\phi}_m - \nabla \tilde{\phi}\|_{L^2(\tilde{\Omega}_t)} \leq c \|\chi_m - \chi_{\Omega_t}\|_{L^2(\tilde{\Omega}_t)}$ .

**PROOF:** From (4.1.12) it is clear that  $\tilde{\phi}$  is in  $H^1(\tilde{\Omega}_t)$  so integration by parts is justified. Similarly, for  $\tilde{\phi}_m$ . Now,

$$\|\nabla \tilde{\phi}_m - \nabla \tilde{\phi}\|_{L^2(\tilde{\Omega}_t)}^2 = \int_{\tilde{\Omega}_t} (\partial_j \tilde{\phi}_m - \partial_j \tilde{\phi})(\partial^j \tilde{\phi}_m - \partial^j \tilde{\phi}) dx \quad (4.1.13)$$

$$= \int_{\partial\tilde{\Omega}_t} N^j (\partial_j \tilde{\phi}_m - \partial_j \tilde{\phi})(\tilde{\phi}_m - \tilde{\phi}) dS(x) - \int_{\tilde{\Omega}_t} (\chi_m - \chi_{\Omega_t})(\tilde{\phi}_m - \tilde{\phi}) dx. \quad (4.1.14)$$

To control the first term in (4.1.14) we note that there is  $\delta > 0$  such that  $\text{dist}(\partial\tilde{\Omega}_t, \partial\Omega_t) > \delta$ . This means that for all  $x$  on  $\partial\tilde{\Omega}_t$  and for all  $z$  in  $\Omega_t$  we have  $|x - z| > \delta$ . Hence for  $x$  on  $\partial\tilde{\Omega}_t$ ,

$$|\nabla \tilde{\phi}_m(x) - \nabla \tilde{\phi}(x)| \leq \int_{\tilde{\Omega}_t} |\chi_m(z) - \chi_{\Omega_t}(z)| |\Phi'(|x - z|)| dz \leq \|\chi_m - \chi_{\Omega_t}\|_{L^2(\tilde{\Omega}_t)}.$$

Also for  $x$  in  $\partial\tilde{\Omega}_t$  and for  $x$  in  $\tilde{\Omega}_t$ ,  $|\tilde{\phi}_m(x) - \tilde{\phi}(x)| \leq \|\chi_m - \chi_{\Omega_t}\|_{L^2(\tilde{\Omega}_t)} \|\Phi(|x - \cdot|)\|_{L^2(\tilde{\Omega}_t)} \leq c\|\chi_m - \chi_{\Omega_t}\|_{L^2(\tilde{\Omega}_t)}$ , so we can control the first and second term in (4.1.14) appropriately.  $\blacksquare$

Let  $\tilde{\phi}_{m,n} = \tilde{\phi}_m - \tilde{\phi}_n$  and let  $\chi_{m,n} = \chi_m - \chi_n$ . We will now show that  $(\zeta\nabla\partial_\theta^j\nabla\tilde{\phi}_{m,n})$  is a Cauchy sequence in  $L^2(\tilde{\Omega}_t)$ .

**PROPOSITION 4.1.5** *For  $0 \leq j \leq 7$ , we have*

$$\|\zeta\nabla\partial_\theta^j\nabla\tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8] \|\chi_{m,n}\|_{L^6(\tilde{\Omega}_t)}, \quad (4.1.15)$$

where  $P$  is linear in  $\kappa^{-0.5}\|x_\kappa\|_8$ .

We begin by proving a lemma which says that we need only be concerned with tangential derivatives:

**LEMMA 4.1.6** *Let  $f$  satisfy  $\Delta f = g$  in  $\tilde{\Omega}_t$  where  $\partial_\theta g = 0$ . For  $0 \leq j \leq 6$ ,*

$$\|\zeta\nabla\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8] \sum_{k=0}^{j+1} \|\zeta\partial_\theta^k\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|g\|_{L^6(\tilde{\Omega}_t)} \quad (4.1.16)$$

and

$$\|\zeta\nabla\partial_\theta^7\nabla f\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8] \sum_{k=0}^{j+1} \|\zeta\partial_\theta^k\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|g\|_{L^6(\tilde{\Omega}_t)} \quad (4.1.17)$$

where  $P$  is linear in  $\kappa^{-0.5}\|x_\kappa\|_8$ . Also for  $0 \leq j \leq 4$ ,

$$\|\zeta\nabla\partial_\theta^j\nabla f\|_{L^6(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8] \sum_{k=0}^{j+3} \|\zeta\partial_\theta^k\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|g\|_{L^6(\tilde{\Omega}_t)} \quad (4.1.18)$$

**PROOF:** We prove this result by induction. For  $j = 0$  we have  $\|\zeta\nabla^2 f\|_{L^2(\tilde{\Omega}_t)} \leq \|\zeta\Delta f\|_{L^2(\tilde{\Omega}_t)} + \|\zeta\partial_\theta\nabla f\|_{L^2(\tilde{\Omega}_t)}$ , by lemma B.5.1, which can be found in appendix B. Now suppose that  $1 \leq j \leq 5$  and that we have (4.1.16) for smaller  $j$ . Then  $\|\zeta\nabla\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} \leq \|\zeta\operatorname{div}\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|\zeta\operatorname{curl}\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} + \|\zeta\partial_\theta^{j+1}\nabla f\|_{L^2(\tilde{\Omega}_t)}$ . Now  $\zeta\operatorname{div}\partial_\theta^j\nabla f = \zeta\partial_\theta^j\Delta f + \sum(\partial_\theta^k\tilde{A})(\zeta\nabla\partial_\theta^l\nabla f)$  where the sum is over  $k+l=j$  such that  $l \leq j-1 \leq 1$ . Since  $\partial_\theta^j\Delta f = 0$ , we have  $\|\zeta\operatorname{div}\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8] \|\zeta\nabla\partial_\theta^l\nabla f\|_{L^2(\tilde{\Omega}_t)}$  which we control by induction. Similarly, we also control  $\|\zeta\operatorname{curl}\partial_\theta^j\nabla f\|_{L^2(\tilde{\Omega}_t)}$ . Now we prove (4.1.18) for  $j = 0$ :

$$\|\zeta\nabla^2 f\|_{L^6(\tilde{\Omega}_t)} \leq \|g\|_{L^6(\tilde{\Omega}_t)} + \|\zeta\partial_\theta\nabla f\|_{L^6(\tilde{\Omega}_t)} \leq \|g\|_{L^6(\tilde{\Omega}_t)} + \|\zeta\nabla\partial_\theta\nabla f\|_{L^2(\tilde{\Omega}_t)} \quad (4.1.19)$$

using Sobolev's inequality and proposition 4.1.2. Also, for  $1 \leq j \leq 4$  we have

$$\|\zeta\nabla\partial_\theta^j\nabla f\|_{L^6(\tilde{\Omega}_t)} \leq \sum\|(\partial_\theta^k\tilde{A})(\nabla\partial_\theta^l\nabla f)\|_{L^6(\tilde{\Omega}_t)} + \|\zeta\partial_\theta^{j+1}\nabla f\|_{L^6(\tilde{\Omega}_t)} \quad (4.1.20)$$

$$\leq \sum\|(\partial_\theta^k\tilde{A})(\nabla\partial_\theta^l\nabla f)\|_{L^6(\tilde{\Omega}_t)} + \|\zeta\nabla\partial_\theta^{j+1}\nabla f\|_{L^2(\tilde{\Omega}_t)} \quad (4.1.21)$$

which we control appropriately. Now we can prove (4.1.16) for  $6 \leq j \leq 7$ .  $\blacksquare$

Using lemma 4.1.6, we see that it is enough to control  $\|\zeta\partial_\theta^j\nabla\tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)}$  appropriately for  $0 \leq j \leq 8$  which is the content of the following proposition:

**PROPOSITION 4.1.7** For  $0 \leq j \leq 6$  we have

$$\|\zeta \partial_\theta^j \nabla \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8] \|\chi_{m,n}\|_{L^6(\tilde{\Omega}_t)}. \quad (4.1.22)$$

and for  $7 \leq j \leq 8$  we have

$$\|\zeta \partial_\theta^j \nabla \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5} \|x_\kappa\|_8] \|\chi_{m,n}\|_{L^6(\tilde{\Omega}_t)}. \quad (4.1.23)$$

where  $P$  is linear in  $\kappa^{-0.5} \|x_\kappa\|_8$ .

**PROOF:** We prove that (4.1.22) and (4.1.23) hold by induction on the order. The start of the induction is similar to lemma 4.1.4. Since the proof of (4.1.22) is similar to the proof of (4.1.23) we now suppose that we have  $j = 8$  and suppose that we have already have appropriate control of the lower order cases. We have

$$\|\zeta \partial_\theta^8 \nabla \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)}^2 = \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^8 \partial_i \tilde{\phi}_{m,n}) (\zeta \partial_\theta^8 \partial^i \tilde{\phi}_{m,n}) dx \quad (4.1.24)$$

$$= \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^8 \partial_i \tilde{\phi}_{m,n}) (\zeta \partial^i \partial_\theta^8 \tilde{\phi}_{m,n}) dx \quad (4.1.25)$$

$$- \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^8 \partial_i \tilde{\phi}_{m,n}) (\partial^i \partial_\theta^8 x_\kappa^l) (\zeta \partial_l \tilde{\phi}_{m,n}) dx \quad (4.1.26)$$

$$+ \sum \int_{\tilde{\Omega}_t} (\zeta \partial_\theta^8 \nabla \tilde{\phi}_{m,n}) (\nabla \partial_\theta^{l_1} x_\kappa) \dots (\nabla \partial_\theta^{l_{s-1}} x_\kappa) (\zeta \partial_\theta^{l_s} \nabla \tilde{\phi}_{m,n}) dx \quad (4.1.27)$$

where  $l_1 + \dots + l_s = 8$  and  $l_1, \dots, l_s \leq 7$ . For  $l_1, \dots, l_{s-1} \leq 5$  we control the third term in (4.1.25) by  $P[\|x_\kappa\|_8] \|\zeta \partial_\theta^8 \nabla \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)} \|\zeta \partial_\theta^{l_s} \nabla \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)}$ , which we control by induction. Suppose that  $6 \leq l_1 \leq 7$ . Then  $\|\nabla \partial_\theta^{l_1} x_\kappa\|_{L^3(\tilde{\Omega}_t)} \leq \kappa^{-0.5} \|x_\kappa\|_8$ ; and  $l_2, \dots, l_{s-1} \leq 2$ , so we control the other terms containing  $x$ . We also have  $0 \leq l_s \leq 2$  and therefore  $\|\zeta \partial_\theta^{l_s} \nabla \tilde{\phi}_{m,n}\|_{L^6(\tilde{\Omega}_t)} \leq \|\zeta \nabla \partial_\theta^{l_s} \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)}$ , which we control appropriately by lemma 4.1.6. Thus we control the third term in (4.1.25). Integrating the first two terms in (4.1.25) by parts gives

$$\int_{\tilde{\Omega}_t} (\zeta \partial_\theta^8 \partial_i \tilde{\phi}_{m,n}) (\partial_\theta^8 x_\kappa^l) (\zeta \partial^i \partial_l \tilde{\phi}_{m,n}) dx - \sum \int_{\tilde{\Omega}_t} (\zeta \partial^i \partial_\theta^8 \partial_i \tilde{\phi}_{m,n}) (\partial_\theta^{j_1} x_\kappa) (\zeta \partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n}) dx \quad (4.1.28)$$

$$+ \sum \int_{\tilde{\Omega}_t} (N^i \partial_\theta^8 \partial_i \tilde{\phi}_{m,n}) (\partial_\theta^{j_1} x_\kappa) (\partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n}) dS(x) \quad (4.1.29)$$

where the sums are over all  $j_1 + j_2 = 8$  such that  $j_1, j_2 \leq 7$ . Here we are ignoring the terms which arise from the derivative falling on  $\zeta$  because in this case we can use proposition 4.1.2. We control the first term in (4.1.28). Also,

$$\zeta \partial^i \partial_\theta^8 \partial_i \tilde{\phi}_{m,n} = (\nabla \partial_\theta^8 x_\kappa) (\zeta \nabla^2 \tilde{\phi}_{m,n}) + \sum (\nabla \partial_\theta^{l_1} x_\kappa) \dots (\nabla \partial_\theta^{l_{s-1}} x_\kappa) (\zeta \nabla \partial_\theta^{l_s} \nabla \tilde{\phi}_{m,n}) \quad (4.1.30)$$

where  $l_1 + \dots + l_s = 8$  and  $l_1, \dots, l_s \leq 7$ , since  $\partial_\theta^8 \Delta \tilde{\phi}_{m,n} = 0$ . For  $l_1, \dots, l_{s-1} \leq 5$  we control the second term in (4.1.30) appropriately by lemma 4.1.6. If  $6 \leq l_1 \leq 7$  then we control  $\nabla \partial_\theta^{l_1} x_\kappa$  as before and  $0 \leq l_2, \dots, l_{s-1} \leq 2$  so we can control the other terms containing  $x_\kappa$  in  $L^\infty(\tilde{\Omega}_t)$ . We also have  $0 \leq l_s \leq 2$  so we can control  $\|\zeta \nabla \partial_\theta^{l_s} \nabla \tilde{\phi}_{m,n}\|_{L^6(\tilde{\Omega}_t)}$  by lemma 4.1.6. We therefore



control the second term in (4.1.30). Let us now consider the first term in (4.1.30): Commute one  $\partial_\theta$  to the outside to obtain, in addition to a lower order term,

$$\int_{\tilde{\Omega}_t} (\partial_\theta \nabla \partial_\theta^7 x_\kappa) (\zeta \nabla^2 \tilde{\phi}_{m,n}) (\partial_\theta^{j_1} x_\kappa) (\zeta \partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n}) dx_\kappa \quad (4.1.31)$$

$$= - \int_{\tilde{\Omega}_t} (\nabla \partial_\theta^7 x_\kappa) (\zeta \partial_\theta \nabla^2 \tilde{\phi}_{m,n}) (\partial_\theta^{j_1} x_\kappa) (\zeta \partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n}) dx \quad (4.1.32)$$

$$- \int_{\tilde{\Omega}_t} (\nabla \partial_\theta^7 x_\kappa) (\zeta \nabla^2 \tilde{\phi}_{m,n}) (\partial_\theta^{j_1+1} x_\kappa) (\zeta \partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n}) dx \quad (4.1.33)$$

$$- \int_{\tilde{\Omega}_t} (\nabla \partial_\theta^7 x_\kappa) (\zeta \nabla^2 \tilde{\phi}_{m,n}) (\partial_\theta^{j_1} x_\kappa) (\zeta \partial_\theta^{j_2+1} \nabla \tilde{\phi}_{m,n}) dx \quad (4.1.34)$$

where no boundary terms arise because the components of  $\partial_\theta$  are orthogonal to the normal on  $\partial\tilde{\Omega}_t$ . In all of the terms in (4.1.32) we control the first two factors in each integrand using lemma 4.1.6. In the second term in (4.1.32) we also have, for  $j_1 \leq 5$ ,  $\|(\partial_\theta^{j_1+1} x_\kappa) (\zeta \partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n})\|_{L^2(\tilde{\Omega}_t)} \leq \|x_\kappa\|_8 \|\zeta \partial_\theta^{j_2} \nabla \tilde{\phi}_{m,n}\|_{L^2(\tilde{\Omega}_t)}$  which we control. For  $6 \leq j_1 \leq 7$  we have  $1 \leq j_2 \leq 2$  and therefore we control the second term in (4.1.32) in this case also. The third term in (4.1.32) follows similarly.

Now we control the boundary term in (4.1.28):

$$\sum \int_{\partial\tilde{\Omega}_t} (\partial_\theta^{k_1} x_\kappa) \dots (\partial_\theta^{k_s} x_\kappa) (\partial_i \partial_{k_1} \dots \partial_{k_s} \tilde{\phi}_{m,n}) N^i (\partial_\theta^{l_1} x_\kappa) \dots (\partial_\theta^{l_s} x_\kappa) (\partial_{l_1} \dots \partial_{l_s} \tilde{\phi}_{m,n}) dS(x) \quad (4.1.35)$$

where the sum is over  $k_1 + \dots + k_s = 8$  and  $l_1 + \dots + l_s = 8$  such that  $l_1, \dots, l_s \leq 7$ . As was mentioned above, there is  $\delta > 0$  such that for all  $x$  on  $\partial\tilde{\Omega}_t$  and  $z$  in  $\Omega_t$  we have  $|x - z| > \delta$ . Therefore  $|\nabla^s \tilde{\phi}_{m,n}(x)| \leq \|\chi_{m,n}\|_{L^2(\tilde{\Omega}_t)}$ . The highest order term of the above terms is

$$\int_{\partial\tilde{\Omega}_t} (\partial_\theta^8 x_\kappa) (\nabla \tilde{\phi}_{m,n}) N (\partial_\theta^7 x_\kappa) (\partial_\theta x_\kappa) (\nabla^2 \tilde{\phi}_{m,n}) dS(x) \quad (4.1.36)$$

which is controlled by  $P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8]$  using the trace theorem. This concludes the proof.  $\blacksquare$

By lemma 4.1.6, therefore, we have proposition 4.1.5. By the dominated convergence theorem,  $(\zeta \nabla \partial_\theta^j \nabla \tilde{\phi}_m)_{m=1}^\infty$  is therefore a Cauchy sequence in  $L^2(\tilde{\Omega}_t)$ , for  $0 \leq j \leq 7$ . This means that  $\zeta \nabla \partial_\theta^j \nabla \tilde{\phi}_m \rightarrow \zeta \nabla \partial_\theta^j \nabla \tilde{\phi}$  in  $L^2(\tilde{\Omega}_t)$ . From lemma 4.1.6 and proposition 4.1.7 we have  $\|\zeta \nabla \partial_\theta^j \nabla \tilde{\phi}_m\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8]$ , for  $0 \leq j \leq 7$  and therefore have  $\|\zeta \nabla \partial_\theta^j \nabla \tilde{\phi}\|_{L^2(\tilde{\Omega}_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8]$  and hence

$$\|\nabla \partial_\theta^j \nabla \phi\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8, \kappa^{-0.5}\|x_\kappa\|_8], \quad (4.1.37)$$

for  $0 \leq j \leq 7$ , where  $P$  is a polynomial which is linear in  $\kappa^{-0.5}\|x_\kappa\|_8$ . From now on we no longer consider the extended domain and all norms are now the usual, non-extended, norms. Now we build some more regularity for  $\phi$ .

**LEMMA 4.1.8** For  $1 \leq s \leq 8$ ,

$$\|\zeta \nabla^{s+1} \phi\|_{L^2(\Omega_t)} \leq P[\|x_\kappa\|_8] + \sum_{0 \leq j \leq s} \|\zeta \partial_\theta^j \nabla f\|_{L^2(\Omega_t)}. \quad (4.1.38)$$

**PROOF:** We prove this by induction on  $s$ . For  $s = 1$  we have, according to lemma 2.5.1,  $|\zeta \nabla^2 \phi| \leq 1 + |\zeta \partial_\theta \nabla \phi|$  which is of the correct form for (4.1.38). Now suppose that  $s = 8$  and that we have (4.1.38) for  $1 \leq s \leq 7$ . Again by lemma 2.5.1, we have  $|\zeta \nabla^{s+1} \phi| \leq |\zeta \nabla^{s-1} \text{curl} \nabla \phi| + |\zeta \nabla^{s-1} \text{div} \nabla \phi| + |\zeta \partial_\theta \nabla^s \phi|$ . Here we have  $\text{div} \nabla \phi = \Delta \phi = 1$  and therefore pointwise on  $\Omega_t$  we have  $|\zeta \nabla^{s-1} \text{div} \nabla \phi| = 0$ . We also have

$$\zeta \nabla^{s-1} \partial_\theta \nabla \phi = \zeta \nabla^{s-1} [(\partial_\theta x_\kappa)(\nabla^2 \phi)] = \zeta \partial_\theta \nabla^s \phi + \sum_{k_1+k_2=s-1, k_2 \leq s-2} (\nabla^{k_1} \partial_\theta x_\kappa) \zeta (\nabla^{k_2+2} \phi). \quad (4.1.39)$$

For  $0 \leq k_1 \leq 4$ , we have  $\|\nabla^{k_1} \partial_\theta x_\kappa\|_{L^\infty(\Omega_t)} \leq \|x_\kappa\|_8$  and  $\|\zeta \nabla^{k_2+2} \phi\|_{L^2(\Omega_t)}$  is controlled by the inductive hypothesis since  $k_2 + 2 \leq s = 8$ . For  $5 \leq k_1 \leq 7$ , we have  $\|\nabla^{k_1} \partial_\theta x_\kappa\|_{L^2(\Omega_t)} \leq \|x_\kappa\|_8$  and in this case  $0 \leq k_2 \leq 2$ , and therefore  $\|\zeta \nabla^{k_2+2} \phi\|_{L^\infty(\Omega_t)} \leq \|\zeta \nabla^{k_2+2+3} \phi\|_{L^2(\Omega_t)}$ , in addition to terms which we control by proposition 4.1.2. We control  $\|\zeta \nabla^{k_2+2+3} \phi\|_{L^2(\Omega_t)}$  by the inductive hypothesis because  $k_2 + 2 + 3 \leq 7$ . Now  $|\zeta \nabla^{s-1} \partial_\theta \nabla \phi| \leq |\zeta \nabla^{s-2} \text{curl} \partial_\theta \nabla \phi| + |\zeta \nabla^{s-2} \text{div} \partial_\theta \nabla \phi| + |\zeta \partial_\theta \nabla^{s-2} \partial_\theta \nabla \phi|$  and  $\zeta \nabla^{s-2} \text{div} \partial_\theta \nabla \phi = \zeta \nabla^{s-2} [(\nabla \partial_\theta x_\kappa)(\nabla^2 \phi) + \partial_\theta \Delta \phi]$ . We have

$$\zeta \nabla^{s-2} [(\nabla \partial_\theta x_\kappa)(\nabla^2 \phi)] = \sum_{k_1+k_2=s-2} (\nabla^{k_1+1} \partial_\theta x_\kappa) \zeta (\nabla^{k_2+2} \phi). \quad (4.1.40)$$

For  $0 \leq k_1 \leq 3$ , we have  $\|\nabla^{k_1+1} \partial_\theta x_\kappa\|_{L^\infty(\Omega_t)} \leq \|x_\kappa\|_8$  and  $\|\zeta (\nabla^{k_2+2} \phi)\|_{L^2(\Omega_t)}$  is controlled by the inductive hypothesis. For  $4 \leq k_1 \leq 6$  we have  $\|\nabla^{k_1+1} \partial_\theta x_\kappa\|_{L^2(\Omega_t)} \leq \|x_\kappa\|_8$  and in this case  $0 \leq k_2 \leq 2$  and we control  $\|\zeta (\nabla^{k_2+2} \phi)\|_{L^\infty(\Omega_t)} \leq \|\zeta (\nabla^{k_2+2+3} \phi)\|_{L^2(\Omega_t)}$  using the inductive hypothesis. Also,  $\zeta \nabla^{s-2} \partial_\theta \Delta \phi = 0$ . This concludes the proof.  $\blacksquare$

Using lemma 4.1.8, we control  $\|\zeta \nabla^s \phi\|_{L^2(\Omega_t)}$  for  $0 \leq s \leq 8$  and hence we obtain theorem 4.1.1.

## 4.2 Estimates to show that $\Lambda$ is a contraction.

To show that  $\Lambda$  is a contraction, fix  $U_1$  and  $U_2$  in  $C(T)$ . Smooth  $U_1$  and  $U_2$  to obtain  $(U_1)_\kappa$  and  $(U_2)_\kappa$ . Let  $x_1$  and  $x_2$  be the flows of  $(U_1)_\kappa$  and  $(U_2)_\kappa$  respectively. Let  $\Omega_{k,t} = x_k(t, \Omega)$  for  $k = 1, 2$  and define

$$\phi_k(t, x) = -\chi_{\Omega_{k,t}} * \Phi(x). \quad (4.2.1)$$

Just as the results in section 3.2 followed the methods similar to those employed in section 3.1, so too do the estimates in this section follow the methods employed in section 4.1. Thus we have the following theorem:

### THEOREM 4.2.1

$$\|(\nabla_1 \phi_1) \circ u_1 - (\nabla_2 \phi_2) \circ u_2\|_8 \leq \kappa^{-0.5} \|U_1 - U_2\|_8 P[\|U_1\|_8, \|U_2\|_8, \kappa^{-0.5} \|U_1\|_8, \kappa^{-0.5} \|U_2\|_8].$$

### 4.3 Estimates for chapter 5 and chapter 7.

Let  $\phi$  be defined as in (4.0.1). To prove the energy estimates in chapter 5 we need the following theorem which follows similarly to theorem 4.1.1.

**THEOREM 4.3.1**  $\|\nabla\phi\|_{H^{7.5}(\Omega_t)} \leq P[\|x_\kappa\|_8]$ .

To obtain the optimal regularity results in chapter 7 we also need the following estimate. Note that this estimate requires estimates from chapter 3.

**THEOREM 4.3.2** *We have*

$$\|\partial_t^2 \nabla \phi\|_{H^{6.5}(\Omega_t)} \leq P[\|x_\kappa\|_{7.5}, \|v\|_{6.5}] \quad (4.3.1)$$

*and*

$$\|\partial_t \nabla \phi\|_{H^{6.5}(\Omega_t)} \leq P[\|x_\kappa\|_{7.5}, \|v\|_{6.5}]. \quad (4.3.2)$$

# Chapter 5

## *A priori* estimates for smoothed Euler.

In this chapter  $\kappa > 0$  is fixed. Using the smoothing defined in chapter 2 we can write down the following smoothed version of the equations (1.0.1), (1.0.3) and (1.0.4):

$$(\partial_t + v_\kappa^i \partial_i) v_j = -\partial_j p - \partial_j \phi \text{ in } \Omega_t \quad (5.0.1)$$

$$\operatorname{div} v = 0 \text{ in } \Omega_t \quad (5.0.2)$$

$$p = 0 \text{ on } \partial\Omega_t \quad (5.0.3)$$

together with smoothed versions of the conditions (1.0.5), (1.0.7) and (1.0.6):

$$\partial_t + v_\kappa^i \partial_i \text{ is in the tangent-space of } \cup_{t \in [0, T]} [\Omega_t \times \{t\}] \quad (5.0.4)$$

$$\nabla p \cdot N \leq -c_0 < 0 \text{ on } \partial\Omega \quad (5.0.5)$$

$$v = v_0 \text{ on } \Omega \quad (5.0.6)$$

where  $\Omega = \Omega_0$ , the initial domain. Suppose that  $v$  satisfies (5.0.1) - (5.0.6) and define

$$E_\kappa(t) = \sup_{[0, t]} [\|V\|_{7.5} + \|x_\kappa\|_8 + \kappa \|V\|_8] \quad (5.0.7)$$

where  $V(t, y) = v(t, x(t, y))$  and  $x$  is the flow of  $v$  defined by the differential equation (1.2.1). Here,  $x_\kappa$  is the smoothed flow of  $V$ . In this chapter we prove the following theorem:

**THEOREM 5.0.3** *There is  $T_1 > 0$  independent of  $\kappa$  such that  $E_\kappa(T_1) \leq E_0$ , where  $E_0 = E_0(E_\kappa(0), \|\Omega\|_9, \|v_0\|_8)$ .*

As a result of theorem 5.0.3, we have  $\sup_{[0, T_1]} \|V\|_8 \leq \kappa^{-1} E_0$ . We will prove this theorem by proving that there is  $T_1 > 0$  such that

$$E_\kappa(T_1) \leq P[E_\kappa(0), \|\Omega\|_9, \|v_0\|_8] + T_1 P[E_\kappa(T_1)]. \quad (5.0.8)$$

To control  $\|V\|_{7.5}$  we use lemma 2.5.2 and lemma 2.5.3 together with,

$$E_1 = \|\operatorname{curl}[v]\|_{H^{6.5}(\Omega_t)} \text{ and } E_2(t) = \|\zeta\langle\partial\theta\rangle^{7.5}v\|_{L^2(\Omega_t)}. \quad (5.0.9)$$

To control  $\|x_\kappa\|_8$  we use

$$E_3(t) = \sum_{k=1}^{\mu} \int_{\partial\Omega_t} (-\nabla p \cdot N) \left[ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle\partial\theta\rangle^{7.5} x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \cdot N \right]^2 \det(A) \circ x_\kappa^{-1} dS(x), \quad (5.0.10)$$

where  $N$  is the external unit normal to  $\partial\Omega_t$ . In the term in (5.0.10), we will suppress  $\det(A) \circ x_\kappa^{-1}$  when it does not play an important role, since it will not produce any terms of higher order and its inclusion produces more cumbersome computations. To control  $\|x_\kappa\|_8$  we will also use

$$E_4(t) = \sum_{k=1}^{\mu} \left\| \operatorname{div} \left[ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \right\|_{H^6(\Omega_t)}^2 \quad (5.0.11)$$

$$+ \sum_{k=1}^{\mu} \left\| \operatorname{curl} \left[ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \right\|_{H^6(\Omega_t)}^2. \quad (5.0.12)$$

We have

$$\left\| \left( \langle\partial\theta\rangle^{6.5} \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle\partial\theta\rangle x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right) \cdot N \right\|_{L^2(\partial\Omega_t)} \quad (5.0.13)$$

$$= \int_{\partial\Omega_t} \left[ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle\partial\theta\rangle^{7.5} x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \cdot N \right]^2 dS(x) \quad (5.0.14)$$

$$\leq \frac{1}{c_0 c_1} \int_{\partial\Omega_t} (-\nabla p \cdot N) \left[ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle\partial\theta\rangle^{7.5} x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \cdot N \right]^2 dS(x) \quad (5.0.15)$$

$$\leq \frac{E_3}{c_0 c_1}. \quad (5.0.16)$$

From lemma 2.5.3, we have, by (5.0.13)

$$\left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle\partial\theta\rangle x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right\|_{H^7(\Omega_t)}^2 \leq P[\|x_\kappa\|_7] \left[ E_4 + \frac{E_3}{c_0 c_1} \right]. \quad (5.0.17)$$

This means that we similarly control

$$\left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial \langle\partial\theta\rangle x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right\|_{H^6(\Omega_t)}^2 \quad (5.0.18)$$

and therefore

$$\left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right\|_{H^{6.5}(\partial\Omega_t)}^2. \quad (5.0.19)$$

Using  $E_4$  and (5.0.19) we therefore control

$$\left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right\|_{H^7(\Omega_t)}^2 \quad (5.0.20)$$

and therefore

$$\left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \right\|_7^2 \text{ and } \left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \right\|_8^2. \quad (5.0.21)$$

In addition we will use

$$E_5(t) = \sum_{k=\mu+1}^{\nu} \|\operatorname{div} [\partial[\xi_k x] \circ x_\kappa^{-1}]\|_{H^6(\Omega_t)} + \sum_{k=\mu+1}^{\nu} \|\operatorname{curl} [\partial[\xi_k x] \circ x_\kappa^{-1}]\|_{H^6(\Omega_t)}, \quad (5.0.22)$$

as well as

$$\left\| \partial_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right\|_{L^2(\Omega_t)} \quad (5.0.23)$$

which together with lemma 2.5.3 gives control of  $\|x_\kappa\|_8$ . To build regularity for  $\kappa\|V\|_8$  we use lemma 2.5.3 and the following asymptotic energy components:

$$E_6(t) = \kappa^2 \sum_{k=1}^{\mu} \sum_{i=1}^2 \int_{\partial\Omega_t \cap U_k} \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V \right) \circ x_\kappa^{-1} \cdot T_{k,i} \right]^2 dS(x)$$

where  $T_{k,i} = \frac{\partial(x_\kappa \circ \Psi_k)}{\partial z^i} \circ \Psi_k^{-1} \circ x_\kappa^{-1}$  and  $E_7(t) = \kappa^2 \|\operatorname{div} [[\partial V] \circ x_\kappa^{-1}]\|_{H^6(\Omega_t)}^2 + \kappa^2 \|\operatorname{curl} [[\partial V] \circ x_\kappa^{-1}]\|_{H^6(\Omega_t)}^2$ .

## 5.1 Control of $\dot{E}_1$ .

We have  $[\partial_t, \partial_i] x_\kappa^j = -(\partial_i v_\kappa^j)$  and therefore  $\partial_t \operatorname{curl} [v] = (\nabla v_\kappa)(\nabla v)$ . Since  $H^{6.5}(\Omega_t)$  is an algebra we can control  $\|\operatorname{curl} [v]\|_{H^{6.5}(\Omega_t)}$ .

## 5.2 Control of $\dot{E}_2$ .

The time derivative of  $E_2$  is equal to

$$2 \int_{\Omega} (\zeta \langle \partial_\theta \rangle^{7.5} V^i) (\zeta \langle \partial_\theta \rangle^{7.5} \partial_t V_i) dy = -2 \int_{\Omega} (\zeta \langle \partial_\theta \rangle^{7.5} V^i) (\zeta \langle \partial_\theta \rangle^{7.5} \partial_i p) dy \quad (5.2.1)$$

$$-2 \int_{\Omega} (\zeta \langle \partial_\theta \rangle^{7.5} V^i) (\zeta \langle \partial_\theta \rangle^{7.5} \partial_i \phi) dy. \quad (5.2.2)$$

We control the second term in (5.2.1) using theorem 4.3.1. In the first term in (5.2.1) we commute the  $\partial_i$  through the  $\langle \partial_\theta \rangle$  and obtain

$$-2 \int_{\Omega} (\zeta \langle \partial_\theta \rangle^{7.5} V^i) (\zeta \partial_i \langle \partial_\theta \rangle^{7.5} p) dy + 2 \int_{\Omega} (\zeta \langle \partial_\theta \rangle^{7.5} V^i) (\zeta \partial_i \langle \partial_\theta \rangle^{7.5} x_\kappa^j) (\partial_j p) dy + \text{lower order terms} \quad (5.2.3)$$

The first term in (5.2.3) we integrate by parts to obtain, in addition to lower order terms,

$$-2 \int_{\Omega} (\zeta \langle \partial_\theta \rangle \partial_i \langle \partial_\theta \rangle^{6.5} V^i) (\zeta \langle \partial_\theta \rangle^{7.5} p) dy \leq \| \langle \partial_\theta \rangle^{\frac{1}{2}} \operatorname{div} [\langle \partial_\theta \rangle^{6.5} v] \| \| \langle \partial_\theta \rangle^8 p \| \quad (5.2.4)$$

using proposition A.0.3. We control the first factor in (5.2.4) because  $\operatorname{div} [v] = 0$  and the second by theorem 3.3.1. We leave the second term in (5.2.3) until later.

### 5.3 Control of $\dot{E}_3$ .

Let  $x_k = x_\kappa \circ \Psi_k$ . The time derivative of  $E_3$  is equal to

$$\int_{\partial\Omega_t} \partial_t |\nabla p| \left[ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x \right) \circ \Psi_k \right] \circ x_k^{-1} \cdot N \right]^2 dS(x) \quad (5.3.1)$$

$$+ 2 \int_{\partial\Omega_t} |\nabla p| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x \right) \circ \Psi_k \right] \circ x_k^{-1} \cdot (\partial_t N) \left[ \vartheta_\kappa * \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x \right) \circ \Psi_k \right] \circ x_k^{-1} \cdot N dS(x) \quad (5.3.2)$$

$$+ 2 \int_{\partial\Omega_t} |\nabla p| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V \right) \circ \Psi_k \right] \circ x_k^{-1} \cdot N \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x \right) \circ \Psi_k \right] \circ x_k^{-1} \cdot N dS(x). \quad (5.3.3)$$

We control the first two terms in (5.3.1). Because  $N_i = \frac{-\partial_i p}{|\nabla p|}$ , the third term in (5.3.1) is equal to

$$- 2 \int_{\partial\Omega_t} \zeta^2 \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k \right] \circ x_k^{-1} N_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ x_k^{-1} (\partial_l p) dS(x) \quad (5.3.4)$$

$$= -2 \int_{\Omega_t} \zeta^2 \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k \right] \circ x_k^{-1} \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ x_k^{-1} (\partial_l p) dx \quad (5.3.5)$$

$$- 2 \int_{\Omega_t} \zeta^2 \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k \right] \circ x_k^{-1} \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ x_k^{-1} (\partial_l p) dx \quad (5.3.6)$$

$$- 2 \int_{\Omega_t} \zeta^2 \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k \right] \circ x_k^{-1} \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ x_k^{-1} (\partial_i \partial_l p) dx \quad (5.3.7)$$

using the divergence theorem. We control the third term in (5.3.5). In the first term in (5.3.5) we commute one of the  $\langle \partial_\theta \rangle$  out. By proposition A.0.3 the first term in (5.3.5) is therefore controlled by

$$\left\| \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^7 V^i \right) \circ \Psi_k \right] \circ x_k^{-1} \right\| \left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^8 x^l \right) \circ \Psi_k \right] \circ x_k^{-1} (\partial_l p) \right\|. \quad (5.3.8)$$

We control the second factor in (5.3.8). To deal with the first factor we have lemma 5.3.1.

**LEMMA 5.3.1** *For a function  $f$  on  $\Omega$  we have*

$$\left| \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} f \right) \circ \Psi_k \right] \circ x_k^{-1} - \vartheta_\kappa * \left[ \xi_k^{\frac{1}{2}} \circ \Psi_k \partial_i [f \circ x_\kappa^{-1}] \circ x_\kappa \circ \Psi_k \right] \circ x_k^{-1} \right| \quad (5.3.9)$$

$$\leq \kappa \|A\|_4 \|f\|_1. \quad (5.3.10)$$

**PROOF:** We have

$$\begin{aligned} \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} f \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} &= A \left( \frac{\partial z}{\partial y} \right) \frac{\partial}{\partial z} \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} f \right) \circ \Psi_k \right] \\ &= \left( \frac{\partial y}{\partial x} \right) \left( \frac{\partial z}{\partial y} \right) \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \circ \Psi_k \frac{\partial}{\partial z} [f \circ \Psi_k] \right) \right]. \end{aligned}$$

Now for  $g$  and  $h$  defined on  $(0, 1)^2$ ,

$$|g\vartheta_\kappa * [h] - \vartheta_\kappa * [gh]| = \left| \int_{(0,1)^2} \vartheta_\kappa(z'') [g(z' - z'') - g(z')] h(z' - z'') dz'' \right| \quad (5.3.11)$$

$$\leq \kappa \|g\|_4 \int_{(0,1)^2} \vartheta_\kappa(z'') |h|(z' - z'') dz'' \quad (5.3.12)$$

$$\leq \kappa \|g\|_4 \|h\|. \quad (5.3.13)$$

■

Therefore

$$\partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^7 V^i \right) \circ \Psi_k \right] \circ x_k^{-1} = \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \circ \Psi_k [\partial_i \partial_\theta^7 v^i] \circ x_\kappa \circ \Psi_k \right) \right] \circ x_k^{-1} + R \quad (5.3.14)$$

where the remainder is controlled by  $\kappa \|A\|_4 \|V\|_8$ , according to lemma 5.3.1. Since  $\operatorname{div} [v] = 0$ ,  $0 = \langle \partial_\theta \rangle^7 \operatorname{div} [v] = \sum (\langle \partial_\theta \rangle^i A) (\langle \partial_\theta \rangle^{j+1} v) + \operatorname{div} \langle \partial_\theta \rangle^7 v$  where the sum is over  $i + j = 7$  and  $j \leq 6$ . Thus we can control the first factor in (5.3.8). Let  $S = (0, 1)^2 \times (-1, 0]$ . Changing variables in the second term in (5.3.5), using the fact that the integral contains the term  $\det(A) \circ x_\kappa^{-1}$ , which had been suppressed, gives

$$- \int_S \zeta^2 \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \Phi_{lk} \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k \right] \quad (5.3.15)$$

$$\times \det \left( \frac{\partial \Psi_k}{\partial z} \right) \circ \Psi_k dz. \quad (5.3.16)$$

where  $\Phi_{lk} = (\partial_l p) \circ x_k$ . Let us again suppress the Jacobian. For the above term we have the following lemma:

**LEMMA 5.3.2** ( $\alpha\mu\nu\delta\alpha$ ) *Let  $f$  and  $g$  be functions defined on  $S$ . Then*

$$\int_S f \vartheta_\kappa * [g] dz = \int_S \vartheta_\kappa * [f] g dz.$$

**PROOF:**

$$\begin{aligned} \int_S f \vartheta_\kappa * [g] dz &= \int_{(-1,0]} \int_{(0,1)^2} \int_{(0,1)^2} f(z', z^3) \vartheta_\kappa(z' - z'') g(z'', z^3) dz'' dz' dz^3 \\ &= \int_S \vartheta_\kappa * [f] g dz, \end{aligned}$$

because  $\vartheta_\kappa$  is even. ■



Using lemma 5.3.2 the term in (5.3.15) is equal to

$$- \int_S \zeta^2 \vartheta_\kappa * \left[ \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \Phi_{lk} \right] \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz \quad (5.3.17)$$

$$= - \int_S \zeta^2 \Phi_{lk} \vartheta_\kappa * \left[ \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \right] \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz \quad (5.3.18)$$

$$+ \int_S \zeta^2 \Phi_{lk} \vartheta_\kappa * \left[ \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \right] \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz \quad (5.3.19)$$

$$- \int_S \zeta^2 \vartheta_\kappa * \left[ \Phi_{lk} \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \right] \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz. \quad (5.3.20)$$

Using (5.3.11) we see that the last two integrals in (5.3.18) can be estimated by

$$\kappa \| \langle \partial_\theta \rangle^{7.5} V \| \| \nabla p \|_4 \| \nabla \vartheta_\kappa * [\zeta^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x] \| \quad (5.3.21)$$

which we can control since  $\vartheta_\kappa * [\langle \partial_\theta \rangle h] = \langle \partial_\theta \rangle \vartheta_\kappa * [h] = \kappa^{-1} \vartheta_\kappa * [h]$ . The first term in (5.3.18) is equal to

$$- \int_S \zeta^2 \Phi_{lk} \left[ \partial_i \vartheta_\kappa * \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz \quad (5.3.22)$$

$$+ \int_S \zeta^2 \Phi_{lk} \left[ \partial_i \vartheta_\kappa * \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz \quad (5.3.23)$$

$$- \int_S \zeta^2 \Phi_{lk} \vartheta_\kappa * \left[ \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \circ x_\kappa \circ \Psi_k \right] \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V^i \right) \circ \Psi_k dz \quad (5.3.24)$$

Using lemma 5.3.1 we can control the two last terms in (5.3.22) by

$$\begin{aligned} & \kappa \| \langle \partial_\theta \rangle^{7.5} V \| \| \nabla p \|_4 \| A \|_4 \left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} x \right) \circ \Psi_k \right] \right\|_1 \\ & \leq \| \langle \partial_\theta \rangle^{7.5} V \| \| \nabla p \|_4 \| A \|_4 \left\| \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{6.5} x \right) \circ \Psi_k \right] \right\|_1 \end{aligned}$$

using, again, the smoothing properties of the convolution. Thus from (5.3.22) we obtain, in addition to a collection of lower order terms

$$- \sum_{k=1}^{\mu} \int_{\Omega} \zeta^2 (\partial_l p) \circ x_\kappa \left[ \partial_i \langle \partial_\theta \rangle^{7.5} \left[ \xi_k^{\frac{1}{2}} \circ x_\kappa^{-1} \vartheta_\kappa * \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} x^l \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \right] \circ x_\kappa \quad (5.3.25)$$

$$\times \langle \partial_\theta \rangle^{7.5} V^i dy \quad (5.3.26)$$

$$= - \int_{\Omega} \zeta^2 (\partial_l p) \circ x_\kappa (\partial_i \langle \partial_\theta \rangle^{7.5} x_\kappa^l) \circ x_\kappa \langle \partial_\theta \rangle^{7.5} V^i dy \quad (5.3.27)$$

$$+ \int_{\Omega} (\partial_l p) \circ x_\kappa (\partial_i \langle \partial_\theta \rangle^{7.5} x_m^l) \circ x_\kappa \langle \partial_\theta \rangle^{7.5} V^i dy. \quad (5.3.28)$$

Here we control the second term in (5.3.27) by integrating by parts and using proposition A.0.3. The first term in (5.3.27) combines with the second term in (5.2.3) to give us

$$- \int_{\Omega_t} \zeta^2 (\langle \partial_\theta \rangle^{7.5} v^i) (\langle \partial_\theta \rangle^{7.5} \partial_i p) \det(A) \circ x_\kappa^{-1} dx \quad (5.3.29)$$

$$- \int_{\Omega_t} \zeta^2 (\langle \partial_\theta \rangle^{7.5} v^i) (\partial_i \langle \partial_\theta \rangle^{7.5} x_\kappa^l) (\partial_l p) \det(A) \circ x_\kappa^{-1} dx \quad (5.3.30)$$

$$= ([\nabla, \langle \partial_\theta \rangle^{\frac{1}{2}}] \langle \partial_\theta \rangle^7 x_\kappa) (\nabla p) \quad (5.3.31)$$

$$+ \sum \int_{\Omega_t} \zeta^2 (\langle \partial_\theta \rangle^{7\frac{1}{2}} v^i) .5rac12 [(\nabla \langle \partial_\theta \rangle^{j_1} x_\kappa) \dots (\nabla \langle \partial_\theta \rangle^{j_{s-1}} x_\kappa) (\langle \partial_\theta \rangle^{j_s} \nabla p)] dx \quad (5.3.32)$$

where the sum is over  $j + \dots + j_s = 7$  and  $j_1, \dots, j_s \leq 6$ . We control the terms above using proposition A.0.1.

## 5.4 Control of $\dot{E}_4$ .

First we deal with the divergence term. Let  $\alpha_k = \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1}$ . We have

$$\partial_t \operatorname{div} \alpha_k = (\nabla v_\kappa) (\nabla \alpha_k) + \operatorname{div} \partial_t \alpha_k \quad (5.4.1)$$

$$= (\nabla v_\kappa) (\nabla \alpha_k) + \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla \partial x_\kappa) (\nabla v) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.2)$$

$$+ \operatorname{div} \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial v \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} - \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \operatorname{div} \partial v \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1}. \quad (5.4.3)$$

Therefore we have an equation of the form  $\partial_t f = g$  which we can integrate with respect to time to obtain  $f(t) = f(0) + \int_{[0,t]} g(s) ds$ . Since  $H^6(\Omega_t)$  is a Banach algebra for  $\Omega_t \subseteq \mathbf{R}^3$  we control the first and second terms in (5.4.2) can be controlled by  $\|V\|_7 \|\alpha_k\|_7$  and  $\|x_\kappa\|_8 \|V\|_7$  respectively. The last two terms in (5.4.2) can, according to lemma 5.3.1, be controlled by  $\kappa \|x_\kappa\|_7 \|V\|_8$ . Now we deal with the curl term. Let  $\alpha_k$  be defined as above. We now consider the two time derivatives on  $\operatorname{curl} \alpha_k$ :

$$\partial_t^2 \operatorname{curl} \alpha_k = \partial_t [(\nabla v_\kappa) (\nabla \alpha_k) + \operatorname{curl} \partial_t \alpha_k] \quad (5.4.4)$$

$$= \partial_t [(\nabla v_\kappa) (\nabla \alpha_k)] + [(\nabla v_\kappa) (\nabla \partial_t \alpha_k)] + \operatorname{curl} \partial_t^2 \alpha_k \quad (5.4.5)$$

$$= \partial_t [(\nabla v_\kappa) (\nabla \alpha_k)] - [(\nabla \partial_t v_\kappa) (\nabla \alpha_k)] + \operatorname{curl} \partial_t^2 \alpha_k. \quad (5.4.6)$$

Now

$$(\operatorname{curl} \partial_t^2 \alpha_k)_{ij} = \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial \partial_t v_j \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} - \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial_i \partial \partial_t v_j \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.7)$$

$$- \partial_j \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial \partial_t v_i \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} + \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial_j \partial \partial_t v_i \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.8)$$

$$+ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial_i \partial \partial_t v_j \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} - \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial_j \partial \partial_t v_i \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.9)$$

$$= (\nabla v_\kappa) \nabla \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial v \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.10)$$

$$+ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla v_\kappa) (\nabla \partial v) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.11)$$

$$+ \partial_t \left[ \partial_i \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial v_j \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} - \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial_i \partial v_j \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \quad (5.4.12)$$

$$+ \partial_t \left[ \partial_j \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial v_i \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} - \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial_j \partial v_i \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \right] \quad (5.4.13)$$

$$+ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla \partial x_\kappa) (\nabla \partial_t v) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1}, \quad (5.4.14)$$

using (6.0.27). The first term in (5.4.10) is equal to

$$\partial_t [(\nabla v_\kappa) \nabla \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1}] - (\nabla \partial_t v_\kappa) \nabla \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} \partial x \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.15)$$

and the second term in (5.4.10) is equal to

$$\partial_t \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla v_\kappa) (\nabla \partial x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} - \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla \partial_t v_\kappa) (\nabla \partial x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1}. \quad (5.4.16)$$

Together (5.4.6), (5.4.10), (5.4.15) and (5.4.16) give an equation which is of the form  $\partial_t [\partial_t f + g] = h$ . Integrating with respect to time once yields  $\partial_t f = (\partial_t f_1)(0) - g(t) + g(0) + \int_{[0,t]} h(u) du$ . Another integration with respect to time again gives

$$f(t) = f(0) + t(\partial_t f)(0) - \int_{[0,t]} g(u) du + tg(0) + \int_{[0,t]} \int_{[0,u_2]} h(u_1) du_1 du_2.$$

In  $H^6(\Omega_t)$  we control  $f(0)$  by  $E(0)$  and  $(\partial_t f)(0)$  by  $\|v_0\|_8$ . We have already seen that we can control the first term in (5.4.6). The second term in (5.4.6) and the first and second term in (5.4.15) can all be controlled using the fact that  $H^6(\Omega_t)$  is a Banach algebra. Applying six

derivatives to the first term in (5.4.16) we have

$$\partial^6 \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla v_\kappa) (\nabla \partial x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.17)$$

$$= \sum \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\partial^{i+1} v_\kappa) (\partial^{j+2} x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.18)$$

$$+ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla v_\kappa) (\partial^8 x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.19)$$

where the sum is over all  $i$  and  $j$  such that  $i + j = 6$  and  $j \leq 5$ . We can control the first of these terms. We also have

$$\vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla v_\kappa) (\partial^8 x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} = (\nabla v_\kappa) \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\partial^8 x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.20)$$

$$+ \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\nabla v_\kappa) (\partial^8 x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.21)$$

$$- (\nabla v_\kappa) \vartheta_\kappa * \left[ \left( \xi_k^{\frac{1}{2}} (\partial^8 x) \right) \circ \Psi_k \right] \circ \Psi_k^{-1} \circ x_\kappa^{-1} \quad (5.4.22)$$

Here we control the first term in (5.4.20) by  $\|V\|_4 \|\alpha_k\|_7$ . Also, using (5.3.11) from lemma 5.3.1 we control the last two terms in (5.4.20) by  $\kappa \|V\|_5 \|V\|_8$ . Similarly, we control the second term in (5.4.16). Terms three and four, and five and six in (5.4.10) can be controlled by  $\kappa \|x_\kappa\|_7 \|V\|_8$ . The last term in (5.4.10) can be controlled.

## 5.5 Control of $\dot{E}_5$ .

This follows similarly to the result in section 5.4.

## 5.6 Control of $\dot{E}_6$ .

The time derivative of  $E_6$  is equal to

$$- 2\kappa^2 \int_{\partial\Omega_t} \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V \right) \circ x_\kappa^{-1} \cdot T_{k,i} \right] \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} [\nabla p] \circ x_\kappa \right) \circ x_\kappa^{-1} \cdot T_{k,i} \right] dS(x) \quad (5.6.1)$$

$$- 2\kappa^2 \int_{\partial\Omega_t} \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V \right) \circ x_\kappa^{-1} \cdot T_{k,i} \right] \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} [\nabla \phi] \circ x_\kappa \right) \circ x_\kappa^{-1} \cdot T_{k,i} \right] dS(x) \quad (5.6.2)$$

$$+ 2\kappa^2 \int_{\partial\Omega_t} \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V \right) \circ x_\kappa^{-1} \cdot T_{k,i} \right] \left[ \left( \xi_k^{\frac{1}{2}} \langle \partial_\theta \rangle^{7.5} V \right) \circ x_\kappa^{-1} \cdot (\partial_t T_{k,i}) \right] dS(x). \quad (5.6.3)$$

We control the second term in (5.6.1) by theorem 4.1.1. We also control the third term in (5.6.1). Now  $\nabla p \cdot T_{k,i} = 0$  for all  $k$  and  $i = 1, 2$ . Thus

$$0 = \langle \partial_\theta \rangle^{7.5} [\nabla p \cdot T_{k,i}] \quad (5.6.4)$$

$$= [\langle \partial_\theta \rangle^{7.5} [\nabla p] \circ x_\kappa] \circ x_\kappa^{-1} \cdot T_{k,i} + \langle \partial_\theta \rangle^{\frac{1}{2}} [ [\langle \partial_\theta \rangle^7 [\nabla p] \circ x_\kappa] \circ x_\kappa^{-1} \cdot T_{k,i} ] \quad (5.6.5)$$

$$- [\langle \partial_\theta \rangle^{7.5} [\nabla p] \circ x_\kappa] \circ x_\kappa^{-1} \cdot T_{k,i} + \langle \partial_\theta \rangle^{\frac{1}{2}} \left[ \sum (\langle \partial_\theta \rangle^{j_1} \nabla p) \cdot (\langle \partial_\theta \rangle^{j_2} T_{k,i}) \right] \quad (5.6.6)$$

$$+ \langle \partial_\theta \rangle^{\frac{1}{2}} [ (\nabla p) (\langle \partial_\theta \rangle^7 T_{k,i}) ] \quad (5.6.7)$$

where the sum is over  $j_1 + j_2 = 7$  and  $j_1, j_2 \leq 6$ . The second and third term in (5.6.5) can be controlled by  $\|T_{k,i}\|_{H^2(\partial\Omega_t)} \|[(\partial_\theta)^7[\nabla p] \circ x_\kappa] \circ x_\kappa^{-1}\|_{L^2(\partial\Omega_t)} \leq \|x_\kappa\|_4^2 \|[(\partial_\theta)^{\frac{1}{2}}\nabla p] \circ x_\kappa\|_7$  using the trace theorem. According to theorem 3.3.1, we can control  $\kappa \|[(\partial_\theta)^{\frac{1}{2}}\nabla p] \circ x_\kappa\|_7$ . We control the fourth term in (5.6.5) by  $\|[\nabla p] \circ x_\kappa\|_7 \|x_\kappa\|_8$ . The last term in (5.6.5) can be controlled by  $\|[\nabla p] \circ x_\kappa\|_4 \|T_{k,i}\|_{H^{7.5}(\partial\Omega_t)} + \|[\nabla p] \circ x_\kappa\|_5 \| \|T_{k,i}\|_{H^7(\partial\Omega_t)}$ . We have  $\kappa \|T_{k,i}\|_{H^{7.5}(\partial\Omega_t)} \leq \|\Omega\|_9 \|x_\kappa\|_8$ , which we control.

## 5.7 Control of $\dot{E}_7$ .

This follows similarly to 5.1.

## 5.8 The constant $c_0$ in (5.0.5).

Using elliptic estimates for  $\partial_t \nabla p$  from theorem 3.3.2, we can show that assuming that we have  $N^i \partial_i p \leq -c_0 < 0$  on  $\partial\Omega$ , then we also have that estimate for a smaller  $c_0$  on some time-interval  $[0, T]$ .

# Chapter 6

## A fixed point formulation.

Fix  $\kappa > 0$  and suppose that we control  $\|\Omega\|_9$  and  $\|v_0\|_8$ . As was mentioned in the introduction, in this chapter we give a fixed point formulation of the smoothed version of Euler's equation (5.0.1) - (5.0.6), defined in chapter 5. We will look for the fixed point solution in the space

$$C_\kappa(T) = \left\{ f \in L^2([0, T], H^8(\Omega)) : \sup_{t \in [0, T]} \|f\|_8(t) \leq \kappa^{-1} E_0 + 1 \right\} \quad (6.0.1)$$

where  $E_0 = E_0(E_\kappa(0), \|\Omega\|_9, \|v_0\|_8)$ , obtained in theorem 5.0.3 is such that  $\|v_0\|_8 \leq \kappa^{-1} E_0$ . Let  $U, V$  and  $W$  be points in  $C(T)$ . Smooth  $U$  and  $V$  to obtain  $U_\kappa$  and  $V_\kappa$ . Let  $x_\kappa$  be the flow of  $U_\kappa$  and let  $\Omega_t = x_\kappa(t, \Omega)$ . We now define a function  $p$  which because it depends on  $U, V$ , and  $W$  will sometimes be denoted by  $p[U, V, W]$ :

$$\Delta p = - (\partial_i [V_\kappa^j \circ x_\kappa^{-1}]) (\partial_j [W^i \circ x_\kappa^{-1}]) + 1 \text{ on } \Omega_t, \quad (6.0.2)$$

where  $\partial_i = \frac{\partial}{\partial x_\kappa^i}$ , with boundary condition

$$p = 0 \text{ on } \partial\Omega_t. \quad (6.0.3)$$

We also define a function  $\phi$  which depends on  $U$ , and will therefore sometimes be denoted  $\phi[U]$ :

$$\phi(t, x) = -\chi_{\Omega_t} * \Phi(x). \quad (6.0.4)$$

Define, for  $y$  in  $\Omega$ ,

$$\Lambda^i(U)(V)(W)(t, y) = v_0^i(y) - \int_{[0, t]} (\partial^i p)(s, x_\kappa(s, y)) ds - \int_{[0, t]} (\partial^i \phi)(s, x_\kappa(s, y)) ds.$$

Using results from chapter 3 and chapter 4 we now prove that  $\Lambda$  is invariant and contractive on  $C(T)$ .

**LEMMA 6.0.1** *Fix  $U$  and  $V$  in  $C(T)$ . Then  $\Lambda(U)(V) : C(T) \rightarrow C(T)$  has a unique fixed-point.*

**PROOF:** By theorem 3.1.2 and theorem 4.1.1 we have estimates for  $p$  and  $\phi$  and thus

$$\|\Lambda(U)(V)(W)\|_s \leq \|v_0\|_s + \int_{[0,t]} P[\|U\|_s, \kappa^{-1}\|U\|_s, \|V\|_s, \|W\|_s] ds \quad (6.0.5)$$

$$+ \int_{[0,t]} P[\|U\|_s, \kappa^{-0.5}\|U\|_s] ds \quad (6.0.6)$$

and therefore

$$\sup_{[0,T]} \|\Lambda(U)(V)(W)\|_s \leq \kappa^{-1}E_0 + T \sup_{[0,T]} P[\kappa^{-1}E_0 + 1, \kappa^{-1}[\kappa^{-1}E_0 + 1], \kappa^{-0.5}[\kappa^{-1}E_0 + 1]]. \quad (6.0.7)$$

For  $T$  small enough, therefore,  $\Lambda(U)(V)$  maps  $C(T)$  into  $C(T)$ . Let  $W_1$  and  $W_2$  be two points in  $C(T)$ . Then

$$\Lambda^i(U)(V)(W_1)(t, y) - \Lambda^i(U)(V)(W_2)(t, y) = \int_{[0,t]} \partial^i (p_2 - p_1)(s, x_\kappa(s, y)) ds$$

where  $p_k = p[U, V, W_k]$ . Let  $q = p_2 - p_1$ . Then  $q$  satisfies

$$\Delta q = -(\partial_i[V_\kappa^j \circ x_\kappa^{-1}]) (\partial_j[W_2^i - W_1^i] \circ x_\kappa^{-1}) \quad \text{on } \Omega_t \quad (6.0.8)$$

with boundary condition  $q = 0$  on  $\partial\Omega_t$ . Using theorem 3.2.1 we have

$$\sup_{[0,T]} \|\Lambda(U)(V)(W_2) - \Lambda(U)(V)(W_1)\|_s \quad (6.0.9)$$

$$\leq T \sup_{[0,T]} P[\|U\|_s, \kappa^{-1}\|U\|_s] \|V\|_s \sup_{[0,T]} \|W_2 - W_1\|_s \quad (6.0.10)$$

$$\leq TP[[\kappa^{-1}E_0 + 1], \kappa^{-1}[\kappa^{-1}E_0 + 1]] \sup_{[0,T]} \|W_2 - W_1\|_s. \quad (6.0.11)$$

Since it is possible to pick  $T$  small enough that

$$\alpha(T) = TP[[\kappa^{-1}E_0 + 1], \kappa^{-1}[\kappa^{-1}E_0 + 1]] < 1, \quad (6.0.12)$$

$\Lambda(U)(V)$  is a contraction mapping.  $C(T)$  is a non-empty Banach-space so by the Banach contraction mapping theorem,  $\Lambda(U)(V)$  has therefore a unique fixed-point.  $\blacksquare$

Let  $\Lambda_1(U)(V)$  denote the unique fixed-point obtained in lemma 6.0.1.

**LEMMA 6.0.2** *Fix  $U$  in  $C(T)$ . Then  $\Lambda_1(U) : C(T) \rightarrow C(T)$  has a unique fixed-point.*

**PROOF:** By lemma 6.0.1,  $\Lambda_1(U) : C(T) \rightarrow C(T)$  is a well-defined map. Let  $V_1$  and  $V_2$  be two points in  $C(T)$ . Then

$$\sup_{[0,T]} \|\Lambda_1(U)(V_2) - \Lambda_1(U)(V_1)\|_s \quad (6.0.13)$$

$$= \sup_{[0,T]} \|\Lambda(U)(V_2)(\Lambda_1(U)(V_2)) - \Lambda(U)(V_1)(\Lambda_1(U)(V_1))\|_s \quad (6.0.14)$$

$$\leq \sup_{[0,T]} \|\Lambda(U)(V_2)(\Lambda_1(U)(V_2)) - \Lambda(U)(V_2)(\Lambda_1(U)(V_1))\|_s \quad (6.0.15)$$

$$+ \sup_{[0,T]} \|\Lambda(U)(V_2)(\Lambda_1(U)(V_1)) - \Lambda(U)(V_1)(\Lambda_1(U)(V_1))\|_s. \quad (6.0.16)$$

Let  $W = \Lambda_1(U)(V_1)$  and let  $p_k = p[U, V_k, W]$ . Define  $q = p_1 - p_2$ , then  $q$  satisfies  $\Delta q = -(\partial_i[(V_1)_\kappa^j - (V_2)_\kappa^j] \circ x_\kappa^{-1})(\partial_j[W^i \circ x_\kappa^{-1}])$  on  $\Omega_t$  with boundary condition  $q = 0$  on  $\partial\Omega_t$ . Theorem 3.2.1 provides the following estimate of the second term in (6.0.13):

$$T \sup_{[0, T]} P[\|U\|_8, \kappa^{-1}\|U\|_8] \sup_{[0, T]} \|V_2 - V_1\|_8 \sup_{[0, T]} \|\Lambda_1(U)(V_1)\|_8. \quad (6.0.17)$$

Thus the estimate from (6.0.11) of the first term in (6.0.13) gives

$$\sup_{[0, T]} \|\Lambda_1(U)(V_2) - \Lambda_1(U)(V_1)\|_8 \leq \alpha(T) \sup_{[0, T]} \|\Lambda_1(U)(V_2) - \Lambda_1(U)(V_1)\|_8 + \alpha(T) \sup_{[0, T]} \|V_2 - V_1\|_8$$

and therefore

$$\sup_{[0, T]} \|\Lambda_1(U)(V_2) - \Lambda_1(U)(V_1)\|_8 \leq \frac{\alpha(T)}{1 - \alpha(T)} \|V_2 - V_1\|_8. \quad (6.0.18)$$

We can pick  $T$  small enough that  $\alpha(T)$  is small enough that  $\frac{\alpha(T)}{1 - \alpha(T)} < 1$  and therefore  $\Lambda_1(U)$  is a contraction mapping. By the Banach contraction mapping theorem,  $\Lambda_1(U)$  has therefore a unique fixed-point.  $\blacksquare$

Let  $\Lambda_2(U)$  denote the unique fixed-point obtained in lemma 6.0.2,.

**LEMMA 6.0.3**  $\Lambda_2 : C(T) \rightarrow C(T)$  has a unique fixed-point.

**PROOF:** We deduce by lemma 6.0.2, that  $\Lambda_2 : C(T) \rightarrow C(T)$  is a well-defined map. Let  $U_1, U_2$  and  $V$  be points in  $C(T)$ . By the definition of  $\Lambda, \Lambda_1$  and  $\Lambda_2$  we have  $\Lambda_2(U) = \Lambda_1(U)(\Lambda_2(U))$  and  $\Lambda_1(U)(V) = \Lambda(U)(V)(\Lambda_1(U)(V))$ . Thus

$$\Lambda_2(U) = \Lambda_1(U)(\Lambda_2(U)) = \Lambda(U)\left(\Lambda_2(U)\right)\left(\Lambda_1(U)(\Lambda_2(U))\right) \quad (6.0.19)$$

and

$$\sup_{[0, T]} \|\Lambda_2(U_2) - \Lambda_2(U_1)\|_8 \quad (6.0.20)$$

$$= \sup_{[0, T]} \left\| \Lambda(U_2)\left(\Lambda_2(U_2)\right)\left(\Lambda_1(U_2)(\Lambda_2(U_2))\right) - \Lambda(U_1)\left(\Lambda_2(U_1)\right)\left(\Lambda_1(U_1)(\Lambda_2(U_1))\right) \right\|_8 \quad (6.0.21)$$

$$\leq \sup_{[0, T]} \left\| \Lambda(U_2)\left(\Lambda_2(U_2)\right)\left(\Lambda_1(U_2)(\Lambda_2(U_2))\right) - \Lambda(U_2)\left(\Lambda_2(U_2)\right)\left(\Lambda_1(U_1)(\Lambda_2(U_1))\right) \right\|_8 \quad (6.0.22)$$

$$+ \sup_{[0, T]} \left\| \Lambda(U_2)\left(\Lambda_2(U_2)\right)\left(\Lambda_1(U_1)(\Lambda_2(U_1))\right) - \Lambda(U_2)\left(\Lambda_2(U_1)\right)\left(\Lambda_1(U_1)(\Lambda_2(U_1))\right) \right\|_8 \quad (6.0.23)$$

$$+ \sup_{[0, T]} \left\| \Lambda(U_2)\left(\Lambda_2(U_1)\right)\left(\Lambda_1(U_1)(\Lambda_2(U_1))\right) - \Lambda(U_1)\left(\Lambda_2(U_1)\right)\left(\Lambda_1(U_1)(\Lambda_2(U_1))\right) \right\|_8. \quad (6.0.24)$$

The first of the above terms can be controlled by

$$\alpha(T) \sup_{[0, T]} \|\Lambda_1(U_2)(\Lambda_2(U_2)) - \Lambda_1(U_1)(\Lambda_2(U_1))\|_8 = \alpha(T) \sup_{[0, T]} \|\Lambda_2(U_2) - \Lambda_2(U_1)\|_8$$

according to (6.0.11). The second of the above terms can be controlled by

$$\alpha(T) \sup_{[0, T]} \|\Lambda_2(U_2) - \Lambda_2(U_1)\|_8$$



according to (6.0.17). The third term can, according to theorem 3.2.2 and theorem 4.2.1, be controlled by  $\alpha(T) \sup_{[0,t]} \|U_2 - U_1\|_8$ . Thus we have

$$\sup_{[0,T]} \|\Lambda_2(U_2) - \Lambda_2(U_1)\|_8 \leq \frac{\alpha(T)}{1 - 2\alpha(T)} \sup_{[0,T]} \|\Lambda_2(U_2) - \Lambda_2(U_1)\|_8.$$

This shows that  $\Lambda_2$  is a contraction and hence it has a unique fixed-point.  $\blacksquare$

Thus we have a unique fixed point solution  $V$  in  $C(T_2)$ , where  $T_2 > 0$  depends on  $E_0$  and  $\kappa$ , which on  $[0, T_2]$  satisfies

$$V(t, y) = \Lambda(V)(V)(t, y) \tag{6.0.25}$$

$$= v_0(y) - \int_{[0,t]} (\nabla p)(s, x_\kappa(s, y)) ds - \int_{[0,t]} (\nabla \phi)(s, x_\kappa(s, y)) ds. \tag{6.0.26}$$

Therefore we have

$$\partial_t V = -(\nabla p) \circ x_\kappa - (\nabla \phi) \circ x_\kappa \tag{6.0.27}$$

in  $\Omega$  where  $x_\kappa$  is the flow of  $V_\kappa$ . Let  $\Omega_t = x_\kappa(t, \Omega)$  then  $p$  above is defined on  $\Omega_t$  by

$$\Delta p = -(\partial_i v_\kappa^j)(\partial_j v^i) + 1 \text{ on } \Omega_t, \tag{6.0.28}$$

where  $v_\kappa = V_\kappa \circ x_\kappa^{-1}$ ,  $v = V \circ x_\kappa^{-1}$  and  $\partial_i = \frac{\partial}{\partial x_\kappa^i}$ , with boundary condition

$$p = 0 \text{ on } \partial\Omega_t \tag{6.0.29}$$

and  $\phi$  is defined by

$$\phi(t, x) = -\chi_{\Omega_t} * \Phi(x). \tag{6.0.30}$$

Also, we have  $\text{div } \partial_t v = -\Delta p - \Delta \phi = (\partial_i v_\kappa^j)(\partial_j v^i)$  and  $\text{div } \partial_t v = \partial_i \partial_t v^i = (\partial_i v_\kappa^j)(\partial_j v^i) + \partial_t \text{div } v$ . Thus  $\partial_t \text{div } v = 0$ . And  $\text{div} [\Lambda(V, V, V)] \circ x_\kappa^{-1}(0, x) = \text{div } v_0 = 0$ . Thus  $\text{div } v = 0$ . This means that  $v$  satisfies (5.0.1) - (5.0.6) and therefore we have estimates for  $v$  in  $L^\infty[[0, T_1], H^8(\Omega)]$  via theorem 5.0.3. Suppose now that the time interval for existence,  $[0, T_2]$ , is shorter than the time interval on which we have *a priori* estimates,  $[0, T_1]$  and that  $T_2$  is the largest such  $T_2$ . By theorem 5.0.3, we have  $\sup_{[0, T_1]} \|V\|_8 \leq \kappa^{-1} E_0$  and in particular,  $\|V(T_2, \cdot)\|_8 \leq \kappa^{-1} E_0$ . Now define

$$C_\kappa(T) = \left\{ f \in L^2([T_2, T], H^8(\Omega)) : \sup_{t \in [T_2, T]} \|f\|_8(t) \leq \kappa^{-1} E_0 + 1 \right\} \tag{6.0.31}$$

and a map

$$\Lambda^i(U)(V)(W)(t, y) = V^i(T_2, y) - \int_{[T_2, t]} (\partial^i p)(s, x_\kappa(s, y)) ds - \int_{[T_2, t]} (\partial^i \phi)(s, x_\kappa(s, y)) ds. \tag{6.0.32}$$

Using the above argument we can show that we also have existence on  $[T_2, 2T_2]$  which contradicts the fact that  $T_2$  was the largest such  $T_2$ . This must mean that  $T_2 = T_1$  and therefore we have existence on  $[0, T_1]$ , an interval independent of  $\kappa$ . Existence for Euler's equation then follows a standard compactness argument which can be found, for example, in [9].

# Chapter 7

## Optimal regularity and uniqueness.

In previous chapters we have supposed that we controlled  $\|\Omega\|_9$  and  $\|v_0\|_8$ . We now relax those assumptions and suppose instead that we control  $\|\Omega\|_8$  and  $\|v_0\|_{7.5}$ . We can regularise this initial data using a standard convolution to obtain  $\Omega_\varepsilon$  in  $H^9$  and  $v_{0,\varepsilon}$  in  $H^8(\Omega_\varepsilon)$ . From the previous sections we obtain a sequence of solutions  $\partial_t x_\varepsilon$  with flows  $x_\varepsilon$  under such initial conditions. Let us suppress the  $\varepsilon$  and define the following energy:

$$E(t) = \sup_{[0,t]} [\|\partial_t^3 x\|_{6.5} + \|\partial_t^2 x\|_7 + \|\partial_t x\|_{7.5} + \|N\|_{H^{6.5}(\partial\Omega_t)}] \quad (7.0.1)$$

where  $N$  is the outward unit normal to  $\partial\Omega_t$ . The theorem we prove in this section is the following:

**THEOREM 7.0.4** *There is  $T$  such that  $E(T) \leq P[E(0)]$ .*

To build regularity for the first term in (7.0.1) we define  $E_1 = \|\eta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x\|^2$  and  $E_2 = \|\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x\|^2$ , where  $\eta = \eta_1$  and  $\zeta = \zeta_1$  are cut off functions defined in chapter 2. And to build regularity for the second term in (7.0.1) we define

$$E_3(t) = \frac{1}{2} \int_{\partial\Omega_t} (-\nabla p \cdot N) (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x) \cdot (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x) dS(x). \quad (7.0.2)$$

Establishing control of the third and fourth term in (7.0.1) follows once we have control the first and second. Now we control the time derivatives of  $E_1$ ,  $E_2$  and  $E_3$ .

### 7.1 Control of $\dot{E}_1$ .

The time derivative of  $E_1$  is

$$-2 \int_{\Omega_t} (\eta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\eta \langle \partial_\theta \rangle^{6.5} \partial_t^2 \partial_i p) dx - 2 \int_{\Omega_t} (\eta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\eta \langle \partial_\theta \rangle^{6.5} \partial_t^2 \partial_i \phi) dx \quad (7.1.1)$$

where the first term in (7.1.1) can be controlled using theorem 3.3.3 and the second term in (7.1.1) can be controlled using theorem 4.3.2.

## 7.2 Control of $\dot{E}_2$ .

We have

$$\dot{E}_2(t) = -2 \int_{\Omega_t} (\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^2 \partial_i p) dx - 2 \int_{\Omega_t} (\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^2 \partial_i \phi) dx. \quad (7.2.1)$$

The second term in (7.2.1) is controlled using theorem 4.3.2. Commuting  $\partial_i$  to the outside in the first term in (7.2.1) gives

$$-2 \int_{\Omega_t} (\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\zeta \partial_i \langle \partial_\theta \rangle^{6.5} \partial_t^2 p) dx + 2 \int_{\Omega_t} (\zeta \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\zeta \partial_i \langle \partial_\theta \rangle^{6.5} \partial_t^2 x^k) (\partial_k p) dx \quad (7.2.2)$$

$$+ \text{lower order terms} \quad (7.2.3)$$

We integrate the first term in (7.2.2) by parts – the result can be controlled using half integration by parts and the fact that  $\text{div} [\partial_t x] = 0$ . We leave the second term from (7.2.2) until section 7.3.

## 7.3 Control of $\dot{E}_3$ .

The time derivative of  $E_3$  is

$$\frac{1}{2} \int_{\partial\Omega_t} \partial_t |\nabla p| (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x) \cdot (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x) dS(x) \quad (7.3.1)$$

$$+ \int_{\partial\Omega_t} (-\nabla p \cdot N) (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x) \cdot (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x) dS(x). \quad (7.3.2)$$

We can control the first term in (7.3.1). We apply the divergence theorem to the second term in (7.3.1) to obtain

$$- \int_{\Omega_t} (\Delta p) (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx - \int_{\Omega_t} (\partial_j p) (\partial^j \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx \quad (7.3.3)$$

$$- \int_{\Omega_t} (\partial_j p) (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\partial^j \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx. \quad (7.3.4)$$

We control the first term in (7.3.3). Since  $\text{curl} [\partial_t^2 x] = 0$  we have

$$\partial^j \langle \partial_\theta \rangle^{6.5} \partial_t^3 x_i = \langle \partial_\theta \rangle^{6.5} \partial_t \partial^j \partial_t^2 x_i - (\partial^j \langle \partial_\theta \rangle^{6.5} \partial_t x^k) (\partial_k \partial_t^2 x_i) + \text{lower order terms} \quad (7.3.5)$$

$$= \langle \partial_\theta \rangle^{6.5} \partial_t \partial_i \partial_t^2 x^j - (\partial^j \langle \partial_\theta \rangle^{6.5} \partial_t x^k) (\partial_k \partial_t^2 x_i) + \text{lower order terms} \quad (7.3.6)$$

$$= \partial_i \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j + (\partial_i \langle \partial_\theta \rangle^{6.5} \partial_t x^k) (\partial_k \partial_t^2 x^j) + \text{lower order terms} \quad (7.3.7)$$

and, of course, similarly for  $\partial^j \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i$ . This means that the second term in (7.3.3) is, in addition to lower order terms, equal to

$$- \int_{\Omega_t} (\partial_j p) (\partial^i \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx \quad (7.3.8)$$

$$= \int_{\Omega_t} p (\partial_j \partial^i \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx + \int_{\Omega_t} p (\partial^i \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\partial_j \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx \quad (7.3.9)$$

$$= - \int_{\Omega_t} (\partial^i p) (\partial_j \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx - \int_{\Omega_t} p (\partial_j \langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\partial^i \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx \quad (7.3.10)$$

$$- \int_{\Omega_t} (\partial^i p) (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\partial_j \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx - \int_{\Omega_t} p (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\partial^i \partial_j \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx. \quad (7.3.11)$$

The first term in (7.3.10) we control using half integration by parts and the fact that  $\operatorname{div} [\partial_t x] = 0$ . The second term in (7.3.10) we integrate by parts to obtain

$$\int_{\Omega_t} (\partial_j p) (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\partial^i \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx + \int_{\Omega_t} p (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^j) (\partial_j \partial^i \langle \partial_\theta \rangle^{6.5} \partial_t^2 x_i) dx \quad (7.3.12)$$

and where we control the first term in (7.3.12) using the fact that  $\operatorname{div} [\partial_t x] = 0$  and the second term in (7.3.12) cancels the fourth term in (7.3.10). We will shortly deal with the third term in (7.3.10). First we use (7.3.7) on the third term in (7.3.3) which is therefore equal to a collection of lower order terms as well as

$$- \int_{\Omega_t} (\partial_j p) (\langle \partial_\theta \rangle^{6.5} \partial_t^3 x^i) (\partial_i \langle \partial_\theta \rangle^{6.5} \partial_t^2 x^j) dx \quad (7.3.13)$$

which in addition to the third term in (7.3.10) cancels the second term from (7.2.2).

## 7.4 Controlling the third and fourth term in (7.0.1).

Now we have  $(\nabla p) \circ x = \partial_t^2 x - (\nabla \phi) \circ x$  which means that  $\nabla p$  and therefore

$$N = \frac{\nabla p}{|\nabla p|} \quad (7.4.1)$$

are controlled in  $H^{6.5}(\partial\Omega_t)$ , using the sign condition. Finally, to control  $\|\partial_t x\|_{7.5}$  we use the relation  $\partial_t^2 x_i = -\partial_i p - \partial_i \phi$ . Thus we have  $\partial_t^3 x_i = (\partial_i \partial_t x^k) (\partial_k p) - \partial_i \dot{p} - \partial_t \partial_i \phi$  which we can dot with a vector  $T_{k,j} = \frac{\partial(x \circ \Psi_k)}{\partial z^j}$  tangential to  $\partial\Omega_t$ . Since  $\dot{p} = 0$  on  $\partial\Omega_t$  we have  $T_{k,j} \cdot \nabla \dot{p} = 0$  and therefore  $(T_{k,j}^i \partial_i \partial_t x^k) \cdot N |\nabla p| = -T_{k,j}^i \partial_t^3 x_i + T_{k,j}^i \partial_t \partial_i \phi$ . This means that we control  $\|(\partial_\theta \partial_t x^k) \cdot N\|_{H^6(\partial\Omega_t)}$  which together with lemma 2.5.3 provides control of  $\|\partial_t x\|_{7.5}$ .

## 7.5 Uniqueness.

Suppose that two solutions  $v_1$  and  $v_2$  arise from the same initial data. The above estimates then show that their difference is zero in some time interval.

# Appendix A

## Properties of $\langle \partial_\theta \rangle$ .

**PROPOSITION A.0.1** *Let  $f$  and  $g$  be functions on  $\Omega$ . Then*

$$\|\langle \partial_\theta \rangle^{\frac{1}{2}}[fg] - \langle \partial_\theta \rangle^{\frac{1}{2}}[f]g\|^2 \leq c\|f\|^2\|g\|_2^2. \quad (\text{A.0.1})$$

**PROOF:** Let  $h_j = (\zeta_j f) \circ \Psi_j$ . We have

$$f_j(z) = \int_{\mathbf{R}^2} \hat{f}_j(\alpha'_1, z^3) e^{i\alpha'_1 \cdot z'} d\alpha'_1 \text{ and } g_j(z) = \int_{\mathbf{R}^2} \hat{g}_j(\alpha'_2, z^3) e^{i\alpha'_2 \cdot z'} d\alpha'_2.$$

From now on we will suppress the  $z^3$  in the above expressions. Therefore

$$(fg)_j = \int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \hat{f}_j(\alpha'_1) \hat{g}_j(\alpha'_2) e^{i\alpha'_1 \cdot z'} e^{i\alpha'_2 \cdot z'} d\alpha'_1 d\alpha'_2.$$

Substitute  $\alpha' = \alpha'_1 + \alpha'_2$ . Then  $\alpha'_1 = \alpha' - \alpha'_2$  and

$$f_j g_j = \int_{\mathbf{R}^2} \left[ \int_{\mathbf{R}^2} \hat{f}_j(\alpha' - \alpha'_2) \hat{g}_j(\alpha'_2) d\alpha'_2 \right] e^{i\alpha' \cdot z'} d\alpha'.$$

Hence, a half derivative of the above is equal to

$$\int_{\mathbf{R}^2} \langle \alpha' \rangle^{\frac{1}{2}} \left[ \int_{\mathbf{R}^2} \hat{f}_j(\alpha - \alpha_2) \hat{g}_j(\alpha_2) d\alpha_2 \right] e^{i\alpha' \cdot z'} d\alpha'. \quad (\text{A.0.2})$$

And half derivative of  $f_j$  is equal to

$$\int_{\mathbf{R}^2} \langle \alpha'_1 \rangle^{\frac{1}{2}} \hat{f}_j(\alpha'_1) e^{i\alpha'_1 \cdot z'} d\alpha'_1$$

Therefore the product of a half derivative of  $f_j$ , and  $g_j$  is equal to

$$\int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \langle \alpha'_1 \rangle^{\frac{1}{2}} \hat{f}_j(\alpha'_1) \hat{g}_j(\alpha'_2) e^{i\alpha'_1 \cdot z'} e^{i\alpha'_2 \cdot z'} d\alpha'_1 d\alpha'_2 \quad (\text{A.0.3})$$

$$= \int_{\mathbf{R}^2} \left[ \int_{\mathbf{R}^2} \langle \alpha' - \alpha'_2 \rangle^{\frac{1}{2}} \hat{f}_j(\alpha' - \alpha'_2) \hat{g}_j(\alpha'_2) d\alpha'_2 \right] e^{i\alpha' \cdot z'} d\alpha'. \quad (\text{A.0.4})$$

The difference between (A.0.2) and (A.0.4) is

$$\int_{\mathbf{R}^2} \int_{\mathbf{R}^2} \left[ \langle \alpha' \rangle^{\frac{1}{2}} - \langle \alpha' - \alpha'_2 \rangle^{\frac{1}{2}} \right] \hat{f}_j(\alpha' - \alpha'_2) \hat{g}_j(\alpha'_2) d\alpha'_2 e^{i\alpha' \cdot z'} d\alpha'. \quad (\text{A.0.5})$$

To control the above we have the following lemma.

**LEMMA A.0.2** *Let  $\eta_1$  and  $\eta_2$  be points in  $\mathbf{R}^n$ . Then*

$$\left| \langle \eta_1 + \eta_2 \rangle^{\frac{1}{2}} - \langle \eta_1 \rangle^{\frac{1}{2}} \right| \leq c \langle \eta_2 \rangle^{\frac{1}{2}}.$$

**PROOF:** Suppose that  $\eta_1$  and  $\eta_2$  are such that  $0 \leq |\eta_1| \leq |\eta_2|$ . Then

$$\left| \langle \eta_1 + \eta_2 \rangle^{\frac{1}{2}} - \langle \eta_1 \rangle^{\frac{1}{2}} \right| \leq C \langle \eta_2 \rangle^{\frac{1}{2}}.$$

Now suppose that  $\eta_1$  and  $\eta_2$  are such that  $0 \leq |\eta_2| < |\eta_1|$ . Then

$$\langle \eta_1 \rangle^{\frac{1}{2}} \left| \frac{(1 + (\eta_1 + \eta_2) \cdot (\eta_1 + \eta_2))^{\frac{1}{4}}}{(1 + \eta_1 \cdot \eta_1)^{\frac{1}{4}}} - 1 \right| = \langle \eta_1 \rangle^{\frac{1}{2}} \left| \left( \frac{1 + \eta_1 \cdot \eta_1 + 2\eta_1 \cdot \eta_2 + \eta_2 \cdot \eta_2}{1 + \eta_1 \cdot \eta_1} \right)^{\frac{1}{4}} - 1 \right| \quad (\text{A.0.6})$$

$$= \langle \eta_1 \rangle^{\frac{1}{2}} \left| \left( 1 + \frac{2\eta_1 \cdot \eta_2 + \eta_2 \cdot \eta_2}{1 + \eta_1 \cdot \eta_1} \right)^{\frac{1}{4}} - 1 \right|. \quad (\text{A.0.7})$$

Define  $c(x) = (1 + x)^{\frac{1}{4}} - 1$ . And there is a constant  $C$  which bounds  $\frac{c(x)}{|x|}$ , for all  $x$  in  $(-4, 4)$ . Therefore,

$$\langle \eta_1 \rangle^{\frac{1}{2}} \left| \left( 1 + \frac{2\eta_1 \cdot \eta_2 + \eta_2 \cdot \eta_2}{1 + \eta_1 \cdot \eta_1} \right)^{\frac{1}{4}} - 1 \right| \leq C \langle \eta_1 \rangle^{\frac{1}{2}} \frac{2\eta_1 \cdot \eta_2 + \eta_2 \cdot \eta_2}{\langle \eta_1 \rangle^2} \quad (\text{A.0.8})$$

$$\leq C \langle \eta_1 \rangle^{\frac{1}{2}} \frac{\langle \eta_1 \rangle \langle \eta_2 \rangle + \langle \eta_2 \rangle^2}{\langle \eta_1 \rangle^2} \quad (\text{A.0.9})$$

$$\leq \frac{\langle \eta_2 \rangle}{\langle \eta_1 \rangle^{\frac{1}{2}}}. \quad (\text{A.0.10})$$

Since  $|\eta_2| < |\eta_1|$ ,  $\frac{\langle \eta_1 \rangle^{\frac{1}{2}}}{\langle \eta_2 \rangle^{\frac{1}{2}}} > 1$ , from where the result follows. ■

And from lemma A.0.2 we see that (A.0.5) is controlled by

$$\int_{\mathbf{R}^2} \left[ \int_{\mathbf{R}^2} \langle \alpha'_2 \rangle^{\frac{1}{2}} |\hat{f}_j(\alpha' - \alpha'_2)| |\hat{g}_j(\alpha'_2)| d\alpha'_2 \right]^2 d\alpha' \quad (\text{A.0.11})$$

$$\leq c \int_{\mathbf{R}^2} \left[ \int_{\mathbf{R}^2} \frac{\langle \alpha'_2 \rangle^{\frac{1}{2}} \langle \alpha'_2 \rangle^a}{\langle \alpha'_2 \rangle^a} |\hat{f}_j(\alpha' - \alpha'_2)| |\hat{g}_j(\alpha'_2)| d\alpha'_2 \right]^2 d\alpha' \quad (\text{A.0.12})$$

$$\leq c \int_{\mathbf{R}^2} \left[ \int_{\mathbf{R}^2} \langle \alpha'_2 \rangle^{2(\frac{1}{2}+a)} |\hat{g}_j(\alpha'_2)|^2 d\alpha'_2 \right] \left[ \int_{\mathbf{R}^2} \frac{|\hat{f}_j(\alpha' - \alpha'_2)|^2}{\langle \alpha'_2 \rangle^{2a}} d\alpha'_2 \right] d\alpha' \quad (\text{A.0.13})$$

$$\leq c \left[ \int_{\mathbf{R}^2} \langle \alpha'_2 \rangle^{2(\frac{1}{2}+a)} |\hat{g}_j(\alpha'_2)|^2 d\alpha'_2 \right] \left[ \int_{\mathbf{R}^2} \frac{1}{\langle \alpha'_2 \rangle^{2a}} \int_{\mathbf{R}^2} |\hat{f}_j(\alpha' - \alpha'_2)|^2 d\alpha' d\alpha'_2 \right]. \quad (\text{A.0.14})$$

For  $a$  such that  $2a > 2$  the last integral converges in three dimensions. Therefore (A.0.14) is controlled by  $\|\langle \partial_\theta \rangle^2 [g]\| \|f\|_2$ . ■

A similar result holds on  $\partial\Omega$ .

**PROPOSITION A.0.3** *Let  $f$  and  $g$  be functions on  $\Omega$ . Let  $(\cdot, \cdot)$  be the  $L^2(\Omega)$ -innerproduct. Then  $|(f, \partial_\theta [g])| \leq c \|\langle \partial_\theta \rangle^{\frac{1}{2}} [f]\| \|\langle \partial_\theta \rangle^{\frac{1}{2}} [g]\|$ .*

**PROOF:** We have

$$\left| \int_{(-1,0)} \int_{(-1,1)^2} f_k(\partial g_l / \partial z^j) dz' dz^3 \right| = \left| \int_{(-1,0)} \int_{\mathbf{R}^2} \hat{f}_k \alpha'_i \hat{g}_l d\alpha' dz^3 \right| \quad (\text{A.0.15})$$

$$= \left| \int_{(-1,0)} \int_{\mathbf{R}^2} \alpha_j^{\frac{1}{2}} \hat{f}_k \alpha_j^{\frac{1}{2}} \hat{g}_i d\alpha' dz^3 \right| \quad (\text{A.0.16})$$

$$\leq \left| \int_{(-1,0)} \left[ \int_{\mathbf{R}^2} \langle \alpha' \rangle |f_k|^2 d\alpha' \right]^{\frac{1}{2}} \left[ \int_{\mathbf{R}^2} \langle \alpha' \rangle |\hat{g}_i|^2 d\alpha' \right]^{\frac{1}{2}} dz^3 \right| \quad (\text{A.0.17})$$

$$\leq c \|\langle \partial_\theta \rangle^{\frac{1}{2}} [f]\| \|\langle \partial_\theta \rangle^{\frac{1}{2}} [g]\|. \quad (\text{A.0.18})$$

■

# Appendix B

## Hodge-decomposition inequalities.

In this section we prove the Hodge-decomposition inequalities which were stated in chapter 2 and is used throughout this paper.

### B.1 The first one.

The first controls all derivatives in terms of the curl, the divergence and derivatives which act near the boundary, and which are tangential to the boundary.

**LEMMA B.1.1** *Let  $\alpha$  be a vector-field on  $\tilde{\Omega}_t$ . Define  $(\text{curl}\alpha)_{jk} = \partial_j\alpha_k - \partial_k\alpha_j$  and  $\text{div}\alpha = \partial_j\alpha^j$ . Then we have the following pointwise estimate on  $\Omega_t$ :*

$$|\zeta\nabla\alpha| \leq |\zeta\text{curl}\alpha| + |\zeta\text{div}\alpha| + |\zeta\partial_\theta\alpha|, \quad (\text{B.1.1})$$

where  $|\cdot|$  denotes the usual Euclidean distance.

**PROOF:** Here we will suppress the index on  $\zeta$ , letting it be denoted simply by  $\zeta$ . Define  $(\text{def}\alpha)_{jk} = \partial_j\alpha_k + \partial_k\alpha_j$ . Thus  $2\nabla\alpha = \text{curl}\alpha + \text{def}\alpha$ . Let  $\beta = \text{diag}(\partial_1\alpha_1, \dots, \partial_n\alpha_n)$  and define  $\gamma = \zeta\text{def}\alpha - \zeta\beta$ . Then  $|\zeta\nabla\alpha| \leq |\zeta\text{curl}\alpha| + |\zeta\text{div}\alpha| + |\gamma|$ . It remains to control  $\gamma$ . Also define

$$Q^{jk} = \delta^{jk} - N^jN^k, \quad (\text{B.1.2})$$

the projection onto tangential vector-fields. Hence

$$\begin{aligned} |\gamma|^2 &= \delta^{ij}\delta^{kl}\gamma_{ik}\gamma_{jl} \\ &= (Q^{ij} + N^iN^j)(Q^{kl} + N^kN^l)\gamma_{ik}\gamma_{jl} \\ &= Q^{ij}Q^{kl}\gamma_{ik}\gamma_{jl} + Q^{ij}N^kN^l\gamma_{ik}\gamma_{jl} + N^iN^jQ^{kl}\gamma_{ik}\gamma_{jl} \\ &+ N^iN^jN^kN^l\gamma_{ik}\gamma_{jl}. \end{aligned}$$



Since  $\gamma$  is symmetric,  $N^i N^j Q^{kl} \gamma_{ik} \gamma_{jl} = Q^{ij} N^k N^l \gamma_{ik} \gamma_{jl}$ . Also,

$$N^i N^j N^k N^l \gamma_{ik} \gamma_{jl} = [N^i N^k \gamma_{ik}]^2 = [\delta^{ik} \gamma_{ik} - Q^{ik} \gamma_{ik}]^2 = [Q^{ik} \gamma_{ik}]^2 \leq Q^{ij} Q^{kl} \gamma_{ik} \gamma_{jl}, \quad (\text{B.1.3})$$

since for a symmetric matrix  $M$  we have  $[\text{Tr}(M)]^2 \leq c \text{Tr}(M^2)$ . From (B.1.3),

$$\begin{aligned} Q^{ij} Q^{kl} \gamma_{ik} \gamma_{jl} + Q^{ij} N^k N^l \gamma_{ik} \gamma_{jl} &+ N^i N^j Q^{kl} \gamma_{ik} \gamma_{jl} + N^i N^j N^k N^l \gamma_{ik} \gamma_{jl} \\ &\leq Q^{ij} Q^{kl} \gamma_{ik} \gamma_{jl} + 2Q^{ij} N^k N^l \gamma_{ik} \gamma_{jl} + cQ^{ij} Q^{kl} \gamma_{ik} \gamma_{jl} \\ &\leq 2cQ^{ij} (Q^{kl} + N^k N^l) \gamma_{ik} \gamma_{jl} \\ &= 2cQ^{ij} \delta^{kl} \gamma_{ik} \gamma_{jl}. \end{aligned}$$

Using the fact that  $\gamma = \zeta \text{def } \alpha - \zeta \beta$  we have

$$Q^{ij} \delta^{kl} \gamma_{ik} \gamma_{jl} = Q^{ij} \delta^{kl} (\zeta \text{def } \alpha)_{ik} (\zeta \text{def } \alpha)_{jl} + Q^{ij} \delta^{kl} (\zeta \text{def } \alpha)_{ik} \zeta \beta_{jl} + Q^{ij} \delta^{kl} \zeta \beta_{ik} (\zeta \text{def } \alpha)_{jl} \quad (\text{B.1.4})$$

$$+ Q^{ij} \delta^{kl} \zeta \beta_{ik} \zeta \beta_{jl} \quad (\text{B.1.5})$$

where the second and third term can be controlled by  $\varepsilon |\zeta \nabla \alpha|^2 + \frac{1}{\varepsilon} |\zeta \text{div } \alpha|^2$  and the fourth term can be controlled by  $|\zeta \text{div } \alpha|^2$ . The first term in (B.1.4) can be controlled as follows:

$$Q^{ij} \delta^{kl} (\zeta \text{def } \alpha)_{ik} (\zeta \text{def } \alpha)_{jl} = Q^{ij} \delta^{kl} (\zeta \partial_i \alpha_k + \zeta \partial_k \alpha_i) (\zeta \partial_j \alpha_l + \zeta \partial_l \alpha_j) \quad (\text{B.1.6})$$

$$= Q^{ij} \delta^{kl} (\zeta \partial_i \alpha_k) (\zeta \partial_j \alpha_l) + Q^{ij} \delta^{kl} (\zeta \partial_i \alpha_k) (\zeta \partial_l \alpha_j) \quad (\text{B.1.7})$$

$$+ Q^{ij} \delta^{kl} (\zeta \partial_k \alpha_i) (\zeta \partial_j \alpha_l) + Q^{ij} \delta^{kl} (\zeta \partial_k \alpha_i) (\zeta \partial_l \alpha_j). \quad (\text{B.1.8})$$

Let  $\nabla_Q^i = Q^{ij} \partial_j$ . Since  $Q^{ij} = \delta_{mn} Q^{im} Q^{jn}$ , the first term in (B.1.7) can be bounded by  $|\zeta \nabla_Q [\alpha]|^2$ . The second and third term in (B.1.7) can be bounded by  $\varepsilon |\zeta \nabla \alpha|^2 + \frac{1}{\varepsilon} |\zeta \nabla_Q [\alpha]|^2$ . The fourth term we manipulate as follows:  $Q^{ij} \delta^{kl} (\zeta \partial_k \alpha_i) (\zeta \partial_l \alpha_j) = \delta_{mn} Q^{mi} (\zeta \partial_k \alpha_i) Q^{nj} (\zeta \partial^k \alpha_j)$  and

$$Q^{mi} (\zeta \partial_k \alpha_i) = Q^{mi} (\zeta \partial_i \alpha_k) + Q^{mi} [\zeta \partial_k \alpha_i - \zeta \partial_i \alpha_k] \quad (\text{B.1.9})$$

$$= \zeta \nabla_Q^m [\alpha_k] + Q^{mi} (\zeta \text{curl } \alpha)_{ki}. \quad (\text{B.1.10})$$

Thus the fourth term in (B.1.7) can be controlled by  $(1 + \frac{1}{\varepsilon}) |\zeta \nabla_Q [\alpha]|^2 + |\zeta \text{curl } \alpha|^2 + \varepsilon |\zeta \nabla \alpha|^2$ . This concludes the proof.  $\blacksquare$

## B.2 The second one.

From the lemma B.1.1 we prove the following corollary:

**LEMMA B.2.1** For  $1 \leq s \leq 8$ ,

$$\|\zeta \alpha\|_{H^s(\Omega_t)} \leq P[\|x\|_8] [\|\zeta \alpha\|_{L^2(\Omega_t)} + \|\eta \alpha\|_{H^{s-1}(\Omega_t)} + \|\zeta \text{curl } \alpha\|_{H^{s-1}(\Omega_t)} + \|\zeta \text{div } \alpha\|_{H^{s-1}(\Omega_t)}] \quad (\text{B.2.1})$$

$$+ P[\|x\|_8] \|\zeta \langle \partial_\theta \rangle^s \alpha\|_{L^2(\Omega_t)}. \quad (\text{B.2.2})$$

**PROOF:** The base case is when  $s = 1$  we have on  $\Omega_t$ , according to lemma 2.5.1,  $\|\zeta \nabla \alpha\| \leq \|\zeta \operatorname{curl} \alpha\|_{L^2(\Omega_t)} + \|\zeta \operatorname{div} \alpha\|_{L^2(\Omega_t)} + \|\zeta \partial_\theta \alpha\|_{L^2(\Omega_t)}$ , which means that (B.2.1) holds. Now suppose that  $2 \leq s \leq 5$  and that we have the result for smaller  $s$ . Then, by lemma 2.5.1 we see that

$$\|\zeta \nabla^s \alpha\|_{L^2(\Omega_t)} \leq \|\zeta \operatorname{curl} \alpha\|_{H^{s-1}(\Omega_t)} + \|\zeta \operatorname{div} \alpha\|_{H^{s-1}(\Omega_t)} + \|\zeta \partial_\theta \nabla^{s-1} \alpha\|_{L^2(\Omega_t)} + \|\eta \alpha\|_{H^{s-1}(\Omega_t)}. \quad (\text{B.2.3})$$

To manipulate the second to last term in (B.2.3) we write

$$\nabla^{s-1} \partial_\theta \alpha - \partial_\theta \nabla^{s-1} \alpha = \sum_{j+k=s-1, k \leq s-2} (\nabla^j \partial_\theta x)(\nabla^{k+1} \alpha). \quad (\text{B.2.4})$$

We have  $\|\nabla^j \partial_\theta x\|_{L^\infty(\Omega_t)} \leq \|x\|_8$  and we control  $\|\nabla^{k+1} \alpha\|_{L^2(\Omega_t)}$  by induction. Now

$$\|\zeta \nabla^{s-1} \partial_\theta \alpha\|_{L^2(\Omega_t)} \leq \|\zeta \operatorname{curl} \nabla^{s-2} \partial_\theta \alpha\|_{L^2(\Omega_t)} + \|\zeta \operatorname{div} \nabla^{s-2} \partial_\theta \alpha\|_{L^2(\Omega_t)} + \|\zeta \partial_\theta \nabla^{s-2} \partial_\theta \alpha\|_{L^2(\Omega_t)} \quad (\text{B.2.5})$$

$$\leq \|\zeta \nabla^{s-2} [(\nabla \partial_\theta x)(\nabla \alpha)]\|_{L^2(\Omega_t)} + \|\zeta \nabla^{s-2} \partial_\theta \operatorname{curl} \alpha\|_{L^2(\Omega_t)} \quad (\text{B.2.6})$$

$$+ \|\zeta \nabla^{s-2} \partial_\theta \operatorname{div} \alpha\|_{L^2(\Omega_t)} + \|\zeta \partial_\theta \nabla^{s-2} \partial_\theta \alpha\|_{L^2(\Omega_t)}. \quad (\text{B.2.7})$$

The first term in (B.2.6) is controlled by

$$\sum \|(\nabla^{j+1} \partial_\theta x)(\zeta \nabla^{k+1} \alpha)\|_{L^2(\Omega_t)} \quad (\text{B.2.8})$$

where the sums is over  $j + k = s - 2$ . This term can be controlled by  $\|x\|_8 \|\zeta \nabla \alpha\|_{H^{s-1}(\Omega_t)}$ . We control the second term in (B.2.6) by

$$\sum \|(\nabla^j \partial_\theta x)(\zeta \nabla^{k+1} \operatorname{curl} \alpha)\|_{L^2(\Omega_t)} \quad (\text{B.2.9})$$

where the sum is over  $j + k = s - 2$ . We control this term by  $\|x\|_8 \|\zeta \operatorname{curl} \alpha\|_{H^{s-1}(\Omega_t)}$ . Similarly for the third term in (B.2.6).

Now suppose that  $s = 8$  and that we have the result for smaller  $s$ . For  $0 \leq j \leq 4$  we control the commutator term in (B.2.4) as above. For  $4 \leq j \leq 7$  we have  $\|\nabla^j \partial_\theta x\|_{L^2(\Omega_t)} \leq \|x\|_8$  and  $\|\zeta \nabla^{k+1} \alpha\|_{L^\infty(\Omega_t)} \leq \|\zeta \alpha\|_{H^7(\Omega_t)}$  which we control by induction. We control the term in (B.2.8) as follows: For  $0 \leq j \leq 3$  we have  $\|\nabla^{j+1} \partial_\theta x\|_{L^\infty(\Omega_t)} \|\zeta \nabla^{k+1} \alpha\|_{L^2(\Omega_t)} \leq \|x\|_8 \|\zeta \alpha\|_{H^7(\Omega_t)}$ . For  $4 \leq j \leq 6$  we have  $\|\nabla^{j+1} \partial_\theta x\|_{L^2(\Omega_t)} \|\zeta \nabla^{k+1} \alpha\|_{L^\infty(\Omega_t)} \leq \|x\|_8 \|\zeta \alpha\|_{H^6(\Omega_t)}$ . The term in (B.2.9) we control by  $\|x\|_8 \|\zeta \operatorname{curl} \alpha\|_{H^7(\Omega_t)}$  and similarly for the third term in (B.2.6). ■

### B.3 The third one.

The third Hodge-decomposition inequality controls all derivatives in terms of the curl, the divergence and boundary derivatives which are tangential to the boundary.

**PROPOSITION B.3.1** *Let  $\alpha$  be a 1-form on  $\Omega_t$ . Let  $\operatorname{div} \alpha$  and  $\operatorname{curl} \alpha$  be defined as in lemma 2.5.1. Then*

$$\|\alpha\|_{H^s(\Omega_t)} \leq P[\|x\|_8] [\|\alpha\|_{L^2(\Omega_t)} + \|\operatorname{div} \alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{curl} \alpha\|_{H^{s-1}(\Omega_t)}] \quad (\text{B.3.1})$$

$$+ P[\|x\|_8] \|(\langle \partial_\theta \rangle^{s-\frac{1}{2}} \alpha) \cdot N\|_{L^2(\partial\Omega_t)}, \quad (\text{B.3.2})$$

where  $N$  is the outward unit normal to  $\partial\Omega_t$  and where  $p(s)$  is a polynomial which depends on  $s$ . Also,

$$\|\alpha\|_{H^s(\Omega_t)} \leq P[\|x\|_8] [\|\alpha\|_{L^2(\Omega_t)} + \|\operatorname{div} \alpha\|_{H^{s-1}(\Omega_t)} + \|\operatorname{curl} \alpha\|_{H^{s-1}(\Omega_t)}] \quad (\text{B.3.3})$$

$$+ P[\|x\|_8] \sum_{k=1}^{\mu} \sum_{l=1}^{n-1} \|(\langle \partial_\theta \rangle^{s-\frac{1}{2}} \alpha) \cdot T_{k,l}\|_{L^2(U_k \cap \partial\Omega_t)}. \quad (\text{B.3.4})$$

**PROOF:** First we prove (B.3.1) and (B.3.3) for  $s = 1$ , then we will use lemma B.2.1 to obtain the higher order results. Finally, we will use interpolation to obtain the result for real  $s$ . Now

$$\|\nabla \alpha\|_{L^2(\Omega_t)} = \int_{\Omega_t} \partial_j \alpha_i \partial^j \alpha^i dx = \int_{\partial\Omega_t} \alpha_i N_j \partial^j \alpha^i dS(x) - (\alpha, \Delta \alpha)_{\Omega_t}$$

where we define  $(\alpha, \Delta \alpha)_{\Omega_t} = \int_{\Omega_t} \alpha_i \partial_j \partial^j \alpha^i dx$ . And

$$\begin{aligned} -(\alpha, \Delta \alpha)_{\Omega_t} &= - \int_{\Omega_t} \alpha_i [\partial^i \partial^j \alpha_j + \partial_j \partial^j \alpha^i - \partial^i \partial^j \alpha_j] dx \\ &= \int_{\Omega_t} \alpha_i [-\partial^i \operatorname{div} \alpha + \partial^j (\operatorname{curl} \alpha)_j^i] dx \\ &= - \int_{\partial\Omega_t} N^i \alpha_i \operatorname{div} \alpha dS(x) + \int_{\Omega_t} [\operatorname{div} \alpha]^2 dx \\ &+ \int_{\partial\Omega_t} \alpha_i N^j (\operatorname{curl} \alpha)_j^i dS(x) - \int_{\Omega_t} \partial^j \alpha_i (\operatorname{curl} \alpha)_j^i dx. \end{aligned}$$

Also,

$$- \int_{\Omega_t} \partial^j \alpha_i (\operatorname{curl} \alpha)_j^i dx = - \int_{\Omega_t} (\operatorname{curl} \alpha)_i^j (\operatorname{curl} \alpha)_j^i dx - \int_{\Omega_t} \partial_i \alpha^j (\operatorname{curl} \alpha)_j^i dx$$

and

$$- \int_{\Omega_t} \partial_i \alpha^j (\operatorname{curl} \alpha)_j^i dx = - \int_{\partial\Omega_t} N_i \alpha^j (\operatorname{curl} \alpha)_j^i dS(x) + \int_{\Omega_t} \alpha^j \partial_i (\operatorname{curl} \alpha)_j^i dx.$$

Moreover,

$$\begin{aligned} \int_{\Omega_t} \alpha^j \partial_i (\operatorname{curl} \alpha)_j^i dx &= \int_{\Omega_t} \alpha^j \partial_i [\partial^i \alpha_j - \partial_j \alpha^i] dx \\ &= (\alpha, \Delta \alpha)_{\Omega_t} - \int_{\Omega_t} \alpha^j \partial_i \partial_j \alpha^i dx \\ &= (\alpha, \Delta \alpha)_{\Omega_t} - \int_{\partial\Omega_t} \alpha^j N_j \partial_i \alpha^i dS(x) + \int_{\Omega_t} [\operatorname{div} \alpha]^2 dx. \end{aligned}$$

From the above we see that

$$\begin{aligned} -2(\alpha, \Delta \alpha)_{\Omega_t} &= 2 \int_{\Omega_t} [\operatorname{div} \alpha]^2 dx - \int_{\Omega_t} (\operatorname{curl} \alpha)_i^j (\operatorname{curl} \alpha)_j^i dx \\ &- 2 \int_{\partial\Omega_t} \alpha \cdot N \operatorname{div} \alpha dS(x) + \int_{\partial\Omega_t} (\alpha_i N^j - N_i \alpha^j) (\operatorname{curl} \alpha)_j^i dS(x). \end{aligned}$$

The boundary terms are

$$\int_{\partial\Omega_t} \alpha_i N_j \partial^j \alpha^i dS(x) - \int_{\partial\Omega_t} \alpha \cdot N \operatorname{div} \alpha dS(x) + \frac{1}{2} \int_{\partial\Omega_t} (\alpha_i N^j - N_i \alpha^j) (\operatorname{curl} \alpha)_j^i dS(x). \quad (\text{B.3.5})$$

The second term in (B.3.5) can be manipulated using  $Q$ : On  $\partial\Omega_t$ ,  $\alpha = \alpha \cdot NN + Q\alpha$  and therefore

$$-\alpha \cdot N \operatorname{div} \alpha = -(\partial_i N^i) [\alpha \cdot N]^2 - \alpha \cdot N \nabla_N [\alpha \cdot N] - \alpha \cdot N N_i \nabla_N [Q^i \alpha] - \alpha \cdot N \nabla_{Q_i} [Q^i \alpha] \quad (\text{B.3.6})$$

where  $\nabla_N = N^i \partial_i$ . In the above,

$$-\alpha \cdot N N_i \nabla_N [Q^i \alpha] = -[\alpha \cdot N]^2 N_i \nabla_N [Q^{ij}] N_j - \alpha \cdot N N_i \nabla_N [Q^{ij}] Q_j \alpha. \quad (\text{B.3.7})$$

And the third term from (B.3.5) we manipulate as follows:

$$\frac{1}{2} (\alpha_i N^j - N_i \alpha^j) (\operatorname{curl} \alpha)_j^i = \frac{1}{2} (\alpha_i N^j - N_i \alpha^j (\partial^i \alpha_j - \partial_j \alpha^i)) \quad (\text{B.3.8})$$

$$= \frac{1}{2} [\alpha_i N^j \partial^i \alpha_j - \alpha_i N^j \partial_j \alpha^i - N_i \alpha^j \partial^i \alpha_j + N_i \alpha^j \partial_j \alpha^i] \quad (\text{B.3.9})$$

$$= \alpha_i N^j \partial^i \alpha_j - \alpha_i N^j \partial_j \alpha^i. \quad (\text{B.3.10})$$

The second term in (B.3.10) cancels the first term in (B.3.5). The first term in (B.3.10) we deal with as follows:

$$\alpha_i N^j \partial^i \alpha_j = \alpha_i \partial^i [\alpha \cdot N] - \alpha_i \alpha_j (\partial^i N^j) = \alpha \cdot N \nabla_N [\alpha \cdot N] + Q_i \alpha \nabla_Q^i [\alpha \cdot N] - \alpha_i \alpha_j (\partial^i N^j). \quad (\text{B.3.11})$$

The first term in (B.3.11) cancels the second term in (B.3.6). The remaining terms therefore are

$$\int_{\partial\Omega_t} [ -(\partial_i N^i) [\alpha \cdot N]^2 - [\alpha \cdot N]^2 N_i \nabla_N [Q^{ij}] N_j - \alpha \cdot N N_i \nabla_N [Q^{ij}] Q_j \alpha ] dS(x) \quad (\text{B.3.12})$$

$$+ \int_{\partial\Omega_t} [ -\alpha \cdot N \nabla_{Q_i} [Q^i \alpha] + Q_i \alpha \nabla_Q^i [\alpha \cdot N] - \alpha_i \alpha_j (\partial^i N^j) ] dS(x). \quad (\text{B.3.13})$$

To get the lower order terms into the form we want we use the fact that we can trade normal and tangential boundary components: Define  $\tau_{ij} = 2\alpha_i \alpha_j - \delta_{ij} (\alpha^k) (\alpha_k)$ . Then

$$\begin{aligned} \left| \|\alpha \cdot N\|_{L^2(\partial\Omega_t)}^2 - \|Q\alpha\|_{L^2(\partial\Omega_t)}^2 \right| &= \left| \int_{\partial\Omega_t} [N^i N^j \alpha_i \alpha_j - Q^{ij} \alpha_i \alpha_j] dS(x) \right| \\ &= \left| \int_{\partial\Omega_t} [2N^i N^j - \delta^{ij}] [\alpha_i \alpha_j] dS(x) \right| \\ &\leq \sum_{k=1}^{\mu} \left| \int_{U_k \cap \partial\Omega_t} \xi_k N^i N^j \tau_{ij} dS(x) \right| \\ &\leq \sum_{k=1}^{\mu} \left| \int_{U_k \cap \Omega_t} \xi_k (\partial^i N^j) \tau_{ij} dx \right| + \sum_{k=1}^{\mu} \left| \int_{U_k \cap \Omega_t} \xi_k N^j \partial^i \tau_{ij} dx \right|. \end{aligned}$$

Now

$$\begin{aligned} \partial^i \tau_{ij} &= 2 \operatorname{div} \alpha \alpha_j + 2\alpha_i (\partial_i \alpha_j) - (\partial_j \alpha_k) (\alpha^k) - (\alpha_k) (\partial_j \alpha^k) \\ &= 2 \operatorname{div} \alpha \alpha_j + 2\alpha_i (\partial_i \alpha_j) + (-(\partial_j \alpha_k) (\alpha^k) + (\partial_k \alpha_j) (\alpha^k)) - (\partial_k \alpha_j) (\alpha^k) \\ &+ (-(\alpha^k) (\partial_j \alpha^k) + (\alpha^k) (\partial_k \alpha^j)) - (\alpha^k) (\partial_k \alpha^j). \end{aligned}$$

Thus,

$$\left| \|\alpha \cdot N\|_{L^2(\partial\Omega_t)}^2 - \|Q\alpha\|_{L^2(\partial\Omega_t)}^2 \right| \leq \|x\|_5 \left[ \|\alpha\|_{L^2(\Omega_t)}^2 + \|\operatorname{div} \alpha\|_{L^2(\Omega_t)} + \|\operatorname{curl} \alpha\|_{L^2(\Omega_t)} \right]. \quad (\text{B.3.14})$$

Hence all the lower order terms in (B.3.12) can be controlled by

$$\|\alpha \cdot N\|_{L^2(\partial\Omega_t)}^2 + \|\alpha\|_{L^2(\Omega_t)}^2 + \|\operatorname{div} \alpha\|_{L^2(\Omega_t)}^2 + \|\operatorname{curl} \alpha\|_{L^2(\Omega_t)}^2$$

or

$$\|Q\alpha\|_{L^2(\partial\Omega_t)}^2 + \|\alpha\|_{L^2(\Omega_t)}^2 + \|\operatorname{div} \alpha\|_{L^2(\Omega_t)}^2 + \|\operatorname{curl} \alpha\|_{L^2(\Omega_t)}^2. \quad (\text{B.3.15})$$

In (B.3.15) above, we have

$$\|Q\alpha\|_{L^2(\partial\Omega_t)}^2 \leq \sum_{k=1}^{\mu} \|\xi_k\|_{\infty} \sum_{j=1}^{n-1} \int_{U_k \cap \partial\Omega_t} \delta_{l_1 m_1} \frac{T_{k,j}^{l_1}}{|T_{k,j}|} \frac{T_{k,j}^{l_2}}{|T_{k,j}|} \alpha_{l_2} \frac{T_{k,j}^{m_1}}{|T_{k,j}|} \frac{T_{k,j}^{m_2}}{|T_{k,j}|} \alpha_{m_2} dS(x) \quad (\text{B.3.16})$$

$$\leq \sum_{k=1}^{\mu} \sum_{j=1}^{n-1} \|\alpha \cdot T_{k,j}\|_{L^2(U_k \cap \partial\Omega_t)}^2. \quad (\text{B.3.17})$$

Also,  $\|\alpha \cdot N\|_{L^2(\partial\Omega_t)} \leq \|(\langle \partial_\theta \rangle^{\frac{1}{2}} \alpha) \cdot N\|_{L^2(\partial\Omega_t)}$  and similarly  $\|\alpha \cdot T_{k,j}\|_{L^2(U_k \cap \partial\Omega_t)}$ . To control the fourth term in (B.3.12) we have

$$\begin{aligned} \int_{\partial\Omega_t} \alpha \cdot N \nabla_{Q_i} [Q^i \alpha] dS(x) &= \sum_{k=1}^{\mu} \int_{U_k \cap \partial\Omega_t} \xi_k \alpha \cdot N \nabla_{Q_i} [Q^i \alpha] dS(x) \\ &= \sum_{k=1}^{\mu} \sum_{j=1}^{n-1} \int_{U_k \cap \partial\Omega_t} \xi_k \alpha \cdot N \frac{T_{k,j}^i}{|T_{k,j}|^2} S_j [Q_i \alpha] dS(x) \\ &\leq \left\| \xi_k \alpha \cdot N \frac{T_{k,j}^i}{|T_{k,j}|^2} \right\|_{H^{\frac{1}{2}}(U_k \cap \partial\Omega_t)} \|Q_i \alpha\|_{H^{\frac{1}{2}}(U_k \cap \partial\Omega_t)}. \end{aligned}$$

Using proposition A.0.1 we have

$$\begin{aligned} \left\| \xi_k \alpha \cdot N \frac{T_{k,j}^i}{|T_{k,j}|^2} \right\|_{H^{\frac{1}{2}}(U_k \cap \partial\Omega_t)} &\leq \left\| (\langle \partial_\theta \rangle^{\frac{1}{2}} \alpha) \cdot N \right\|_{L^2(U_k \cap \partial\Omega_t)} \left\| \xi_k \frac{T_{k,j}^i}{|T_{k,j}|^2} \right\|_{L^2(U_k \cap \partial\Omega_t)} \\ &\quad + \|\alpha\|_{L^2(U_k \cap \partial\Omega_t)} \left\| \langle \partial_\theta \rangle^2 \left[ N \xi_k \frac{T_{k,j}^i}{|T_{k,j}|^2} \right] \right\|_{L^2(U_k \cap \partial\Omega_t)} \end{aligned}$$

and

$$\begin{aligned} \|Q_i \alpha\|_{H^{\frac{1}{2}}(U_k \cap \partial\Omega_t)} &\leq \sum_{j=1}^{n-1} \left\| \frac{T_{k,j}^i}{|T_{k,j}|^2} \alpha \cdot T_{k,j} \right\|_{H^{\frac{1}{2}}(U_k \cap \partial\Omega_t)} \\ &\leq \sum_{j=1}^{n-1} \left\| (\langle \partial_\theta \rangle^{\frac{1}{2}} \alpha) \cdot T_{k,j}^i \right\|_{L^2(U_k \cap \partial\Omega_t)} \left\| \frac{T_{k,j}^i}{|T_{k,j}|^2} \right\|_{L^2(U_k \cap \partial\Omega_t)} \\ &\quad + \sum_{j=1}^{n-1} \|\alpha\|_{L^2(U_k \cap \partial\Omega_t)} \left\| \langle \partial_\theta \rangle^2 \left[ \frac{T_{k,j}^i}{|T_{k,j}|^2} \right] \right\|_{L^2(U_k \cap \partial\Omega_t)}. \end{aligned}$$

By using the fact that  $ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$  and the trace theorem we see that the fourth term in (B.3.12) can be controlled appropriately. The fifth term in (B.3.12) can be controlled similarly. This proves (B.3.1) and (B.3.3) for  $s = 1$ . Suppose now that  $s = 8$  and that we have the result for smaller  $s$ . Using lemma B.2.1, we have

$$\|\alpha\|_{H^s(\Omega_t)}^2 = \|\alpha\|_{L^2(\Omega_t)}^2 + \|\nabla\alpha\|_{H^7(\Omega_t)}^2 \quad (\text{B.3.18})$$

$$\leq P[\|x\|_8] [\|\alpha\|_{L^2(\Omega_t)} + \|\nabla\alpha\|_{L^2(\Omega_t)} + \|\text{curl}\alpha\|_{H^7(\Omega_t)} + \|\text{div}\alpha\|_{H^7(\Omega_t)}] \quad (\text{B.3.19})$$

$$+ P[\|x\|_8] \sum_{j=1}^7 \|\langle \partial_\theta \rangle^j \nabla\alpha\|_{L^2(\Omega_t)}. \quad (\text{B.3.20})$$

For  $j = 7$  we have

$$\langle \partial_\theta \rangle^7 \nabla\alpha - \nabla \langle \partial_\theta \rangle^7 \alpha = \sum (\nabla \langle \partial_\theta \rangle^{j_1} x) \dots (\nabla \langle \partial_\theta \rangle^{j_{k-1}} x) (\nabla \langle \partial_\theta \rangle^{j_k} \alpha) \quad (\text{B.3.21})$$

where the sum is over  $j_1 + \dots + j_k \leq 7$  such that  $j_k \leq 6$ . The commutator in (B.3.21) can be controlled by  $\|x\|_8 \|\alpha\|_{H^7(\Omega_t)}$ . Using computation for the case  $s = 1$  we have

$$\|\nabla \langle \partial_\theta \rangle^7 \alpha\|_{L^2(\Omega_t)} \leq P[\|x\|_8] [\|\langle \partial_\theta \rangle^7 \alpha\|_{L^2(\Omega_t)} + \|\text{div} \langle \partial_\theta \rangle^7 \alpha\|_{L^2(\Omega_t)} + \|\text{curl} \langle \partial_\theta \rangle^7 \alpha\|_{L^2(\Omega_t)}] \quad (\text{B.3.22})$$

$$+ P[\|x\|_8] \|(\langle \partial_\theta \rangle^7 \alpha) \cdot N\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \quad (\text{B.3.23})$$

$$\leq P[\|x\|_8] [\|\alpha\|_{H^7(\Omega_t)} + \|\text{div}\alpha\|_{H^7(\Omega_t)} + \|\text{curl}\alpha\|_{H^7(\Omega_t)}] \quad (\text{B.3.24})$$

$$+ P[\|x\|_8] \|(\langle \partial_\theta \rangle^7 \alpha) \cdot N\|_{H^{\frac{1}{2}}(\partial\Omega_t)}. \quad (\text{B.3.25})$$

We have, using corollary A.0.1,

$$\|(\langle \partial_\theta \rangle^6 \alpha) \cdot N\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \leq \|(\langle \partial_\theta \rangle^{6\frac{1}{2}} \alpha) \cdot N\|_{L^2(\partial\Omega_t)} \|N\|_{L^\infty(\partial\Omega_t)} + \|\alpha\|_{H^7(\partial\Omega_t)} \| \langle \partial_\theta \rangle^2 N \|_{L^2(\partial\Omega_t)}.$$

By interpolation we now obtain the result for non-integer  $s$ . This concludes the proof.  $\blacksquare$

## B.4 The fourth one: For differences.

**LEMMA B.4.1** *Let  $x_1$  and  $x_2$  be coordinates on  $\Omega_{t,1} = u_1(t, \Omega)$  and  $\Omega_{t,2} = u_2(t, \Omega)$  respectively. Let  $\alpha_1$  and  $\alpha_2$  be defined on  $\Omega_{t,1}$  and  $\Omega_{t,2}$  respectively. Define*

$$(\text{curl}_1 \alpha_1)_{ij} = \frac{\partial \alpha_{1j}}{\partial x_1^i} - \frac{\partial \alpha_{1i}}{\partial x_1^j}; \quad (\text{def}_1 \alpha_1)_{ij} = \frac{\partial \alpha_{1j}}{\partial x_1^i} + \frac{\partial \alpha_{1i}}{\partial x_1^j}$$

and  $\text{div}_1 \alpha_1 = \frac{\partial \alpha_{1i}}{\partial x_1^i}$ , where  $\alpha_{1i}$  is the  $i$ th component of  $\alpha_1$ . Let  $\text{curl}_2 \alpha_2$ ,  $\text{def}_2 \alpha_2$  and  $\text{div}_2 \alpha_2$  be defined similarly. Let  $N_1(y) = N_1 \circ u_1(y)$  be the exterior unit normal to  $\partial\Omega_{t,1}$  and let  $N_2$  be defined similarly. Also define

$$Q_1^{jk}(y) = Q_1^{jk} \circ u_1(y) = \delta^{jk} - N_1^j(y) N_1^k(y),$$

the projection onto tangential vector-fields on  $\partial\Omega_{t,1}$ . Let  $Q_2$  be defined similarly. Then we have the following pointwise estimate on  $\Omega$ :

$$|(\zeta\nabla_1\alpha_1)\circ u_1 - (\zeta\nabla_2\alpha_2)\circ u_2| \leq |(\zeta\text{curl}_1\alpha_1)\circ u_1 - (\zeta\text{curl}_2\alpha_2)\circ u_2| \quad (\text{B.4.1})$$

$$+ |(\zeta\text{div}_1\alpha_1)\circ u_1 - (\zeta\text{div}_2\alpha_2)\circ u_2| \quad (\text{B.4.2})$$

$$+ |(\zeta\partial_\theta\alpha_1)\circ u_1 - (\zeta\partial_\theta\alpha_2)\circ u_2| \quad (\text{B.4.3})$$

$$+ |Q_1 - Q_2| |(\zeta\nabla_2\alpha_2)\circ u_2|, \quad (\text{B.4.4})$$

where  $|\cdot|$  denotes the usual Euclidean distance.

**PROOF:** Let  $\beta_1 = \text{diag}\left(\left(\frac{\partial\alpha_{11}}{\partial x_1^1}\right)\circ u_1, \dots, \left(\frac{\partial\alpha_{1n}}{\partial x_1^n}\right)\circ u_1\right)$  Define  $\beta_2$  similarly. Also, define  $\gamma_1 = (\text{def}_1\alpha_1)\circ u_1 - \beta_1$  and define  $\gamma_2$  similarly. Then

$$(\nabla_1\alpha_1)\circ u_1 - (\nabla_2\alpha_2)\circ u_2 = [(\text{curl}_1\alpha_1)\circ u_1 - (\text{curl}_2\alpha_2)\circ u_2] + [(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2] \quad (\text{B.4.5})$$

$$= [(\text{curl}_1\alpha_1)\circ u_1 - (\text{curl}_2\alpha_2)\circ u_2] + [\beta_1\circ u_1 - \beta_2\circ u_2] \quad (\text{B.4.6})$$

$$+ [\gamma_1\circ u_1 - \gamma_2\circ u_2]. \quad (\text{B.4.7})$$

Let  $f = \gamma_1\circ u_1 - \gamma_2\circ u_2$ . Then

$$\begin{aligned} |f|^2 &= \delta^{ij}\delta^{kl}f_{ik}f_{jl} \\ &= (Q_1^{ij} + N_1^iN_1^j)(Q_1^{kl} + N_1^kN_1^l)f_{ik}f_{jl} \\ &= Q_1^{ij}Q_1^{kl}f_{ik}f_{jl} + Q_1^{ij}N_1^kN_1^lf_{ik}f_{jl} + N_1^iN_1^jQ_1^{kl}f_{ik}f_{jl} \\ &\quad + N_1^iN_1^jN_1^kN_1^lf_{ik}f_{jl}. \end{aligned}$$

Since  $f$  is symmetric,  $N_1^iN_1^jQ_1^{kl}f_{ik}f_{jl} = Q_1^{ij}N_1^kN_1^lf_{ik}f_{jl}$ . Also,

$$N_1^iN_1^jN_1^kN_1^lf_{ik}f_{jl} = [N_1^iN_1^k f_{ik}]^2 = [\delta^{ik}f_{ik} - Q_1^{ik}f_{ik}]^2 = [Q_1^{ik}f_{ik}]^2 \leq Q_1^{ij}Q_1^{kl}f_{ik}f_{jl}, \quad (\text{B.4.8})$$

since for a symmetric matrix  $M$  we have  $[\text{Tr}(M)]^2 \leq c\text{Tr}(M^2)$ . From (B.4.8),

$$\begin{aligned} Q_1^{ij}Q_1^{kl}f_{ik}f_{jl} + Q_1^{ij}N_1^kN_1^lf_{ik}f_{jl} &+ N_1^iN_1^jQ_1^{kl}f_{ik}f_{jl} + N_1^iN_1^jN_1^kN_1^lf_{ik}f_{jl} \\ &\leq Q_1^{ij}Q_1^{kl}f_{ik}f_{jl} + 2Q_1^{ij}N_1^kN_1^lf_{ik}f_{jl} + cQ_1^{ij}Q_1^{kl}f_{ik}f_{jl} \\ &\leq 2cQ_1^{ij}(Q_1^{kl} + N_1^kN_1^l)f_{ik}f_{jl} \\ &= 2cQ_1^{ij}\delta^{kl}f_{ik}f_{jl}. \end{aligned}$$

Using the fact that  $f = \gamma_1\circ u_1 - \gamma_2\circ u_2 = [(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2] + [\beta_2\circ u_2 - \beta_1\circ u_1]$  we have

$$\begin{aligned} Q_1^{ij}\delta^{kl}f_{ik}f_{jl} &= Q_1^{ij}\delta^{kl}[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{ik}[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{jl} \\ &+ Q_1^{ij}\delta^{kl}[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{ik}[\beta_2\circ u_2 - \beta_1\circ u_1]_{jl} \\ &+ Q_1^{ij}\delta^{kl}[\beta_2\circ u_2 - \beta_1\circ u_1]_{ik}[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{jl} \\ &+ Q_1^{ij}\delta^{kl}[\beta_2\circ u_2 - \beta_1\circ u_1]_{ik}[\beta_2\circ u_2 - \beta_1\circ u_1]_{jl} \end{aligned}$$

where the second and third term can be controlled by  $\varepsilon|(\nabla_1\alpha_1)\circ u_1 - (\nabla_2\alpha_2)\circ u_2|^2 + \frac{1}{\varepsilon}|\beta_2\circ u_2 - \beta_1\circ u_1|^2$  and the fourth term can be controlled by  $|\beta_2\circ u_2 - \beta_1\circ u_1|^2$ . The first term in (B.4.9) can be controlled as follows: Since

$$[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{ik} = \left[ \left( \frac{\partial\alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \right] \quad (\text{B.4.9})$$

$$+ \left[ \left( \frac{\partial\alpha_{1i}}{\partial x_1^k} \right) \circ u_1 - \left( \frac{\partial\alpha_{2i}}{\partial x_2^k} \right) \circ u_2 \right] \quad (\text{B.4.10})$$

we have

$$Q_1^{ij}\delta^{kl}[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{ik}[(\text{def}_1\alpha_1)\circ u_1 - (\text{def}_2\alpha_2)\circ u_2]_{jl} \quad (\text{B.4.11})$$

$$= Q_1^{ij}\delta^{kl} \left[ \left[ \left( \frac{\partial\alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \right] + \left[ \left( \frac{\partial\alpha_{1i}}{\partial x_1^k} \right) \circ u_1 - \left( \frac{\partial\alpha_{2i}}{\partial x_2^k} \right) \circ u_2 \right] \right] \quad (\text{B.4.12})$$

$$\times \left[ \left[ \left( \frac{\partial\alpha_{1l}}{\partial x_1^j} \right) \circ u_1 - \left( \frac{\partial\alpha_{2l}}{\partial x_2^j} \right) \circ u_2 \right] + \left[ \left( \frac{\partial\alpha_{1j}}{\partial x_1^l} \right) \circ u_1 - \left( \frac{\partial\alpha_{2j}}{\partial x_2^l} \right) \circ u_2 \right] \right] \quad (\text{B.4.13})$$

$$= Q_1^{ij}\delta^{kl} \left[ \left( \frac{\partial\alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \right] \left[ \left( \frac{\partial\alpha_{1l}}{\partial x_1^j} \right) \circ u_1 - \left( \frac{\partial\alpha_{2l}}{\partial x_2^j} \right) \circ u_2 \right] \quad (\text{B.4.14})$$

$$+ Q_1^{ij}\delta^{kl} \left[ \left( \frac{\partial\alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \right] \left[ \left( \frac{\partial\alpha_{1j}}{\partial x_1^l} \right) \circ u_1 - \left( \frac{\partial\alpha_{2j}}{\partial x_2^l} \right) \circ u_2 \right] \quad (\text{B.4.15})$$

$$+ Q_1^{ij}\delta^{kl} \left[ \left( \frac{\partial\alpha_{1i}}{\partial x_1^k} \right) \circ u_1 - \left( \frac{\partial\alpha_{2i}}{\partial x_2^k} \right) \circ u_2 \right] \left[ \left( \frac{\partial\alpha_{1l}}{\partial x_1^j} \right) \circ u_1 - \left( \frac{\partial\alpha_{2l}}{\partial x_2^j} \right) \circ u_2 \right] \quad (\text{B.4.16})$$

$$+ Q_1^{ij}\delta^{kl} \left[ \left( \frac{\partial\alpha_{1i}}{\partial x_1^k} \right) \circ u_1 - \left( \frac{\partial\alpha_{2i}}{\partial x_2^k} \right) \circ u_2 \right] \left[ \left( \frac{\partial\alpha_{1j}}{\partial x_1^l} \right) \circ u_1 - \left( \frac{\partial\alpha_{2j}}{\partial x_2^l} \right) \circ u_2 \right] \quad (\text{B.4.17})$$

The first term in (B.4.14) is a product of two things of the form

$$Q_1^{im} \left[ \left( \frac{\partial\alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \right] \quad (\text{B.4.18})$$

$$= Q_1^{im} \left( \frac{\partial\alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - Q_2^{im} \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \quad (\text{B.4.19})$$

$$+ Q_2^{im} \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 - Q_1^{im} \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \quad (\text{B.4.20})$$

$$= (S\alpha_{1k}) \circ u_1 - (S\alpha_{2k}) \circ u_2 + [Q_2^{im} - Q_1^{im}] \left( \frac{\partial\alpha_{2k}}{\partial x_2^i} \right) \circ u_2. \quad (\text{B.4.21})$$



The above can be controlled by  $|(\mathcal{S}\nabla_1\alpha) \circ u_1 - (\mathcal{S}\nabla_2\alpha) \circ u_1| + |Q_2 - Q_1| |(\nabla_2\alpha_2) \circ u_2|$ . From the fourth term in (B.4.14) we obtain a product of two things of the form

$$\begin{aligned}
Q_1^{im} \left( \frac{\partial \alpha_{1i}}{\partial x_1^k} \right) \circ u_1 - Q_1^{im} \left( \frac{\partial \alpha_{2i}}{\partial x_2^k} \right) \circ u_2 &= Q_1^{im} \left( \frac{\partial \alpha_{1k}}{\partial x_1^i} \right) \circ u_1 - Q_1^{im} \left( \frac{\partial \alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \\
&+ Q_1^{im} \left( \frac{\partial \alpha_{1i}}{\partial x_1^k} \right) \circ u_1 - Q_1^{im} \left( \frac{\partial \alpha_{2i}}{\partial x_2^k} \right) \circ u_2 \\
&- Q_1^{im} \left( \frac{\partial \alpha_{1k}}{\partial x_1^i} \right) \circ u_1 + Q_1^{im} \left( \frac{\partial \alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \\
&= (S\alpha_{1k}) \circ u_1 - (S\alpha_{2k}) \circ u_2 \\
&+ [Q_2^{im} - Q_1^{im}] \left( \frac{\partial \alpha_{2k}}{\partial x_2^i} \right) \circ u_2 \\
&+ Q_1^{im} [(\text{curl}_1\alpha_1)_{ki} \circ u_1 - (\text{curl}_2\alpha_2)_{ki} \circ u_2]
\end{aligned}$$

by (B.4.21). ■

## B.5 The fifth one: For the extended domain.

**LEMMA B.5.1** *Let  $\alpha$  be a function on  $\tilde{\Omega}_t$ . Define  $(\text{curl}\alpha)_{ij} = \partial_i\alpha_j - \partial_j\alpha_i$  and  $\text{div}\alpha = \partial_i\alpha^i$ .*

*Then*

$$|\zeta\nabla\alpha| \leq P[\|x\|_8] [|\zeta\text{curl}\alpha| + |\zeta\text{div}\alpha| + |\zeta\partial_\theta\alpha|],$$

*on  $\tilde{\Omega}_t$ , where  $|\cdot|$  denotes the usual Euclidean distance.*

**PROOF:** This result follows similarly to lemma B.1.1. ■

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