WELL-POSEDNESS FOR THE LINEARIZED MOTION OF AN INCOMPRESSIBLE LIQUID WITH FREE SURFACE BOUNDARY

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1. Introduction

We consider Euler's equations describing the motion of a perfect incompressible fluid in vacuum:

$$\big(\partial_t + V^k\partial_k\big)v_j = -\partial_j p, \quad j=1,...,n \quad \text{in} \quad \mathcal{D}, \tag{1.1}$$

$$\operatorname{div} V = \partial_k V^k = 0 \quad \text{in} \quad \mathcal{D} \tag{1.2}$$

where $\partial_i=\partial/\partial x^i$ and $\mathcal{D}=\cup_{0\le t\le T}\,\{t\}\times\mathcal{D}_t$, $\mathcal{D}_t\subset\mathbb{R}^n$. Here $V^k=\delta^{ki}v_i=v_k$ and we use the summation convention over repeated upper and lower indices. The velocity vector field of the fluid is V, p is the pressure and $\mathcal{D}_t$ is the domain the fluid occupies at time t. We also require boundary conditions on the free boundary $\partial\mathcal{D}=\cup_{0\le t\le T}\,\{t\}\times\partial\mathcal{D}_t$;

$$p=0, \quad \text{on} \quad \partial\mathcal{D}, \tag{1.3}$$

$$(\partial_t + V^k\partial_k)|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \tag{1.4}$$

Condition (1.3) says that the pressure p vanishes outside the domain and condition (1.4) says that the boundary moves with the velocity V of the fluid particles at the boundary.

Given a domain $\mathcal{D}_0\subset\mathbb{R}^n$, that is homeomorphic to the unit ball, and initial data v_0, satisfying the constraint (1.2), we want to find a set $\mathcal{D}=\cup_{0\le t\le T}\,\{t\}\times\mathcal{D}_t$, $\mathcal{D}_t\subset\mathbb{R}^n$ and a vector field v solving (1.1)-(1.4) with initial conditions

$$\{x;\,(0,x)\in\mathcal{D}\}=\mathcal{D}_0, \qquad \text{and} \qquad v=v_0, \quad \text{on} \quad \{0\}\times\mathcal{D}_0 \tag{1.5}$$

Let $\mathcal{N}$ be the exterior unit normal to the free surface $\partial\mathcal{D}_t$. Christodoulou[C2] conjectured that the initial value problem (1.1)-(1.5), is well posed in Sobolev spaces if

$$\nabla_{\mathcal{N}}\, p \le -c_0 < 0, \quad \text{on} \quad \partial\mathcal{D}, \qquad \text{where} \quad \nabla_{\mathcal{N}}=\mathcal{N}^i\partial_{x^i}. \tag{1.6}$$

Condition (1.6) is a natural *physical condition* since the pressure p has to be positive in the interior of the fluid. It is essential for the well posedness in Sobolev spaces. A condition related to Rayleigh-Taylor instability in [BHL,W1] turns out to be equivalent to (1.6), see [W2]. Taking the divergence of (1.1) gives:

$$-\triangle p = (\partial_j V^k)\partial_k V^j, \qquad \text{in} \quad \mathcal{D}_t, \qquad p=0, \quad \text{on} \quad \partial\mathcal{D}_t \tag{1.7}$$

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In the irrotational case, when $\operatorname{curl} v_{ij} = \partial_i v_j - \partial_j v_i = 0$, then $\triangle p \leq 0$ so $p \geq 0$ and (1.6) holds by the strong maximum principle. Wu [W1,W2] proved well posedness locally in time, (globally in space), in Sobolev spaces in the irrotational case. Ebin [E1] showed that the equations are ill posed when (1.6) is not satisfied and the pressure is negative and Ebin [E2] announced an existence result when one adds surface tension to the boundary condition. With Christodoulou [CL] we proved *a priori* bounds in Sobolev spaces in the general case of non vanishing curl, assuming (1.6). Usually if one has *a priori* estimates, existence follows from similar estimates for some regularization or iteration scheme for the equation. However, the sharp estimates in [CL] use all the symmetries of the equations and so only hold for perturbations of the equations that preserve the symmetries. Here we show existence in Sobolev spaces for the linearized equations using a new type of estimates.

The incompressible perfect fluid is to be thought of as an idealization of a liquid. For small bodies like water drops surface tension should help holding it together and for larger denser bodies like stars its own gravity should play a role. Here we neglect the influence of such forces. Instead it is the incompressibility condition that prevents the body from expanding and it is the fact that the pressure is positive that prevents the body from breaking up in the interior. Let us also point out that, from a physical point of view one can alternatively think of the pressure as being a small positive constant on the boundary instead of vanishing. What makes this problem difficult is that the regularity of the boundary enters to highest order. Roughly speaking, the velocity tells the boundary where to move and the boundary is the zero set of the pressure that determines the acceleration.

Some existence results in Sobolev spaces are known in the irrotational case, for the closely related water wave problem which describes the motion of the surface of the ocean under the influence of earth's gravity. In that problem, the gravitational field can be considered as uniform, however this problem reduces to our problem by going to an accelerated frame. The domain $\mathcal{D}_t$ is unbounded for the water wave problem coinciding with a half-space in the case of still water. Nalimov[Na] and Yosihara[Y] proved local existence in Sobolev spaces in two space dimensions for initial conditions sufficiently close to still water. Beale, Hou and Lowengrab[BHL] have given an argument to show that problem is linearly well posed in a weak sense in Sobolev spaces, assuming a condition, which can be shown to be equivalent to (1.6). The condition (1.6) prevents the Rayleigh-Taylor instability from occurring when the water wave turns over. Finally Wu[W1,2] proved local existence in general in two and three dimensions for the water wave problem. The method of proofs in these papers uses that the velocity is irrotational and divergence free and hence harmonic to reduce the equations to equations on the boundary only.

The main result here is existence for the linearized equations in the case of non vanishing curl. The irrotational case was proved by Yosihara [Y]. The proof in [Y], see also [W1,W2], reduces the equation to the boundary and it does not generalize. Instead, we project the linearized equation onto an equation in the interior using the orthogonal projection onto divergence free vector fields in the L^2 inner product. This removes a difficult term, the differential of the linearization of the pressure, and reduces a higher order term, the linearization of the moving boundary, to a symmetric unbounded operator on divergence free vector fields. The linearized equation becomes an evolution equation in the interior for this operator, which we call the normal operator. It is basically the differential of the harmonic extension to the interior of the normal component. In the irrotational case it becomes the normal derivative which is elliptic on harmonic functions and our equation reduces to an equation on the boundary similar to those in [Y,W1,W2].

The normal operator is positive due to (1.6) and this will lead to energy bounds. However, existence of regular solutions does not follow from standard energy methods or semi-group methods since the operator is time dependent and non-elliptic in the case of non vanishing curl. Usually one gets equations and estimates for higher derivatives by commuting differential operators through the equation, but we can only use operators whose commutator with the normal operator is controlled by the normal operator. Geometric arguments lead us to use Lie derivatives with respect to divergence free vector fields tangential at the boundary. The commutators of these with the normal operator are controlled by the normal operator and they preserve the divergence free condition. The same considerations apply to time differentiation so one should use the Lie derivative with respect to the material derivative (1.4) which reduces to the time derivative of the vector field in the Lagrangian coordinates. To get estimates for all derivatives we use the fact that we have a better evolution equation for the curl and that any derivative can be controlled by tangential derivatives, the curl and the divergence.

As pointed out above, existence does not follow directly from estimates but one must have existence and uniform estimates for some regularizing sequence. We replace the normal operator by a sequence of bounded operators converging to it which are still symmetric, positive and they uniformly satisfy the same commutator estimates with the differential operators above. Due to the geometric construction of the differential operators there is a natural regularization which corresponds to replacing the boundary by an inhomogeneous term supported in a small neighborhood of it.

Existence for the linearized equations or some modification will be part of any existence proof for the nonlinear problem. The estimates here require more regularity of the solution we linearize around than we get for the linearization. However, we use the techniques presented here in a forthcoming paper [L3], to prove existence for the nonlinear problem with the Nash-Moser technique.

In order to formulate the linearized equations one has to introduce some parametrization of the boundary. Let us therefore first express Euler's equations in the Lagrangian coordinates in which the boundary becomes fixed. Given a domain $\mathcal{D}_0$ in $\mathbf{R}^n$, that is diffeomorphic to the unit ball Ω, we can by a theorem in [DM] find a diffeomorphism $f_0 : \Omega \to \mathcal{D}_0$ that up to a constant factor is volume preserving, i.e. after an additional scaling $\det(\partial f_0/\partial y)=1$. Assume that $\mathcal{D}$ and $v \in C(\mathcal{D})$ are given satisfying (1.4). The Lagrangian coordinates $x=x(t,y)=f_t(y)$ are given by solving

$$\frac{dx}{dt} = V(t, x(t,y)), \qquad x(0,y) = f_0(y), \qquad y \in \Omega. \tag{1.8}$$

Then $f_t : \Omega \to \mathcal{D}_t$ is a volume preserving diffeomorphism, since $\operatorname{div} V = 0$, and the boundary becomes fixed in the new y coordinates. Let us introduce the notation

$$D_t = \frac{\partial}{\partial t}\Big|_{x=const} + V^k \frac{\partial}{\partial x^k} = \frac{\partial}{\partial t}\Big|_{y=const} \qquad \text{and} \qquad \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}, \tag{1.9}$$

for the material derivative and partial differential operators expressed in the Lagrangian coordinates. In these coordinates Euler's equations (1.1), the incompressibility condition (1.2) and the boundary condition (1.3) become

$$D_t^2 x^i = -\partial_i\, p, \qquad \det\left(\partial x/\partial y\right) = 1, \qquad \text{in} \quad [0,T]\times\Omega \qquad \text{and} \qquad p\Big|_{\partial\Omega} = 0. \tag{1.10}$$

where $p=p(t,y)$, ∂_i now is to be thought of as the differential operator in (1.9) in y and D_t is the time derivative. We then define $V = D_t x$. Note that the second equation in (1.10) follows since $D_t \ln\big(\det\left(\partial x/\partial y\right)\big) = \operatorname{div} V = 0$. Taking the divergence of the first equation in (1.10) gives (1.7) so p is determined as functional of (x,V). The initial conditions (1.5) become

$$x\Big|_{t=0} = f_0, \qquad D_t\, x\Big|_{t=0} = V_0 \tag{1.11}$$

subject to the constraints,

$$\det\left(\partial f_0/\partial y\right) = 1, \qquad \text{and} \qquad \operatorname{div} V_0 = 0 \tag{1.12}$$

and Christodoulou's physical condition become

$$\nabla_{\mathcal{N}} p\big|_{\partial\Omega} \leq -c_0 < 0 \tag{1.13}$$

where $\mathcal{N}$ is the exterior unit normal to $\partial\mathcal{D}_t$ parametrized by $x(t,y)$.

Let us now derive the linearized equations of (1.10). We assume that $(x(t,y),p(t,y))$ is a given smooth solution of (1.10) satisfying (1.13) for $0 \leq t \leq T$. Let $\overline{x}(t,y,r)$ and $\overline{p}(t,y,r)$ be smooth functions also of a parameter r, such that $(\overline{x},\overline{p})\big|_{r=0} = (x,p)$ and set $(\delta x, \delta p) = (\partial\overline{x}/\partial r, \partial\overline{p}/\partial r)\big|_{r=0}$. Then the linearized equations is the requirement on $(\delta x,\delta p)$, that $(\overline{x},\overline{p})$ satisfies the equations (1.10) up to terms bounded by r^2 as $r \to 0$. In other words, if

$$\Phi_i(x,p) = D_t^2 x^i + \partial_i p, \quad i=1,...,n, \qquad \Phi_0(x,p) = \det(\partial x/\partial y) - 1, \qquad \Phi_{n+1}(x,p) = p\big|_{\partial\Omega}, \tag{1.14}$$

then then linearized operator is defined by

$$\Phi'(x,p)(\delta x,\delta p) = \frac{\partial \Phi(\overline{x},\overline{p})}{\partial r}\Big|_{r=0}, \qquad \text{where} \quad \overline{x} = x + r\delta x, \quad \overline{p} = p + r\delta p \tag{1.15}$$

Euler's equations (1.10) become $\Phi(x,p)=0$ and the linearized equations are

$$\Phi'(x,p)(\delta x,\delta p) = 0 \tag{1.16}$$

Applying the operator $\delta f = \partial f/\partial r\big|_{r=0}$ to (1.10), using that by (2.8) $[\delta,\partial_i] = -(\partial_i \delta x^k)\partial_k$, gives the linearized equations

$$D_t^2 \delta x^i - (\partial_k p)\partial_i \delta x^k = -\partial_i \delta p, \qquad \operatorname{div}\delta x = 0, \qquad \text{and} \qquad \delta p\Big|_{\partial\Omega} = 0. \tag{1.17}$$

where we used that $\delta \ln\big(\det(\partial x/\partial y)\big) = \operatorname{div}\delta x$, see (2.6). Here δp is determined as a functional of $(\delta x, D_t \delta x)$ since taking the divergence of (1.17) gives an elliptic equation for δp similar to (1.7). We now want to sow existence for (1.17) with initial data

$$\delta x\big|_{t=0} = \delta f_0, \qquad D_t\, \delta x\big|_{t=0} = \delta V_0, \tag{1.18}$$

satisfying the constraints

$$\operatorname{div}\delta f_0 = 0, \qquad \operatorname{div}\delta V_0 = (\partial_i \delta f_0^k)\partial_k V_0^i \tag{1.19}$$

We remark, that the difference between (1.10) and (1.17) is the term $\partial_k p\, \partial_i \delta x^k$ in (1.17). This term is higher order but because of the sign condition (1.6) it will contribute with a positive term to the energy. We also remark that the equation (1.17) also shows up in estimating energies of higher order derivatives for (1.10) in [CL]. In fact, the material derivative D_t corresponds to the variation δ given by time translation. Our main result is:

Theorem 1.1. *Let Ω be the unit ball in $\mathbf{R}^n$ and suppose that (x,p) is a smooth solution of (1.10) satisfying (1.13) for $0 \leq t \leq T$. Suppose that $(\delta f_0, \delta V_0)$ are smooth satisfying the constraints (1.19). Then the linearized equations (1.17) have a smooth solution $(\delta x, \delta p)$ for $0 \leq t \leq T$ satisfying the initial conditions (1.18). Let $\mathcal{N}$ be the exterior unit normal to $\partial\mathcal{D}_t$ parametrized by $x(t,y)$ and let $\delta x_{\mathcal{N}} = \mathcal{N}\cdot\delta x$ be the normal component. Set*

$$E_r(t) = \|D_t \delta x(t,\cdot)\|_{H^r(\Omega)} + \|\delta x(t,\cdot)\|_{H^r(\Omega)} + \|\delta x_{\mathcal{N}}(t,\cdot)\|_{H^r(\partial\Omega)} \tag{1.20}$$

where $H^r(\Omega)$ and $H^r(\partial\Omega)$ are the Sobolev spaces in Ω respectively on $\partial\Omega$. Then there are constants C_r depending only on (x,p), r and T such that

$$E_r(t) \le C_r E_r(0), \qquad \textit{for} \qquad 0 \le t \le T, \qquad r \ge 0. \tag{1.21}$$

Furthermore, let $N^r(\Omega)$ be the completion of $C^\infty(\overline{\Omega})$ divergence free vector fields in the norm $\|\delta x\|_{H^r(\Omega)} + \|\delta x_{\mathcal{N}}\|_{H^r(\partial\Omega)}$. Then if the constraints in (1.19) hold and

$$(\delta f_0, \delta V_0) \in N^r(\Omega) \times H^r(\Omega) \tag{1.22}$$

it follows that (1.17)-(1.18) has a solution

$$(\delta x, D_t \delta x) \in C([0,T], N^r(\Omega) \times H^r(\Omega)). \tag{1.23}$$

As we have argued, any smooth solution of (1.1)-(1.5) with $\mathcal{D}_0$ diffeomorphic to the unit ball can be reduced to a smooth solution of (1.10) where Ω is the unit ball. The term $\|\delta x_{\mathcal{N}}\|_{H^r(\Omega)}$ is equivalent to the variation of the second fundamental form $\theta = \overline{\partial}\mathcal{N}$ of the free boundary $\partial\mathcal{D}_t$ measured in $H^{r-2}(\Omega)$, so our energy is essentially $\|\delta\theta\|_{H^{r-2}(\partial\Omega)} + \|\delta v\|_{H^r(\Omega)}$. This is to be compared with the *a priori* bounds for the nonlinear problem in [CL] for $\|\theta\|_{H^{r-2}(\partial\Omega)} + \|v\|_{H^r(\Omega)}$. A slightly more general theorem holds, see section 2. Let us now outline the main ideas in the proof. We will rewrite the linearized equations (1.17) in a geometrically invariant way and use this to obtain energy bounds and a regularization of the equation which will give existence.

We have defined our functions and vector fields to be functions of the Lagrangian coordinates $(t,y) \in [0,T] \times \Omega$ but we can alternatively think of them as functions of the Eulerian coordinates $(t,x) \in \mathcal{D}$, and we will make this identification without explicitly saying that we compose with the inverse of the change of coordinate $y \to x(t,y)$. The time derivative has a simple expression in the Lagrangian coordinates but the space derivatives have a simpler expression in the Eulerian coordinates, see (1.9). For the most part we will think of our functions and vector fields in the Lagrangian frame but we use the inner product coming from the Eulerian frame, i.e. in the Lagrangian frame we use the pull-back metric of the Euclidean inner product:

$$X \cdot Z = \delta_{ij} X^i Z^j = g_{ab} X^a Z^b, \qquad \text{where} \quad X^a = X^i \frac{\partial y^a}{\partial x^i}, \qquad g_{ab} = \delta_{ij} \frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b}. \tag{1.24}$$

Here X^i refers to the components of the vector X in the Eulerian frame, X^a refers to the components in the Lagrangian frame, g_{ab} is the metric in the Lagrangian frame and δ_{ij} is the Euclidean metric in the Eulerian frame. The letters a,b,c,d,e,f,g will refer to indices in the Lagrangian frame whereas the indices i,j,k,l,m,n will refer to the Eulerian frame. The norms and most of the operators we consider have an invariant interpretation so it does not matter in which frame they are expressed. In the introduction we use express the vector fields in the Eulerian frame but later we express the vector fields in the Lagrangian frame. The L^2 inner product of vector fields is given by

$$\langle X, Z \rangle = \int_{\mathcal{D}_t} X \cdot Z \, dx = \int_\Omega X \cdot Z \, dy \tag{1.25}$$

where the equality follows from the incompressibility condition: $\det(\partial x/\partial y) = 1$.

We now want to derive energy bounds for the linearized equations (1.17). Let us first point out that the boundary condition $p\big|_{\partial\Omega} = 0$ implies that the energy is conserved for a solution of Euler's equations (1.10). We have

$$\frac{d}{dt}\int_{\mathcal{D}_t} |V|^2 dx = \int_{\mathcal{D}_t} D_t |V|^2 dx = -2\int_{\mathcal{D}_t} V^i \partial_i p \, dx = 2\int_{\mathcal{D}_t} \operatorname{div} V \, p \, dx - 2\int_{\partial\mathcal{D}_t} V_{\mathcal{N}} \, p \, dS = 0, \tag{1.26}$$

where $V_{\mathcal{N}} = \mathcal{N}_i V^i$ is the normal component of V. In fact, the first equality follows from the incompressibility condition after expressing the integrals as integrals over Ω as in (1.25), the second is Euler's equations (1.10), the third follows from the divergence theorem and the last is the boundary condition and the divergence free condition.

We will now use the orthogonal projection onto divergence free vector fields to rewrite the linearized equations (1.17) in an invariant way that can be used to derive energy bounds and for which there is a natural regularization. The orthogonal projection onto divergence free vector fields in the inner product (1.25) is given by

$$PX^i = X^i - \delta^{ij}\partial_j\, q, \qquad \text{where} \qquad \triangle q = \operatorname{div} X, \quad q\Big|_{\partial\mathcal{D}_t} = 0. \tag{1.27}$$

We now want to project the first equation in (1.17) onto divergence free vector fields. This removes the right hand side $\partial_i \delta p$, since we project along gradients of functions that vanish on the boundary. The second term in the first equation in (1.17) can be written as $-\partial_i\big((\partial_k p)\delta x^k\big) + (\partial_i\partial_k p)\delta x^k$, where the last part is lower order and the projection of the first part turns out to be a positive symmetric operator on divergence free vector fields. We define the normal operator A to be

$$AX^i = P\big(-\delta^{ij}\partial_j(X^k\partial_k p)\big) = -\delta^{ij}\partial_j\big(X^k\partial_k p - q\big). \tag{1.28}$$

where q is chosen so that the divergence of AX vanishes and q vanishes on the boundary. Then A is a positive symmetric operator on divergence free vector fields, if condition (1.6) holds. In fact, if X and Z are divergence free then

$$\langle X, AZ\rangle = -\int_{\mathcal{D}_t} X^i\partial_i(Z^k\partial_k p)\, dx = \int_{\partial\mathcal{D}_t} X_{\mathcal{N}} Z_{\mathcal{N}}\,(-\nabla_{\mathcal{N}} p)\, dS, \qquad X_{\mathcal{N}} = \mathcal{N}_i X^i \tag{1.29}$$

There is one more issue we have to deal with before writing up the linearized equations (1.17) in a more pleasant form. The time derivative D_t does not preserve the divergence free condition so we have to modify it so it does. The operator

$$\mathcal{L}_{D_t} X^i = D_t X^i - (\partial_k V^i) X^k = \frac{\partial x^i}{\partial y^a} D_t\Big(\frac{\partial y^a}{\partial x^k} X^k\Big) \tag{1.30}$$

preserves the divergence free condition if V is divergence free. This is because it is the space time Lie derivative with respect to the divergence free vector field $D_t = (1, V)$ restricted to the space components. Another way to look at it is that it is just the time derivative of the vector field X expressed in the Lagrangian frame. The divergence is invariant under coordinate changes and the volume form is time independent so it commutes with time differentiation in the Lagrangian coordinates.

We now project the linearized equations (1.17) and get an evolution equation on divergence free vector fields for the normal operator A:

$$\ddot{X}^i + AX^i = -2P\big((\partial_k V^i)\dot{X}^k\big) \qquad \text{where} \qquad X = \delta x, \quad \dot{X} = \mathcal{L}_{D_t}\delta x, \quad \ddot{X} = \mathcal{L}_{D_t}^2 \delta x \tag{1.31}$$

Introducing the orthogonal projection onto divergence free vector fields solved to problems. First it turned the higher order term, the second term in (1.17) into a positive symmetric operator. Secondly it got rid of the third term in (1.17) which caused considerable difficulties in [CL]. In fact, the projection of a gradient of a function that vanishes on the boundary vanishes. The right hand side of (1.31) is lower order since the projection is a bounded operator. Associated with (1.31) is the energy

$$E(t) = \langle \dot{X}, \dot{X}\rangle + \langle X, (A+I)X\rangle \tag{1.32}$$

and one can show an energy estimate $|E'(t)| \le CE(t)$ which gives an energy bound. We remark that for divergence free vector fields (1.32) is equivalent to (1.20) with $r=0$. In order to show this energy bound we must calculate the commutator of the time derivative and the normal operator, which follows from the argument below.

In order to prove the energy bound and similar energy bounds for higher derivatives one has to control the commutator of differential operators with the normal operator. This is however a delicate matter since these commutators have to be controlled by the normal operator itself and only certain geometric operators satisfy this. Let T be a divergence free vector field that is tangential at the boundary and let

$$\mathcal{L}_T X^i = T^k \partial_k X^i - X^k \partial_k T^i \tag{1.33}$$

be the Lie derivative with respect to T applied to a vector field X. Then $\mathcal{L}_T X$ is divergence free if X is divergence free. It turns out that the commutators between $\mathcal{L}_T$ and the normal operator can be controlled by the normal operator:

$$[\mathcal{L}_T, A]X^i = (\mathcal{L}_T \delta^{ij})\delta_{jk} A X^k + A_{Tp} X^i \tag{1.34}$$

where for f vanishing on the boundary we defined

$$A_f X = -P\big(\delta^{ij}\partial_j (X^k \partial_k f)\big). \tag{1.35}$$

(1.34) follows from (1.28) using that the Lie derivative commutes with exterior differentiation and that the tangential derivatives Tp and Tq also vanish on the boundary since p and q do. In view of the physical condition (1.13) it follows from (1.29) that

$$|\langle X, A_f X\rangle| \le C\langle X, AX\rangle, \qquad \text{where} \qquad C = \|\nabla_{\mathcal{N}} f / \nabla_{\mathcal{N}} p\|_{L^\infty(\partial\Omega)}. \tag{1.36}$$

Applying $\mathcal{L}_T$ to the linearized equations (1.31) therefore gives a similar equation for $\mathcal{L}_T \delta x$ for which we also get energy bounds if T is a divergence free vector field that is tangential at the boundary. The second term in the commutator (1.34) can be controlled using (1.36). In order to control the first term in the commutator one has to use that AX can be controlled in terms of $\mathcal{L}_{D_t}^2 \delta x$ through the equation (1.31). Therefore we also have to differentiate the equation with respect to time and include time derivatives up to highest order in the energies. We define energies

$$E_r^{\mathcal{T}}(t) = \sum_{|I|\le r,\, I\in\mathcal{T}} \sqrt{\langle \mathcal{L}_{D_t}\mathcal{L}_T^I X, \mathcal{L}_{D_t}\mathcal{L}_T^I X\rangle + \langle \mathcal{L}_T^I X, A\mathcal{L}_T^I X\rangle}, \qquad X = \delta x \tag{1.37}$$

where $\mathcal{T}$ is a family of divergence free vector fields that are tangential at the boundary and span the tangent space of the boundary including the time derivative D_t and $\mathcal{L}_T^I$ is any product of $r = |I|$ Lie derivatives with respect to these. Then one can prove energy estimates $E_r^{\mathcal{T}}(t) \le CE_r^{\mathcal{T}}(0)$.

The energies (1.37) only contain tangential derivatives. In order to control normal derivatives also we use:

$$|\partial Z| \le C\big(|\operatorname{div} Z| + |\operatorname{curl} \underline{Z}| + \sum_{S\in\mathcal{S}} |SZ|\big) \tag{1.38}$$

where $\mathcal{S}$ is a family of vector fields that span the tangent space of the boundary and $\operatorname{curl} \underline{Z}_{ij} = \partial_i \underline{Z}_j - \partial_j \underline{Z}_i$, where $\underline{Z}_i = \delta_{ij} Z^j$ is the one form corresponding to the vector field Z. The divergence of $\dot{X} =$

$\mathcal{L}_{D_t}\delta x$ vanishes and there is a better evolution equation for curl$\,\underline{\dot{X}}$. In fact the curl of the higher order operator A in (1.31) considered as an operator with values in the one forms vanishes since it is a gradient. For a solution of Euler's equations (1.10) the curl is preserved:

$$\mathcal{L}_{D_t}\,\mathrm{curl}\, v = 0, \tag{1.39}$$

where $\mathcal{L}_{D_t}$ is the space time Lie derivative with respect to $D_t = (1, V)$ of the two form σ:

$$\mathcal{L}_{D_t}\sigma_{ij} = D_t\,\sigma_{ij} + (\partial_i V^l)\sigma_{lj} + (\partial_j V^l)\sigma_{il} = \frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j} D_t\Big(\frac{\partial x^k}{\partial y^a}\frac{\partial x^l}{\partial y^b}\sigma_{kl}\Big), \tag{1.40}$$

restricted to the space components, i.e. it is the time derivative of the two form expressed in the Lagrangian frame. For the linearized equations we have the following identity:

$$\mathcal{L}_{D_t}\,\mathrm{curl}\,\delta z = 0, \qquad \delta z_i = \delta_{ij}\mathcal{L}_{D_t} X^j - \mathrm{curl}\, v_{ij}\, X^j, \qquad X^i = \delta x^i \tag{1.41}$$

Since the Lie derivative commutes with exterior differentiation curl$\,\mathcal{L}_T^I\delta z$ is also conserved.

The above argument gives energy bounds, assuming existence. However, existence does not follow directly from estimates. To show existence we must approximate the linearized equations with some equation for which we know there is existence and prove that we have uniform bounds for the norms as the approximation gets better so that we can construct a sequence that tends to a solution of the linearized equations. For $f > 0$ in $\mathcal{D}_t$ and $f\big|_{\partial\mathcal{D}_t} = 0$ we define the smoothed out normal operator by

$$A_f^\varepsilon X^i = P\big(- \chi_\varepsilon(d)\delta^{ij}\partial_j(f d^{-1} X^k\partial_k d)\big) = P\big(\chi_\varepsilon'(d)\delta^{ij}(\partial_j d) f d^{-1} X^k \partial_k d\big) \tag{1.42}$$

where $d = d(y) = \mathrm{dist}\,(y, \partial\Omega)$ and $\chi_\varepsilon(d) = \chi(d/\varepsilon)$. Here χ is a smooth cut off function, $\chi(s) = 1$, when $s \geq 1$, $\chi(s) = 0$, when $s \leq 0$ and $\chi'(s) \geq 0$. Then A_f^ε is a positive symmetric operator on divergence free vector fields, if condition (1.6) holds. In fact, if X and Z are divergence free then

$$\langle X, A_f^\varepsilon Z\rangle = -\int_{\mathcal{D}_t}\chi_\varepsilon(d) X^i\partial_i(f d^{-1} Z^k\partial_k d)\, dx = \int_{\mathcal{D}_t}(X^i\partial_i d)(Z^k\partial_k d)\,\chi_\varepsilon'(d) f d^{-1}\, dx. \tag{1.43}$$

It follows that A_f^ε is symmetric and positive and satisfies the same commutator properties as A_f and the curl of A_f^ε vanishes when $d \geq \varepsilon$. Furthermore A_f^ε is a bounded operator, i.e. $\|A_f^\varepsilon X\|_r \leq C_{\varepsilon r}\|X\|_r$.

We will actually first obtain energy estimates for the linearized equations with vanishing initial data and an inhomogeneous divergence free term that vanishes to any order as $t \to 0$:

$$\ddot{X}^i + AX^i + 2P\big((\partial_k V^i)\dot{X}^k\big) = \delta\Phi, \qquad \mathcal{L}_{D_t}^k X\big|_{t=0} = 0, \quad k \leq r, \qquad \mathrm{div}\,\delta\Phi = 0, \tag{1.44}$$

of the form

$$E_r^{\mathcal{T}}(t) \leq C_r\int_0^t \|\delta\Phi\|_r^{\mathcal{T}}\, d\tau, \qquad \text{where} \quad \|\delta\Phi\|_r^{\mathcal{T}} = \sum_{|I|\leq r,\, I\in\mathcal{T}}\|\mathcal{L}_T^I\delta\Phi\| \tag{1.45}$$

One can reduce to this situation by subtracting a power series solution in time to (1.31). (1.44) with A replaced by $A^\varepsilon = A_p^\varepsilon$ is just an ordinary differential equation in $H^r(\Omega)$ so existence for this equation follows. Because A^ε uniformly satisfies the same commutator estimates as A we will obtain uniform energy bounds and will be able to pass to the limit as $\varepsilon \to 0$ and obtain a solution for (1.44). The reason we have to first subtract off the initial conditions in this way is that the energy (1.37) contains time derivatives up to highest order and these would have to be obtained from the equation. The operator A^ε is smoothing but only in the tangential directions and in the normal directions it is worse than A so if we had replaced A by A^ε directly in (1.31) the higher order initial conditions would have depended on A^ε in an uncontrollable way. As described above, we will first prove the energy bounds in such a way that we can obtain the same uniform bounds for the smoothed out equation and pass to the limit as $\varepsilon \to 0$ to obtain existence. Once we have existence we can then obtain the more natural energy bounds for the initial value problem in Theorem 1.1.

2. Lagrangian coordinates, the linearized equation and statement of the theorem.

Let us introduce Lagrangian coordinates in which the boundary becomes fixed. Let Ω be a domain in $\mathbf{R}^n$ and let $f_0 : \Omega \to \mathcal{D}_0$ be a diffeomorphism that is volume preserving; $\det(\partial f_0/\partial y) = 1$. For simplicity we will assume that $\mathrm{Vol}(\mathcal{D}_0)$ is the volume of the unit ball in R^n. By a theorem of [DM] we can prescribe the volume form up to a constant for any mapping of one domain into another so we may assume that Ω is the unit ball. Assume that $v(t,x)$ and $p(t,x)$, $(t,x) \in \mathcal{D}$ are given satisfying the boundary conditions (1.3)-(1.4). The Lagrangian coordinates $x = x(t,y) = f_t(y)$ are given by solving

$$dx/dt = V(t, x(t,y)), \qquad x(0,y) = f_0(y), \quad y \in \Omega \tag{2.1}$$

Then $f_t : \Omega \to \mathcal{D}_t$ is a volume preserving diffeomorphism, since $\operatorname{div} V = 0$, and the boundary becomes fixed in the new y coordinates. Let us introduce the notation

$$D_t = \frac{\partial}{\partial t}\Big|_{y=constant} = \frac{\partial}{\partial t}\Big|_{x=constant} + V^k \frac{\partial}{\partial x^k}, \tag{2.2}$$

for the material derivative and

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}. \tag{2.3}$$

In these coordinates Euler's equation (1.1), the incompressibility condition (1.2) and the boundary condition (1.3) become

$$D_t^2 x^i = -\partial_i p, \qquad \kappa = \det(\partial x/\partial y) = 1, \qquad p\big|_{\partial\Omega} = 0 \tag{2.4}$$

where $x{=}x(t,y)$, $p{=}p(t,y)$. The initial conditions (1.5) become

$$x\,\big|_{t=0} = f_0, \qquad D_t\, x\,\big|_{t=0} = V_0. \tag{2.5}$$

In fact, recall that $D_t \det(M) = \det(M)\operatorname{tr}(M^{-1} D_t M)$, for any matrix M depending on t so

$$D_t \det(\partial x/\partial y) = \det(\partial x/\partial y)\,(\partial y^a/\partial x^i)(\partial D_t x^i/\partial y^a) = \partial_i D_t x^i = \operatorname{div} D_t x = \operatorname{div} V = 0. \tag{2.6}$$

Note that p is uniquely determined as a functional of x by (2.4)-(2.5). In fact taking the divergence of Euler's equations (2.4) using (2.6) gives $\triangle p = -(\partial_i D_t x^j)(\partial_j D_t x^i)$.

Let δ be a *variation* with respect to some parameter r, in the Lagrangian coordinates:

$$\delta = \partial/\partial r\big|_{(t,y)=const}, \tag{2.7}$$

We think of $x(t,y,r)$ and $p(t,y,r)$ as depending on r and differentiate with respect to r. Differentiating (2.3) using the formula for the derivative of the inverse of a matrix, $\delta M^{-1} = -M^{-1}(\delta M)M^{-1}$, gives

$$[\delta, \partial_i] = -(\partial_i \delta x^k)\partial_k. \tag{2.8}$$

Differentiating (2.4), using (2.8) and (2.6) with D_t replaced by δ gives the linearized equations:

$$D_t^2 \delta x^i - (\partial_k p)\partial_i \delta x^k = -\partial_i \delta p, \qquad \operatorname{div} \delta x = 0, \qquad \delta p\big|_{\partial\Omega} = 0. \tag{2.9}$$

It is however better to use the fact that v and p are solutions of Euler's equations, $D_t v_i = -\partial_i p$, to arrive at the following equation

$$D_t^2 \delta x^i - \partial_i\big((\partial_k p)\delta x^k\big) = -\partial_i \delta p + (\partial_k D_t v_i)\delta x^k, \qquad \operatorname{div} \delta x = 0, \qquad \delta p\big|_{\partial\Omega} = 0. \tag{2.10}$$

We will now transform the vector field δx to Lagrangian coordinates, because in these coordinates the time derivative preserves the divergence free condition. Let

$$W^a = \delta x^i \, \frac{\partial y^a}{\partial x^i}, \qquad \delta x^i = W^b \frac{\partial x^i}{\partial y^b}, \qquad q = \delta p. \tag{2.11}$$

The letters a, b, c, d, e, f will refer to quantities in the Lagrangian frame whereas the letters i, j, k, l, m, n will refer to ones in Eulerian frame, e.g. $\partial_a = \partial/\partial y^a$ and $\partial_i = \partial/\partial x^i$. With this convention we have

$$\partial_i = \frac{\partial y^a}{\partial x^i}\partial_a, \qquad \partial_a = \frac{\partial x^i}{\partial y^a}\partial_i. \tag{2.12}$$

Multiplying the first equation in (2.10) by $\partial x^i/\partial y^a$ and summing over i gives

$$\delta_{ij}\frac{\partial x^i}{\partial y^a} D_t^2 \delta x^j - \partial_a\big((\partial_c p)W^c\big) = \partial_a q + \frac{\partial x^i}{\partial y^a}(\partial_c D_t v_i)W^c \tag{2.13}$$

since $(\partial_k p)\delta x^k = (\partial_c p)W^c$ and $(\partial_k D_t v_i)\delta x^k = (\partial_c D_t v_i)W^c$. On the other hand

$$D_t \, \delta x^i = (D_t W^b)\frac{\partial x^i}{\partial y^b} + W^b \frac{\partial V^i}{\partial y^b}, \qquad \text{and} \tag{2.14}$$

$$D_t^2 \delta x^i = (D_t^2 W^b)\frac{\partial x^i}{\partial y^b} + 2\frac{\partial V^i}{\partial y^b} D_t W^b + W^b \frac{\partial D_t V^i}{\partial y^b} \tag{2.15}$$

Multiplying (2.15) by $\partial x^i/\partial y^a$, summing over i, and substituting into (2.13) gives

$$\delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} D_t^2 W^b - \partial_a\big((\partial_c p)W^c\big) = \partial_a q - 2\frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}(\partial_k v_i) D_t W^b, \tag{2.16}$$

where

$$g_{ab} = \delta_{ij}\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} \tag{2.17}$$

is the metric δ_{ij} expressed in the Lagrangian coordinates. Let g^{ab} be the inverse of the metric g_{ab},

$$\dot g_{ab} = D_t g_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}\big(\partial_k v_i + \partial_i v_k\big) \qquad \text{and} \qquad \omega_{ab} = \frac{\partial x^i}{\partial y^a}\frac{\partial x^k}{\partial y^b}(\partial_i v_k - \partial_k v_i) \tag{2.18}$$

be the time derivative of the metric and the vorticity in the Lagrangian coordinates. Expression (2.16) becomes

$$g_{ab} D_t^2 W^b - \partial_a\big((\partial_c p)W^c\big) = -\partial_a q - \big(\dot g_{ac} - \omega_{ac}\big) D_t W^c. \tag{2.19}$$

(2.19) can alternatively be expressed, using the inverse g^{ab} of g_{ab}, in the form

$$D_t^2 W^a - g^{ab}\partial_b\big((\partial_c p)W^c\big) = -g^{ab}\partial_b q - g^{ab}\big(\dot g_{bc} - \omega_{bc}\big)D_t W^c. \tag{2.20}$$

The divergence is invariant under coordinate changes so the second condition in (2.10) is

$$\operatorname{div} W = \kappa^{-1}\partial_a(\kappa W^a) = 0, \qquad \text{where} \qquad \kappa = \det(\partial x/\partial y) = 1 \tag{2.21}$$

Finally, the last equation in (2.10) is, since $q = \delta p$,

$$q\big|_{\partial\Omega} = 0 \tag{2.22}$$

Then linearized equations are now the requirement that (2.20), (2.21) and (2.22) hold and we want to find (W, q) satisfying these equations and the initial conditions

$$W\big|_{t=0} = W_0, \qquad \dot W\big|_{t=0} = W_1, \qquad \text{where} \qquad \operatorname{div} W_0 = \operatorname{div} W_1 = 0, \qquad \dot W = D_t W \tag{2.23}$$

We can however express (2.20)-(2.23) in as one equation as follows. First we note that $q = \delta p$ is determined as a functional of W and $D_t W$. In fact, it follows from (2.21) that $\operatorname{div} D_t^2 W = 0$ so taking the divergence of (2.20) using (2.22) gives us an elliptic equation for q:

$$\triangle q = \kappa^{-1}\partial_a\big(\kappa g^{ab}\partial_b q\big) = \kappa^{-1}\partial_a\Big(\kappa g^{ab}\partial_b\big((\partial_c p)W^c\big) - \kappa g^{ab}\big(\dot g_{bc} - \omega_{bc}\big)D_t W^c\Big), \qquad q\big|_{\partial\Omega} = 0. \tag{2.24}$$

We now write $q = q_1 + q_2 + q_3$, where $q_i\big|_{\partial\Omega} = 0$ and $\triangle q_i$ is equal to each of the three terms in the right hand side of (2.24). The equations (2.20)-(2.22) can then be written as one equation, $L_1 W = 0$ where

$$L_1 W = \ddot W + AW + \dot G\dot W - C\dot W \tag{2.25}$$

and

$$AW^a = -g^{ab}\partial_b\big((\partial_c p)W^c - q_1\big), \qquad \triangle q_1 = \triangle\big((\partial_c p)W^c\big). \qquad q_1\big|_{\partial\Omega} = 0 \tag{2.26}$$

$$\dot G\dot W^a = g^{ab}(\dot g_{bc}\dot W^c + q_2), \qquad \triangle q_2 = -\partial_a\big(g^{ab}\dot g_{bc}\dot W^c\big) \qquad q_2\big|_{\partial\Omega} = 0 \tag{2.27}$$

$$C\dot W^a = g^{ab}(\omega_{bc}\dot W^c - q_3), \qquad \triangle q_3 = \partial_a\big(g^{ab}\omega_{bc}\dot W^c\big) \qquad q_3\big|_{\partial\Omega} = 0 \tag{2.28}$$

We will prove the following theorem:

Theorem 2.1. *Suppose that $x, p \in C^\infty([0,T]\times\overline\Omega)$, $p\big|_{\partial\Omega} = 0$, $\nabla_N p\big|_{\partial\Omega} \le -c_0 < 0$ and $\operatorname{div} D_t x = 0$. Suppose that $F \in C^\infty([0,T]\times\overline\Omega)$ and $W_0, W_1 \in C^\infty(\overline\Omega)$ are all divergence free. Then*

$$L_1 W = F, \qquad W\big|_{t=0} = W_0, \qquad \dot W\big|_{t=0} = W_1, \tag{2.29}$$

where L_1 be given by (2.25)-(2.28), has a divergence free solution $W \in C^\infty([0,T]\times\overline\Omega)$.

Let $H^r(\Omega)$ be the Sobolev spaces and let $N^r(\overline\Omega)$ be the completion of $C^\infty(\overline\Omega)$ divergence free vector fields in the norm $\|W\|_{H^r(\Omega)} + \|W_N\|_{H^r(\partial\Omega)}$, where $W_N = W\cdot N$ is the normal component. Then if

$$(W_0, W_1) \in N^r(\Omega)\times H^r(\Omega), \qquad F \in L^1\big([0,T], H^r(\Omega)\big) \tag{2.30}$$

are all divergence free it follows that (2.29) have a a divergence free solution

$$(W, \dot W) \in C\big([0,T], N^r(\Omega)\times H^r(\Omega)\big). \tag{2.31}$$

Moreover, with a constant C depending only on the C^{r+2} norm of x and p and the constant c_0 we have

$$\|\dot W(t)\|_{H^r} + \|W(t)\|_{N^r} \le C\Big(\|\dot W(0)\|_{H^r} + \|W(0)\|_{N^r} + \int_0^t \|F(\tau)\|_{H^r}\, d\tau\Big). \tag{2.32}$$

Remark. The restrictions that $\operatorname{div} V = 0$ and $\operatorname{div} F = 0$ can be removed and in order to use the Nash-Moser technique one indeed needs to show that the linearized operator is invertible away from a solution and outside the divergence free class. In [L3] the techniques presented here are used to show this.

3. The projection onto divergence free vector fields and the normal operator.

Let P be the orthogonal projection onto divergence free vector fields in the inner product

$$\langle W, U\rangle = \int_\Omega g_{ab} W^a U^b \, dy, \tag{3.1}$$

Then the projection P

$$PU^a = U^a - g^{ab}\partial_b \, q, \qquad \triangle q = \partial_a\big(g^{ab}\partial_b \, q\big) = \operatorname{div} U = \partial_a(U^a), \qquad q\Big|_{\partial\Omega} = 0. \tag{3.2}$$

That this is the orthogonal projection follows since $g_{ab}g^{bc} = \delta_a^c$ and

$$\langle W, (I-P)U\rangle = -\int_\Omega g_{ab} W^a g^{bc}\partial_c q \, dy = \int_\Omega (\partial_a W^a)\, q \, dy - \int_{\partial\Omega} N_a W^a \, q \, dS = 0, \quad \text{if} \quad \partial_a W^a = 0 \tag{3.3}$$

where N_a is the exterior unit conormal and dS is the surface measure. The projection of a gradient of a function that vanishes on the boundary vanishes:

$$P\big(g^{ab}\partial_b q\big) = 0, \qquad \text{if} \qquad q\big|_{\partial\Omega} = 0. \tag{3.4}$$

The projection has norm one:

$$\|PU\| \leq \|U\|, \qquad \|(I-P)U\| \leq \|U\|, \qquad \|W\| = \langle W, W\rangle^{1/2} \tag{3.5}$$

The projection is continuous on the Sobolev spaces $H^r(\Omega)$ if the metric is sufficiently regular:

$$\|PU\|_{H^r(\Omega)} \leq C_r \|U\|_{H^r(\Omega)}, \tag{3.6}$$

since it is just a matter of solving the Dirichlet problem:

$$\|q\|_{H^{r+1}(\Omega)} \leq C_r \|U\|_{H^r(\Omega)}, \qquad r \geq 0, \qquad \text{if} \quad \triangle q = \operatorname{div} U, \quad q\big|_{\partial\Omega} = 0. \tag{3.7}$$

For $r \geq 1$ this is the standard estimate for the Dirichlet problem. For $r = 0$ this is obtained by multiplying by q, using that the right hand side is in divergence form, integrating by parts and using that $q\big|_{\partial\Omega} = 0$. Furthermore if the metric also depends smoothly on time t then

$$\sum_{j=0}^{k} \|D_t^j PU\|_{H^r(\Omega)} \leq C_{r,k} \sum_{j=0}^{k} \|D_t^j U\|_{H^r(\Omega)}. \tag{3.8}$$

This follows by induction in k from commuting through time derivatives in (3.2):

$$\triangle D_t^m q = -\sum_{j=0}^{m-1} \binom{m}{j} \partial_a\big((D_t^{m-j} g^{ab})\partial_b D_t^j q\big) + \partial_a\big(D_t^m U^a\big), \qquad D_t^m q\big|_{\partial\Omega} = 0 \tag{3.9}$$

which using (3.7) gives $\|D_t^m q\|_{H^{r+1}(\Omega)} \leq C_{r,m} \sum_{j=0}^{m-1} \|D_t^m q\|_{H^{r+1}(\Omega)} + C_{r,m} \sum_{j=0}^{m} \|D_t^j U\|_{H^r(\Omega)}$.

For functions f vanishing on the boundary we define operators on divergence free vector fields

$$A_f W^a = P\big(- g^{ab}\partial_b((\partial_c f) W^c)\big), \tag{3.10}$$

A_f is symmetric, i.e. $\langle U, A_f W\rangle = \langle A_f U, W\rangle$, since for U and W divergence free it follows from (3.3)

$$\langle U, A_f W\rangle = -\int_\Omega U^a \partial_a\big((\partial_c f) W^c\big)\,\kappa dy = \int_{\partial\Omega} (-\nabla_N f) U_N W_N\,\kappa dS, \qquad U_N = N_a U^a \tag{3.11}$$

If p is the pressure in Euler's equations then normal operator A in (2.26) is

$$A = A_p \geq 0, \qquad \text{i.e.} \qquad \langle W, AW\rangle \geq 0, \qquad \text{if} \qquad \nabla_N p\Big|_{\partial\Omega} \leq 0 \tag{3.12}$$

which is true by our assumption (1.6). It follows from Cauchy Schwartz inequality that

$$|\langle U, A_{f\,p} W\rangle| \leq \langle U, A_{|f|p} U\rangle^{1/2} \langle W, A_{|f|p} W\rangle^{1/2} \leq \|f\|_{L^\infty(\partial\Omega)} \langle U, AU\rangle^{1/2} \langle W, AW\rangle^{1/2} \tag{3.13}$$

since $\nabla_N(P) = f\nabla_N p$ on the boundary. The positivity properties (3.12) and (3.13) are of fundamental importance to us. In particular, since p vanishes on the boundary so does $\dot p = D_t p$ and therefore

$$\dot A = A_{\dot p} \qquad \text{satisfies} \qquad |\langle W, \dot A W\rangle| \leq \|\nabla_N \dot p / \nabla_N p\|_{L^\infty(\partial\Omega)} \langle W, AW\rangle \tag{3.14}$$

$\dot A$ is the time derivative of the operator A, considered as an operator with values in the one forms.

It follows from (3.10) and (3.5) that $\|A_f W\| \leq \|\partial^2 f\|_{L^\infty(\Omega)}\|W\| + \|\partial f\|_{L^\infty(\Omega)}\|\partial W\|$. However, A_f acting on divergence free vector fields by (3.11) depends only on $\nabla_N f\big|_{\partial\Omega}$, i.e. $A_{\tilde f} = A_f$ if $\nabla_N \tilde f\big|_{\partial\Omega} = \nabla_N f\big|_{\partial\Omega}$. We can therefore replace f by the Taylor expansion of order one in the distance to the boundary in polar coordinates multiplied by a smooth function that is one close to the boundary and vanishes close to the origin. It follows that

$$\|A_f W\| \leq C \sum_{S\in\mathcal{S}} \|\nabla_N S f\|_{L^\infty(\partial\Omega)} \|W\| + C\|\nabla_N f\|_{L^\infty(\partial\Omega)}(\|\partial W\| + \|W\|). \tag{3.15}$$

where $\mathcal{S}$ is a set of vector fields that span the tangent space of the boundary, see section 6.

For two forms α we define bounded projected multiplication operators given by

$$M_\alpha W^a = P\big(g^{ab}\alpha_{bc} W^c\big), \qquad \|M_\alpha W\| \leq \|\alpha\|_{L^\infty(\Omega)}\|W\|. \tag{3.16}$$

In particular the operators in (2.27) and (2.28) are bounded projected multiplication operators:

$$G = M_g, \qquad C = M_\omega, \qquad \dot G = M_{\dot g} \tag{3.17}$$

where g is the metric, ω the vorticity and $\dot g$ the time derivative of the metric.

4. The lowest order energy estimate.

Since $\det(\partial x/\partial y)=1$ it follows from introducing Lagrangian coordinates, that for a function f

$$\int_{\mathcal{D}_t} f\,dx=\int_\Omega f\,dy,\qquad \text{so}\quad \frac{d}{dt}\int_{\mathcal{D}_t} f\,dx=\int_{\mathcal{D}_t} D_t f\,dx \tag{4.1}$$

We note that if v is a solution if Euler's equations, $D_t v_i=-\partial_i p$, and p vanish on the boundary then

$$\frac{d}{dt}\int_{\mathcal{D}_t}|V|^2\,dx=2\int_{\mathcal{D}_t}V^iD_tv_i\,dx=-2\int_{\mathcal{D}_t}V^i\partial_i p\,dx=2\int_{\mathcal{D}_t}(\operatorname{div}V)p\,dx-2\int_{\partial\mathcal{D}_t}V_Np\,dS=0 \tag{4.2}$$

We now want to obtain energy estimates for the linearized equations

$$L_1W=\ddot W+AW+\dot G\dot W-C\dot W=F \tag{4.3}$$

where A, $\dot G$ and C are as in section 3 and F is divergence free. Because of the unbounded but positive and symmetric operator A there is an additional term in the energy:

$$E=E(W)=\langle \dot W,\dot W\rangle+\langle W,(A+I)W\rangle \tag{4.4}$$

where the inner product is given by (3.1).

Since $\langle \dot W,\dot W\rangle=\int_\Omega g_{ab}\dot W^a\dot W^b\,dy$ and $D_t\big(g_{ab}\dot W^a\dot W^b\big)=\dot g_{ab}\dot W^a\dot W^b+2g_{ab}\dot W^aD_t\dot W^b$, we have

$$\frac{d}{dt}\langle \dot W,\dot W\rangle=2\langle \dot W,D_t\dot W\rangle+\langle \dot W,\dot G\dot W\rangle \tag{4.5}$$

where $\dot G$ is given by (3.17). By (3.3) and (3.11) $\langle W,AW\rangle=-\int_\Omega W^a\partial_a\big((\partial_c p)W^c\big)\,dy$, and

$$D_t\big(W^a\partial_a\big((\partial_c p)W^c\big)\big)=\dot W^a\partial_a\big((\partial_c p)W^c\big)+W^a\partial_a\big((\partial_c p)\dot W^c\big)+W^a\partial_a\big((\partial_c D_t p)W^c\big). \tag{4.6}$$

Since A is symmetric we get

$$\frac{d}{dt}\langle W,AW\rangle=2\langle \dot W,AW\rangle+\langle W,\dot AW\rangle \tag{4.7}$$

where $\dot A W^i=A_{\dot p}W^i$ is given by (3.10) with $f=\dot p=D_tp$. Hence

$$\begin{aligned}\frac{d}{dt}E(W)&=2\langle \dot W,\ddot W+AW+W\rangle+\langle \dot W,\dot G\dot W\rangle+\langle W,\dot AW\rangle+\langle W,\dot GW\rangle\\&=2\langle \dot W,L_1W\rangle+2\langle \dot W,W\rangle-\langle \dot W,\dot G\dot W\rangle+\langle W,\dot AW\rangle+\langle W,\dot GW\rangle.\end{aligned} \tag{4.8}$$

where we used that $\langle \dot W,C\dot W\rangle$ vanishes since C is antisymmetric. The operator $\dot G$ is bounded by (3.16)-(3.17) and $|\langle W,\dot AW\rangle|$ is bounded by (3.14) so

$$|\dot E|\le\Big(1+\|\dot g\|_{L^\infty(\Omega)}+\|\nabla_N D_tp/\nabla_Np\|_{L^\infty(\partial\Omega)}\Big)E+2\sqrt{E}\|F\|. \tag{4.9}$$

With $n(t)=1+\|\dot g\|_{L^\infty(\Omega)}+\|\nabla_N D_tp/\nabla_Np\|_{L^\infty(\partial\Omega)}$ and $E_0=\sqrt{E}$ we hence have

$$E_0(t)\le e^{\int_0^t n\,d\tau}\Big(E_0(0)+\int_0^t\|F\|\,d\tau\Big). \tag{4.10}$$

5. Turning the initial conditions into an inhomogeneous divergence free term.

As explained in the introduction we want to reduce the initial value problem

$$L_1 W = \ddot{W} + AW + \dot{G}\dot{W} - C\dot{W} = F, \qquad W\big|_{t=0} = W_0, \qquad \dot{W}\big|_{t=0} = W_1 \tag{5.1}$$

to the case of vanishing initial conditions and an inhomogeneous term F that vanishes to any order as $t \to 0$. This is achieved by subtracting off a power series solution in t to (5.1):

$$W_{0r}^a(t, y) = \sum_{s=0}^{r+2} \frac{t^s}{s!} W_s^a(y) \tag{5.2}$$

We note that if W_s are divergence free it follows that W_{0r} is divergence free. Here W_0 and W_1 are the initial conditions, W_2 is obtained form the equation (5.1) at $t = 0$: $W_2 = F - AW_0 - \dot{G}W_1 + CW_1$. Similarly, one gets higher order terms by first differentiating the equation with respect to time. It is clear that doing so we obtain an expression $D_t^{k+2} W = M_k(W, ..., D_t^{k+1} W) + D_t^k F$ and from this we inductively define $W_{k+2} = M_k(W_0, ..., W_{k+1})\big|_{t=0} + D_t^k F\big|_{t=0}$. Here M_k is some linear operator of order at most one and that is all we need to know. However, we are going to calculate the explicit form of M_k since we will do similar calculations later on for other operators and this is a simple model case.

Now it turns out that its easier to differentiate the corresponding operator with values in one forms;

$$\underline{L}_1 W_a = g_{ab} L_1 W^b = g_{ab} \ddot{W}^b - \partial_a\big((\partial_c p) W^c\big) + \partial_a q + (\dot{g}_{ab} - \omega_{ab})\dot{W}^b = g_{ab} F^b \tag{5.3}$$

where q is chosen so the last terms are divergence free, and afterwards project the result to the divergence free vector fields. Let

$$q^s = D_t^s q \qquad p^s = D_t^s p, \qquad g_{ab}^s = D_t^s g_{ab}, \qquad \omega_{ab}^s = D_t^s \omega_{ab}, \qquad F_s = D_t^s F \tag{5.4}$$

In general it follows from applying D_t^r to (5.3), restricting to $t = 0$ gives that

$$\sum_{s=0}^{r} \tbinom{r}{s} \Big(g_{ab}^{r-s} W_{s+2}^b - \partial_a\big((\partial_c p^{r-s}) W_s^c\big)\Big) + \partial_a q^r + \sum_{s=0}^{r} \tbinom{r}{s} (g_{ab}^{r-s+1} - \omega_{ab}^{r-s}) W_{s+1}^b = \sum_{s=0}^{r} \tbinom{r}{s} g_{ab}^{r-s} F_s \tag{5.5}$$

We now want to project each term onto divergence free vector fields. Let

$$A_s W^c = P\big(-g^{ab}\partial_b((\partial_c p^s) W^c)\big), \qquad G_s W^c = P\big(g^{ca} g_{ab}^s W^b\big), \qquad C_s W^c = P\big(g^{ac}\omega_{ab}^s W^b\big) \tag{5.6}$$

We obtain

$$W_{r+2} = -\sum_{s=0}^{r-1} \tbinom{r}{s} G_{r-s} W_{s+2} - \sum_{s=0}^{r} \tbinom{r}{s} \big(G_{r-s+1} W_{s+1} - C_{r-s} W_{s+1} + A_{r-s} W_s - G_{r-s} F_s\big) \tag{5.7}$$

This inductively defines W_{r+2} from $W_0, ..., W_{r+1}$. With W_{0r} given by (5.2) we have hence achieved that

$$D_t^s\big(L_1 W_{0r} - F\big)\big|_{t=0} = 0, \qquad \text{for} \quad s \le r, \qquad W_{0r}\big|_{t=0} = W_0, \quad \dot{W}_{0r}\big|_{t=0} = W_1 \tag{5.8}$$

Replacing W by $W - W_{0r}$ and F by $F - L_1 W_{0r}$ hence reduces (5.1) to the case of vanishing initial data and an inhomogeneous term that vanishes to any order r as $t \to 0$.

We also note that if the initial data are smooth then we can construct a smooth approximate solution $\tilde{W}$ that satisfies the equation to all orders as $t \to 0$. This is obtained by multiplying the k^{th} term in (5.2) by a smooth cutoff $\chi(t/\varepsilon_k)$, to be chosen below, and summing up the infinite series. Here χ is smooth $\chi(s) = 1$ for $|s| \le 1/2$ and $\chi(s) = 0$ for $|s| \ge 1$. The sequence $\varepsilon_k > 0$ can then be chosen small enough so that the series converges in $C^m([0, T], H^m)$ for any m if we take $(\|\tilde{W}_k\|_k + 1)\varepsilon_k \le 1/2$.

6. Construction of the tangential vector fields.

Let us now construct the tangential divergence free vector fields, that are time independent expressed in the Lagrangian coordinates, i.e. that commute with D_t:

$$[D_t, T] = 0. \tag{6.1}$$

This means that in the Lagrangian coordinates they are of the form $T^a(y)\partial/\partial y^a$ and since $\det(\partial x/\partial y) = 1$ the divergence free condition is just

$$\partial_a T^a = 0. \tag{6.2}$$

Since Ω is the unit ball in $\mathbf{R}^n$ the vector fields can be explicitly given. The vector fields

$$y^a \partial/\partial y^b - y^b \partial/\partial y^a \tag{6.3}$$

corresponding to rotations, span the tangent space of the boundary and are divergence free in the interior. Furthermore they span the tangent space of the level sets of the distance function from the boundary in the Lagrangian coordinates

$$d(y) = \operatorname{dist}(y, \partial\Omega) = 1 - |y| \tag{6.4}$$

away from the origin $y \neq 0$. We will denote this set of vector fields by $\mathcal{S}_0$ We also construct a set of divergence free vector fields that span the full tangent space at distance $d(y) \geq d_0$ and that are compactly supported in the interior at a fixed distance $d_0/2$ from the boundary. The basic one is

$$h(y^3, ..., y^n)\Big(f(y^1)g'(y^2)\partial/\partial y^1 - f'(y^1)g(y^2)\partial/\partial y^2\Big), \tag{6.5}$$

which is divergence free. Furthermore we can choose f, g, h such that it is equal to $\partial/\partial y^1$ when $|y^i| \leq 1/4$, for $i = 1, ..., n$ and so that it is 0 when $|y^i| \geq 1/2$ for some i. In fact let f and g be smooth functions such that $f(s) = 1$ when $|s| \leq 1/4$ and $f(s) = 0$ when $|s| \geq 1/2$ and $g'(s) = 1$ when $|s| \leq 1/4$ and $g(s) = 0$ when $|s| \geq 1/2$. Finally let $h(y^3, ..., y^n) = f(y^3)\cdots f(y^n)$. By scaling, translation and rotation of these vector fields we can obviously construct a finite set of vector fields that span the tangent space when $d \geq d_0$ and are compactly supported in the set where $d \geq d_0/2$. We will denote this set of vector fields by $\mathcal{S}_1$. Let $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ denote the family of tangential space vector fields and let $\mathcal{T} = \mathcal{S} \cup \{D_t\}$ denote the family of space time tangential vector fields.

Let the radial vector field be

$$R = c_1 y^a \partial/\partial y^a, \qquad c_1 > 0 \tag{6.6}$$

Now, $\operatorname{div} R = n$ is not 0 but for our purposes it suffices that it is constant since what we need is that if $\operatorname{div} W = 0$ then $\operatorname{div} \mathcal{L}_R W = R \operatorname{div} W - W \operatorname{div} R = 0$, where the Lie derivative $\mathcal{L}_R$ is defined in the next section. Let $\mathcal{R} = \mathcal{S} \cup \{R\}$. Note that $\mathcal{R}$ span the full tangent space of the space everywhere. Let $\mathcal{U} = \mathcal{S} \cup \{R\} \cup \{D_t\}$ denote the family of all the vector fields construct above. Note also that the radial vector field commutes with the rotations;

$$[R, S] = 0, \qquad S \in \mathcal{S}_0 \tag{6.7}$$

Furthermore, the commutators of two vector fields in $\mathcal{S}_0$ is just $\pm$ another vector field in $\mathcal{S}_0$. Therefore, for $i = 0, 1$, let $\mathcal{R}_i = \mathcal{S}_i \cup \{R\}$, $\mathcal{T}_i = \mathcal{S}_i \cup \{D_t\}$ and $\mathcal{U}_i = \mathcal{S}_i \cup \{R\} \cup \{D_t\}$.

Let $\mathcal{U} = \{U_i\}_{i=1}^M$ be some labeling of our family of vector fields. We will also use multindices $I = (i_1, ..., i_r)$ of length $|I| = r$. so $U^I = U_{i_1} \cdots U_{i_r}$ and $\mathcal{L}_U^I = \mathcal{L}_{U_{i_1}} \cdots \mathcal{L}_{U_{i_r}}$. Sometimes we will write $\mathcal{L}_U^I$, where $U \in \mathcal{S}_0$ or $I \in \mathcal{S}_0$, meaning that $U_{i_k} \in \mathcal{S}_0$ for all of the indices in I.

Note also that the vector fields $U^a(y)\partial/\partial y^a$ expressed in the x coordinates are given by $U^i \partial/\partial x^i$ where $U^i = U^a \partial x^i/\partial y^a$. We here use the convention that indices $a, ..., f$ refers to the components in the Lagrangian frame and indices $i, ..., n$ refers to the components in the Eulerian frame.

7. Lie derivatives.

Let us now introduce the Lie derivative of the vector field W with respect to the vector field T;

$$\mathcal{L}_T W^a = TW^a - (\partial_c T^a) W^c \tag{7.1}$$

We will only deal with Lie derivatives with respect to the vector fields T constructed in the previous section. For those vector fields $\operatorname{div} T = 0$ so

$$\operatorname{div} W = 0 \qquad \Longrightarrow \qquad \operatorname{div} \mathcal{L}_T W = T \operatorname{div} W - W \operatorname{div} T = 0. \tag{7.2}$$

The Lie derivative of a one form is defined by

$$\mathcal{L}_T \alpha_a = T\alpha_a + (\partial_a T^c)\alpha_c. \tag{7.3}$$

The Lie derivatives also commute with exterior differentiation, $[\mathcal{L}_T, d] = 0$ so if q is a function,

$$\mathcal{L}_T \partial_a q = \partial_a T q. \tag{7.4}$$

The Lie derivative of a two form is given by

$$\mathcal{L}_T \beta_{ab} = T\beta_{ab} + (\partial_a T^c)\beta_{cb} + (\partial_b T^c)\beta_{ac}. \tag{7.5}$$

Furthermore if w is a one form and $\operatorname{curl} w_{ab} = dw_{ab} = \partial_a w_b - \partial_b w_a$ then since the Lie derivative commutes with exterior differentiation:

$$\mathcal{L}_T \operatorname{curl} w_{ab} = \operatorname{curl} \mathcal{L}_T w_{ab}. \tag{7.6}$$

We will also use that the Lie derivative satisfies Leibnitz rule, e.g.

$$\mathcal{L}_T(\alpha_c W^c) = (\mathcal{L}_T \alpha_c) W^c + \alpha_c \mathcal{L}_T W^c, \qquad \mathcal{L}_T(\beta_{ac} W^c) = (\mathcal{L}_T \beta_{ac}) W^c + \beta_{ac} \mathcal{L}_T W^c. \tag{7.7}$$

Furthermore, we will also treat D_t as if it were a Lie derivative and we will set

$$\mathcal{L}_{D_t} = D_t. \tag{7.8}$$

Now of course this is not a space Lie derivative but rather could be interpreted as a space time Lie derivative in the domain $[0,T] \times \Omega$. But the important thing is that it satisfies all the properties of the other Lie derivatives we are considering, such as $\operatorname{div} W = 0$ implies that $\operatorname{div} D_t W = 0$ and $D_t \operatorname{curl} w = \operatorname{curl} D_t w$, simply because it commutes with partial differentiation with respect to the y coordinates. The reason we use the notation (7.9) is that we will apply products of Lie derivatives and (7.9) and it is more efficient with the same notation. Furthermore

$$[\mathcal{L}_{D_t}, \mathcal{L}_T] = 0 \tag{7.9}$$

this is because this quantity is $\mathcal{L}_{[D_t, T]}$ and $[D_t, T] = 0$ for the vector fields we are considering, or it follows from (7.1) and that $T^a = T^a(y)$ is independent of t.

8. Commutators between Lie derivatives with respect to tangential vector fields and the normal and multiplication operators.

Note that the projection P defined in section 3 almost commutes with the Lie derivative with respect to tangential vector fields. In fact if denote the corresponding operator on one forms by $\underline{P}$

$$\underline{P}u_a = u_a - \partial_a q \tag{8.1}$$

where q is as in (3.2) and $u_a = g_{ab}U^b$, then $\mathcal{L}_T\underline{P}u_a = \mathcal{L}_T u_a - \partial_a Tq$. Since $q = 0$ on the boundary it follows that $Tq = 0$ there so the last term vanishes if we project again:

$$\underline{P}(\mathcal{L}_T\underline{P}u_a) = \underline{P}\mathcal{L}_T u_a \tag{8.2}$$

We will need to calculate commutator between Lie derivatives with respect to tangential vector fields T and the operator A_f defined in section 3. Let $\underline{A}_f$ denote the corresponding operator taking a vector field to the one form

$$\underline{A}_f W_a = g_{ab}A_f W^b = -\partial_a\big((\partial_c f)W^c - q\big), \tag{8.3}$$

Then since

$$\mathcal{L}_T\partial_a\big((\partial_c f)W^c\big) = \partial_a\big((\partial_c Tf)W^k\big) + \partial_a\big((\partial_c f)\mathcal{L}_T W^c\big) \tag{8.4}$$

it follows from (8.2)

$$\underline{P}\mathcal{L}_T\underline{A}_f W_a = \underline{A}_f\mathcal{L}_T W_a + \underline{A}_{Tf}W_a \tag{8.5}$$

Note that if $f = p$ then it follows from (3.13) that the commutator is lower order. In fact $p = 0$ on the boundary implies that $Tp = 0$ on the boundary if T is a tangential vector field. Since $\nabla_N p \neq 0$ it follows Tp/p is a continuous function that is equal to $\nabla_N Tp/\nabla_N p$ on the boundary. Hence by (3.13)

$$|\langle W, A_{Tp}W\rangle| \leq \|\nabla_N Tp/\nabla_N p\|_{L^\infty(\partial\Omega)}\langle W, AW\rangle \tag{8.6}$$

In view of (8.2) it follows that the multiplication operator M_α, defined by (3.16) in section 3, satisfies the commutator relation

$$\underline{P}\mathcal{L}_T\underline{M}_\alpha W = \underline{M}_\alpha\mathcal{L}_T W + \underline{M}_{\mathcal{L}_T\alpha}W, \qquad \text{where} \qquad \underline{M}_\alpha W_a = g_{ab}M_\alpha W^b \tag{8.7}$$

for a two form α. Let

$$G_T = M_{g^T}, \qquad g^T_{ab} = \mathcal{L}_T g_{ab}, \qquad C_T = M_{\omega^T}, \omega^T = \mathcal{L}_T\omega \tag{8.8}$$

We will also use special notation for the time derivatives of G:

$$\dot{G} = G_{D_t} = M_{\dot{g}}, \qquad \dot{g}_{ab} = D_t g_{ab} \tag{8.9}$$

and of A

$$A_T = A_{Tp}, \qquad \dot{A} = A_{D_t} = A_{D_t p} \tag{8.10}$$

In the following sections we will commute through products of vector fields $\mathcal{L}_T^I = \mathcal{L}_{T_{i_1}}\cdots\mathcal{L}_{T_{i_r}}$ where $I = (i_1, ..., i_r)$ and we will use the notation

$$A_I = A_{T^I p}, \qquad G_I = M_{g^I}, \qquad C_I = M_{\omega^I} \qquad \dot{G}_I = M_{\dot{g}^I} \tag{8.11}$$

where $g^I_{ab} = \mathcal{L}_T^I g_{ab}$, $\dot{g}^I_{ab} = \mathcal{L}_T^I D_t g_{ab}$ and $\omega^I_{ab} = \mathcal{L}_T^I\omega_{ab}$.

9. Commutators between the linearized equation and Lie derivatives with respect to tangential vector fields.

We are now ready to commute tangential vector fields through the linearized equation and in the next section get the higher order energy estimates of tangential derivatives. Let $T \in \mathcal{T}$ be a tangential vector fields and recall that $[\mathcal{L}_T, D_t] = 0$ and that if W are divergence free then so is $\mathcal{L}_T W$. Let us now apply Lie derivatives $\mathcal{L}_T^I = \mathcal{L}_{T_{i_1}} \cdots \mathcal{L}_{T_{i_r}}$, where $I = (i_1, ..., i_r)$ is a multi index, to the linearized equation (2.19) with an inhomogeneous divergence free term F vanishing to order r as $t \to 0$:

$$g_{ab}\ddot{W}^b - \partial_a\big((\partial_c p)W^c\big) = -\partial_a q - (\dot{g}_{ca} - \omega_{ca})\dot{W}^c + g_{ab}F^b, \qquad W\big|_{t=0} = \dot{W}\big|_{t=0} = 0. \tag{9.1}$$

which yields

$$\begin{aligned} c^I_{I_1 I_2}(\mathcal{L}_T^{I_1} g_{ab})\mathcal{L}_T^{I_2}\ddot{W}^b - c^I_{I_1 I_2}\partial_a\big((\partial_c T^{I_1} p)\mathcal{L}_T^{I_2} W^c\big) \\ = -\partial_a T^I q - 2c^I_{I_1 I_2}\big(\mathcal{L}_T^{I_1}(\dot{g}_{ca} - \omega_{ca})\big)\mathcal{L}_T^{I_2}\dot{W}^c + c^I_{I_1 I_2}(\mathcal{L}_T^{I_1} g_{ab})\mathcal{L}_T^{I_2} F^b \end{aligned} \tag{9.2}$$

where we sum over all $I_1 + I_2 = I$ and $c^I_{I_1 I_2} = 1$. Let us introduce some new notation

$$W_I = \mathcal{L}_T^I W, \quad F_I = \mathcal{L}_T^I F \quad g^I_{ab} = \mathcal{L}_T^I g_{ab} \quad \omega^I_{ab} = \mathcal{L}_T^I \omega_{ab}, \quad p_I = T^I p, \quad q_I = T^I q \tag{9.3}$$

and $\dot{g}^I_{ab} = D_t \mathcal{L}_T^I g_{ab}$, $\dot{W}_I = D_t W_I$ etc. With this notation (9.2) becomes

$$c^I_{I_1 I_2} g^{I_1}_{ab}\ddot{W}^b_{I_2} - c^I_{I_1 I_2}\partial_a\big((\partial_c p_{I_1})W^c_{I_2}\big) = -\partial_a q_I - c^I_{I_1 I_2}\big(\dot{g}^{I_1}_{ab} - \omega^{I_1}_{ab}\big)\dot{W}^b_{I_2} + c^I_{I_1 I_2} g^{I_1}_{ab} F^b_{I_2} \tag{9.4}$$

Let us now project each term onto divergence free vector fields and also introduce some notation for the resulting operators

$$A_I W^a = A_{T^I p} W^a, \qquad G_I W^a = P\big(g^{ac} g^I_{cb} W^b\big) \tag{9.5}$$

and

$$\dot{G}_I W^a = P\big(g^{ac}\dot{g}^I_{cb} W^b\big), \qquad C_I W^a = P(g^{ac}\omega^I_{cb} W^b) \tag{9.6}$$

From now on we set $\tilde{c}_I^{I_1 I_2} = c^I_{I_1 I_2}$ when $I_2 \neq I$ and $\tilde{c}_I^{I_1 I_2} = 0$ if $I_2 = I$. Projecting each term onto divergence free vector fields we can now write (9.4) as

$$L_1 W_I = \ddot{W}_I + AW_I + \dot{G}\dot{W}_I - C\dot{W}_I = F_I - \tilde{c}_I^{I_1 I_2}\big(A_{I_1} W_{I_2} + \dot{G}_{I_1}\dot{W}_{I_2} - C_{I_1}\dot{W}_{I_2} + G_{I_1}\ddot{W}_{I_2} + G_{I_1} F_{I_2}\big) \tag{9.7}$$

Here G_J, $\dot{G}_J$ and C_J are all bounded operators. By (3.16)-(3.17):

$$\|G_J W\| \leq \|\mathcal{L}_T^J g\|_{L^\infty(\Omega)}\|W\|, \qquad \|C_J W\| \leq \|\mathcal{L}_T^J \omega\|_{L^\infty(\Omega)}\|W\| \tag{9.8}$$

The terms $G_{I_1}\ddot{W}_{I_2}$ are easy to take care of by also including time derivatives up to highest order in our estimates since $|I_2| \leq |I| - 1$. AW_I itself will be included in the higher order energy, which is just going to be a sum of terms of the form (4.4) with W replaced by W_I for $|I| \leq r$. However, we also have to deal with $A_{I_1} W_{I_2}$ since A_{I_1} is an operator of order 1. Since $|I_2| \leq |I| - 1 \leq r - 1$ in the terms $A_{I_1} W_{I_2}$ and since the energy will give us $\dot{W}_I$ for all $|I| \leq r$ we in particular will have an estimate for $\ddot{W}_{I_2}$ which, using the equation (9.7), up to terms of lower order is $-AW_{I_2}$. Since $A_J = A_{T^J p}$ it follows from (3.13) that

$$|\langle U, A_J W\rangle| \leq \|\nabla_N T^J p/\nabla_N p\|_{L^\infty(\partial\Omega)}\langle U, AU\rangle^{1/2}\langle W, AW\rangle^{1/2}, \tag{9.9}$$

However this does not imply that the norm of A_J is bounded by the norm of A. Therefore we have to deal with these terms with A_{I_1} in an indirect way, by including them in the energy and using (9.9).

10. The *a priori* energy bounds for tangential derivatives.

To obtain estimates for higher derivatives we apply tangential vector fields to the equation and get similar equations for higher derivatives. However, there are a some commutators coming up that we have to deal with. One can be dealt with by adding a lower order term to the energy and another commutator one deals with by also considering higher time derivatives. The main point is however that commutators with the normal operator can be controlled by the normal operator through (9.9). Let $W_T = \mathcal{L}_T W$, $F_T = \mathcal{L}_T F$ and let G_T, C_T and A_T be as in (8.8) and (8.10). By (9.7)

$$L_1 W_T = F_T - A_T W - \dot{G}_T W + C_T \dot{W} + G_T \ddot{W} + G_T F \tag{10.1}$$

The terms one has to deal with are $A_T W$ and $G_T \ddot{W}$. Let $E = E(W)$, where $E(W)$ is given by (4.4),

$$E_T = E(W_T) = \langle \dot{W}_T, \dot{W}_T \rangle + \langle W_T, (A+I) W_T \rangle, \qquad \text{and} \qquad D_T = 2\langle W_T, A_T W \rangle. \tag{10.2}$$

D_T is lower order compared to E_T since by (9.9) it is bounded by a constant times $\sqrt{E_T}\sqrt{E}$ and we already have an estimate for E in (4.10). We will add D_T to the energy E_T to pick up the commutator A_T between $\mathcal{L}_T$ and A. By (4.8)

$$\begin{aligned} \dot{E}_T + \dot{D}_T = 2\langle \dot{W}_T, L_1 W_T \rangle + 2\langle \dot{W}_T, W_T \rangle - \langle \dot{W}_T, \dot{G} \dot{W}_T \rangle + \langle W_T, \dot{A} W_T \rangle + \langle W_T, \dot{G} W_T \rangle \\ + 2\langle \dot{W}_T, A_T W \rangle + 2\langle W_T, A_T \dot{W} \rangle + 2\langle W_T, \dot{A}_T W \rangle \\ = 2\langle W_T, A_T \dot{W} \rangle + 2\langle \dot{W}_T, G_T \ddot{W} \rangle + 2\langle \dot{W}_T, F + G_T F \rangle \\ + 2\langle \dot{W}_T, -\dot{G}_T W + C_T \dot{W} + W_T \rangle - \langle \dot{W}_T, \dot{G} \dot{W}_T \rangle + \langle W_T, \dot{A} W_T \rangle + \langle W_T, \dot{G} W_T \rangle + 2\langle W_T, \dot{A}_T W \rangle \end{aligned} \tag{10.3}$$

Here, the terms on the last row are bounded by E_T and E using (9.8) and (9.9). The only terms that remains to control are $2\langle \dot{W}_T, G_T \ddot{W} \rangle$ and $2\langle W_T, A_T \dot{W} \rangle$. These terms are controlled by simultaneously consider one more time derivative, i.e. if $T = D_t$, and estimate energies for these.

Let us now define higher order energies. Let

$$E_I = E(W_I) = \langle \dot{W}_I, \dot{W}_I \rangle + \langle W_I, (A+I) W_I \rangle, \qquad W_I = \mathcal{L}_T^I W. \tag{10.4}$$

With notation as in the previous section we have by (4.8) and (9.7)

$$\begin{aligned} \dot{E}_I = 2\langle \dot{W}_I, D_t \dot{W}_I + A W_I + \dot{G} \dot{W}_I - C \dot{W}_I \rangle \\ + 2\langle \dot{W}_I, W_I \rangle - \langle \dot{W}_I, \dot{G} \dot{W}_I \rangle + \langle W_I, \dot{A} W_I \rangle + \langle W_I, \dot{G} W_I \rangle \\ = -2\tilde{c}_I^{I_1 I_2} \Big(\langle \dot{W}_I, A_{I_1} W_{I_2} \rangle + \langle \dot{W}_I, \dot{G}_{I_1} W_{I_2} \rangle - \langle \dot{W}_I, C_{I_1} \dot{W}_{I_2} \rangle + \langle \dot{W}_I, G_{I_1} \ddot{W}_{I_2} \rangle + \langle \dot{W}_I, G_{I_1} F_{I_2} \rangle \Big) \\ + 2\langle \dot{W}_I, F_I \rangle + 2\langle \dot{W}_I, W_I \rangle - \langle \dot{W}_I, \dot{G} \dot{W}_I \rangle + \langle W_I, \dot{A} W_I \rangle + \langle W_I, \dot{G} W_I \rangle \end{aligned} \tag{10.5}$$

To deal with the term $\langle \dot{W}_I, A_{I_1} W_{I_2} \rangle$ we introduce

$$D_I = 2\, \tilde{c}_I^{I_1 I_2} \langle W_I, A_{I_1} W_{I_2} \rangle \tag{10.6}$$

Then

$$\dot{D}_I = 2\, \tilde{c}_I^{I_1 I_2} \Big(\langle \dot{W}_I, A_{I_1} W_{I_2} \rangle + \langle W_I, A_{I_1} \dot{W}_{I_2} \rangle + \langle W_I, \dot{A}_{I_1} W_{I_2} \rangle \Big) \tag{10.7}$$

and hence

$$
\begin{aligned}
&(10.8)\quad \dot E_I+\dot D_I=\\
&-2\,\tilde c_I^{I_1I_2}\Big(-\langle W_I,A_{I_1}\dot W_{I_2}\rangle-\langle W_I,\dot A_{I_1}W_{I_2}\rangle+\langle \dot W_I,\dot G_{I_1}\dot W_{I_2}\rangle-\langle \dot W_I,C_{I_1}\dot W_{I_2}\rangle+\langle \dot W_I,G_{I_1}\ddot W_{I_2}\rangle+\langle \dot W_I,G_{I_1}F_{I_2}\rangle\Big)\\
&\qquad\qquad +2\langle \dot W_I,F_I\rangle+2\langle \dot W_I,W_I\rangle-\langle \dot W_I,\dot G\dot W_I\rangle+\langle W_I,\dot AW_I\rangle+\langle W_I,\dot GW_I\rangle
\end{aligned}
$$

We have hence replaced the bad term by two terms that we can control by (9.9). Furthermore, we can also bound D_I itself using (9.9).

For a two form α and a function q vanishing on the boundary let

$$
\|\alpha\|_\infty=\|\,|\alpha|\,\|_{L^\infty(\Omega)},\qquad \|\partial q\|_{\infty,\,p^{-1}}=\|\nabla_N q/\nabla_N p\|_{L^\infty(\partial\Omega)}\le \|\partial q\|_\infty/c_0, \tag{10.9}
$$

and for a vector fields W let

$$
\langle W\rangle_A=\langle W,AW\rangle^{1/2},\qquad \|W\|=\langle W,W\rangle^{1/2}. \tag{10.10}
$$

With this notation it now follows from (10.8) and (9.8)-(9.9) that

$$
\begin{aligned}
(10.11)\quad \dot E_I+\dot D_I\le\; & 2\langle W_I\rangle_A\tilde c_I^{I_1I_2}\Big(\|\partial p_{I_1}\|_{\infty,\,p^{-1}}\langle \dot W_{I_2}\rangle_A+\|\partial \dot p_{I_1}\|_{\infty,\,p^{-1}}\langle W_{I_2}\rangle_A\Big)\\
&+2\|\dot W_I\|\,\tilde c_I^{I_1I_2}\Big((\|\dot g^{I_1}\|_\infty+\|\omega^{I_1}\|_\infty)\|\dot W_{I_2}\|+\|g^{I_1}\|_\infty\|\ddot W_{I_2}\|+\|g^{I_1}\|_\infty\|F_{I_2}\|\Big)\\
&+\|\dot W_I\|\big(2\|F_I\|+2\|W_I\|+\|\dot g\|_\infty\|\dot W_I\|\big)+\|W_I\|\,\|\dot g\|_\infty\|W_I\|+\langle W_I\rangle_A\|\partial\dot p\|_{\infty,\,p^{-1}}\langle W_I\rangle_A
\end{aligned}
$$

Furthermore

$$
|D_I|\le 2\langle W_I\rangle_A\,\tilde c_I^{I_1I_2}\|\partial p_{I_1}\|_{\infty,\,p^{-1}}\langle W_{I_2}\rangle_A \tag{10.12}
$$

Definition 10.1. For $\mathcal V$ any of our families of vector fields let

$$
E_s^{\mathcal V}=\sum_{|I|\le s,\,I\in\mathcal V}\sqrt{E_I},\qquad \|W\|_s^{\mathcal V}=\sum_{|I|\le s,\,I\in\mathcal V}\|\mathcal L_T^I W\| \tag{10.13}
$$

where E_I is given by (10.4). For a two form α and a function q vanishing on the boundary let

$$
\|\alpha\|_{s,\infty}^{\mathcal V}=\sum_{|J|\le s,\,J\in\mathcal V}\|\mathcal L_T^J\alpha\|_\infty,\qquad \|\partial q\|_{s,\infty,\,p^{-1}}^{\mathcal V}=\sum_{|J|\le s,\,J\in\mathcal V}\|\partial T^J q\|_{\infty,\,p^{-1}} \tag{10.14}
$$

where the norms are given by (10.9). Furthermore, let

$$
n_s^{\mathcal V}=\|\dot g\|_{s,\infty}^{\mathcal V}+\|\omega\|_{s,\infty}^{\mathcal V}+\|g\|_{s+1,\infty}^{\mathcal V}+\|\partial p\|_{s+1,\infty,p^{-1}}^{\mathcal V}+\|\partial\dot p\|_{s,\infty,p^{-1}}^{\mathcal V} \tag{10.15}
$$

If $I\in\mathcal T$ and $|I|=r$ then with the notation in Definition 10.1 we obtain from (10.11) and (10.12):

$$
|\dot E_I+\dot D_I|\le CE_r^{\mathcal T}\sum_{s=0}^{r}n_s^{\mathcal T}\big(E_{r-s}^{\mathcal T}+\|F\|_{r-s}^{\mathcal T}\big),\qquad |D_I|\le CE_r^{\mathcal T}\sum_{s=0}^{r-1}n_s^{\mathcal T}E_{r-1-s}^{\mathcal T} \tag{10.16}
$$

If we integrate the first inequality from 0 to t using that $E_I(0) = D_I(0) = 0$ and the second inequality we get with a constant depending on $\overline{n}_r = \sup_{0\leq\tau\leq T} n_s^{\mathcal{T}}(\tau)$

$$E_I \leq C E_r^{\mathcal{T}} E_{r-1}^{\mathcal{T}} + C \int_0^t E_r^{\mathcal{T}} \big(E_r^{\mathcal{T}} + \|F\|_r^{\mathcal{T}} \big)\, d\tau \tag{10.17}$$

If we sum over $|I| \leq r$ and divide by $\overline{E}_r(t) = \sup_{0\leq\tau\leq t} E_r^{\mathcal{T}}(\tau)$ we get for some other constant

$$\overline{E}_r \leq C \overline{E}_{r-1} + C \int_0^t \big(\overline{E}_r + \|F\|_r^{\mathcal{T}} \big)\, d\tau. \tag{10.18}$$

Hence with $M_r(t) = \int_0^t \overline{E}_r\, d\tau$, we get

$$\frac{dM_r}{dt} - C M_r \leq C \overline{E}_{r-1} + C \int_0^t \|F\|_r^{\mathcal{T}}\, d\tau \tag{10.19}$$

Multiplying by the integrating factor e^{-Ct} and integrating from 0 to t we see that M_r is bounded by some constant depending on $t \leq T$ times the right hand side and hence it follows that for some other constant

$$\overline{E}_r \leq C \overline{E}_{r-1} + C \int_0^t \|F\|_r^{\mathcal{T}}\, d\tau \tag{10.20}$$

Since we already proved a bound for $\overline{E}_0$ in (4.10) it inductively follows that:

Lemma 10.1. *Suppose that $x, p \in C^{r+2}([0,T] \times \Omega)$, $p\big|_{\partial\Omega} = 0$, $\nabla_N p\big|_{\partial\Omega} \leq -c_0 < 0$ and $\operatorname{div} V = 0$, where $V = D_t x$. Suppose that W is a solution of (9.1) where F is divergence free and vanishing to order r as $t \to 0$. Let $E_s^{\mathcal{T}}$ be defined by (10.14). Then there is a constant C depending only on the norm of (x,p), a lower bound for c_0 and an upper bound for T, such that if $E_s^{\mathcal{T}}(0) = 0$, for $s \leq r$, then*

$$E_r^{\mathcal{T}}(t) \leq C \int_0^t \|F\|_r^{\mathcal{T}}\, d\tau, \qquad \text{for} \quad 0 \leq t \leq T \tag{10.21}$$

11. Estimates of derivatives of a vector field in terms of the curl, the divergence and tangential derivatives.

In this section we show that derivatives of vector fields can be estimated by derivatives of the curl, the divergence and tangential derivatives. First we prove the basic estimate in the Euclidean coordinates in Lemma 11.1 below. This estimate it is not invariant and so in Lemma 11.2 we express it in terms of Lie derivatives which is invariant.

Lemma 11.1. *We have*

$$|\partial\alpha| \leq C_n\big(|\operatorname{curl}\alpha| + |\operatorname{div}\alpha| + \textstyle\sum_{S\in\mathcal{S}} |S\alpha| \big), \qquad \operatorname{curl}\alpha_{ij} = \partial_i\alpha_j - \partial_j\alpha_i \qquad \operatorname{div}\alpha = \delta^{ij}\partial_i\alpha_j \tag{11.1}$$

for a one form α_i in the Eulerian frame, where C_n only depends on the dimension n. Here the norms are the Euclidean norms, $|\partial\alpha| = \sqrt{\sum_{i,j=1}^n |\partial_i\alpha_j|^2}$.

Proof of Lemma 11.1.. Since $\mathcal{S}$ span the full tangent space in the interior when the distance to the boundary $d(y) \geq d_0$ we may assume that $d(y) < d_0$. Let $\Omega^a = \{y;\, d(y) > a\}$ and let $\mathcal{D}_t^a$ be the image of this set under mapping $y \to x(t,y)$. Let N the exterior unit normal to $\partial\mathcal{D}_t^a$. Then $q^{ij} = \delta^{ij} - N^i N^j$ is the inverse of the tangential metric. Since the tangential vector fields span the tangent space of the level sets of the distance function we have $q^{ij} a_i a_j \leq C \sum_{S\in\mathcal{S}} S^i S^j a_i a_j$, where here $S^i = S^a \partial x^i/\partial y^a$. We claim that for any two tensor β_{ij}:

$$\delta^{ij}\delta^{kl}\beta_{ki}\beta_{lj} \leq C_n\big(\delta^{ij}q^{kl}\beta_{ki}\beta_{lj} + |\hat\beta|^2 + (\operatorname{tr}\beta)^2\big) \tag{11.2}$$

where $\hat\beta_{ij} = \beta_{ij} - \beta_{ji}$ is the antisymmetric part and $\operatorname{tr}\beta = \delta^{ij}\beta_{ij}$ is the trace. To prove (11.2) we may assume that β is symmetric and traceless. Writing $\delta^{ij} = q^{ij} + N^i N^j$ we see that the estimate for such tensors follows from the estimate $N^i N^j N^k N^l \beta_{ki}\beta_{lj} = (N^i N^k \beta_{ki})^2 = (q^{ik}\beta_{ki})^2 \leq n q^{ij} q^{kl}\beta_{ki}\beta_{lj}$. (This inequality just says that $(\operatorname{tr}(Q\beta))^2 \leq n \operatorname{tr}(Q\beta Q\beta)$ which is obvious if one writes it out and use the symmetry.) $\square$

The inequality (11.1) is not invariant under changes of coordinates so we want to replace it by an inequality that is, so we can get an inequality that holds also in the Lagrangian frame. After that we want to derive higher order versions of it as well. The divergence and the curl are invariant but the other terms are not. There are two ways to make these terms invariant. One is to replace the differentiation by covariant differentiation and the other is to replace it by Lie derivatives with respect to the our family of vector fields in section 6. Both ways will result in a lower order term just involving the norm of the one form itself multiplied by a constant which depends on two derivatives of the coordinates.

Definition 11.1. Let c_1 be a constant such that

$$\sum_{a,b}\big(|g_{ab}| + |g^{ab}|\big) \leq c_1^2, \qquad |\partial x/\partial y|^2 + |\partial y/\partial x|^2 \leq c_1^2 \tag{11.3}$$

and let K_1 denote a continuous function of c_1.

We note that the bound for the Jacobian of the coordinate and its inverse follows from the bound for the metric and its inverse and the bound for the Jacobian and its inverse implies an equivalent bound for the metric and its inverse with c_1^2 multiplied by n. All our constants in what follows in this section will depend on a bound for c_1 and we will denote such a constants by K_1.

Lemma 11.2. *In the Lagrangian frame we have, with $\underline{W}_a = g_{ab}W^b$,*

$$|\mathcal{L}_U W| \leq K_1\Big(|\operatorname{curl}\underline{W}| + |\operatorname{div} W| + \textstyle\sum_{S\in\mathcal{S}}|\mathcal{L}_S W| + [g]_1|W|\Big), \qquad U \in \mathcal{R}, \tag{11.4}$$

$$|\mathcal{L}_U W| \leq K_1\Big(|\operatorname{curl}\underline{W}| + |\operatorname{div} W| + \textstyle\sum_{T\in\mathcal{T}}|\mathcal{L}_T W| + [g]_1|W|\Big), \qquad U \in \mathcal{U}, \tag{11.5}$$

where $[g]_1 = 1 + |\partial g|$. Furthermore

$$|\partial W| \leq K_1\Big(|\mathcal{L}_R W| + \textstyle\sum_{S\in\mathcal{S}}|\mathcal{L}_S W| + |W|\Big) \tag{11.6}$$

When $d(y) \leq d_0$ we may replace the sums over $\mathcal{S}$ by the sums over $\mathcal{S}_0$ and the sum over $\mathcal{T}$ by the sum over $\mathcal{T}_0$.

Proof of Lemma 11.2. (11.5) follows directly from (11.4) by adding the time derivative to the right hand side. We will show that (11.4) in the Eulerian frame follows from (11.1) and then it follows directly that (11.4) holds in Lagrangian frame as well since everything is invariant. Let $Z^i = \delta^{ij}\alpha_j$. Then $\mathcal{L}_U Z^i = UZ^i - (\partial_k \tilde{U}^i)Z^k$, where $\tilde{U}^i = U^a \partial x^i/\partial y^a$ are the components of the vector field U expressed in the Eulerian frame. Now transforming to the Lagrangian frame, partial differentiation becomes covariant differentiation. $(\partial_k \tilde{U}^i)(\partial x^k/\partial y^a)(\partial y^b/\partial x^i) = \nabla_a U^b$, where $\nabla_a U^b = \partial_a U^b + \Gamma_a{}^b{}_c U^c$, and $\Gamma_{ab}{}^c = g^{cd}\left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right)/2 = (\partial y^c/\partial x^i)\partial_a\partial_b x^i$ are the Christoffel symbols. Since $|\partial_a U^b| \le C$ it follows that $|\partial_k \tilde{U}^i| \le C[g]_1$. That we may replace $\mathcal{S}$ by $\mathcal{S}_0$ close to the boundary follows from the proof of Lemma 11.1. (11.6) follows since $\mathcal{R}$ span the tangent space and $|\mathcal{L}_U W^a - UW^a| = |(\partial_c U^a)W^c| \le C|W|$. $\square$

We are now going to derive higher order versions of the inequality in Lemma 11.2. We want to apply the lemma to W replaced by $\mathcal{L}_U^J W$. Then in our applications the divergence term vanishes and as we shall see later on we will be able to control the curl of $(\mathcal{L}_U^J \underline{W})_a = \mathcal{L}_U^J (g_{ab}W^b)$ which however is not the same as the curl of $(\underline{\mathcal{L}_U^J W})_a = g_{ab}\mathcal{L}_U^J W^b$ but the difference is lower order and can be easily estimated. Let us first introduce some notation:

Definition 11.2. Let β be a function, a one or two form or vector field, let $\mathcal{V}$ be any of our families of vector fields and set

$$|\beta|_s^{\mathcal{V}} = \sum_{|J|\le s,\, J\in\mathcal{V}} |\mathcal{L}_S^J \beta|, \qquad [\beta]_u^{\mathcal{V}} = \sum_{s_1+\ldots+s_k\le u,\, s_i\ge 1} |\beta|_{s_1}^{\mathcal{V}} \cdots |\beta|_{s_k}^{\mathcal{V}}, \qquad [\beta]_0^{\mathcal{V}} = 1. \tag{11.7}$$

In particular $|\beta|_r^{\mathcal{R}}$ is equivalent to $\sum_{|\alpha|\le r} |\partial_y^\alpha \beta|$ and $|\beta|_r^{\mathcal{U}}$ is equivalent to $\sum_{|\alpha|+k\le r} |D_t^k \partial_y^\alpha \beta|$.

Lemma 11.3. *With the convention that* $|\operatorname{curl}\underline{W}|_{-1}^{\mathcal{V}} = |\operatorname{div} W|_{-1}^{\mathcal{V}} = 0$ *we have*

$$|W|_r^{\mathcal{R}} \le K_1\big(|\operatorname{curl}\underline{W}|_{r-1}^{\mathcal{R}} + |\operatorname{div} W|_{r-1}^{\mathcal{R}} + |W|_r^{\mathcal{S}} + \sum_{s=1}^{r} |g|_s^{\mathcal{R}} |W|_{r-s}^{\mathcal{R}}\big), \tag{11.8}$$

$$|W|_r^{\mathcal{R}} \le K_1 \sum_{s=0}^{r} [g]_s^{\mathcal{R}}\big(|\operatorname{curl}\underline{W}|_{r-1-s}^{\mathcal{R}} + |\operatorname{div} W|_{r-1-s}^{\mathcal{R}} + |W|_{r-s}^{\mathcal{S}}\big). \tag{11.9}$$

The same inequalities also holds with $\mathcal{R}$ *replaced by* $\mathcal{U}$ *everywhere and* $\mathcal{S}$ *replaced by* $\mathcal{T}$*:*

$$|W|_r^{\mathcal{U}} \le K_1\big(|\operatorname{curl}\underline{W}|_{r-1}^{\mathcal{U}} + |\operatorname{div} W|_{r-1}^{\mathcal{U}} + |W|_r^{\mathcal{T}} + \sum_{s=1}^{r} |g|_s^{\mathcal{U}} |W|_{r-s}^{\mathcal{U}}\big), \tag{11.10}$$

$$|W|_r^{\mathcal{U}} \le K_1 \sum_{s=0}^{r} [g]_s^{\mathcal{U}}\big(|\operatorname{curl}\underline{W}|_{r-1-s}^{\mathcal{U}} + |\operatorname{div} W|_{r-1-s}^{\mathcal{U}} + |W|_{r-s}^{\mathcal{T}}\big). \tag{11.11}$$

Proof of Lemma 11.3. We will first prove (11.8) We claim that

$$\sum_{|I|=r, U\in\mathcal{R}} |\mathcal{L}_U^I W| \le K_1 \sum_{|J|=r-1, U\in\mathcal{R}} \big(|\operatorname{curl}\underline{\mathcal{L}_U^J W}| + |\operatorname{div}\mathcal{L}_U^J W| + [g]_1 |\mathcal{L}_U^J W|\big) + K_1 \sum_{|I|=r, S\in\mathcal{S}} |\mathcal{L}_S^I W| \tag{11.12}$$

First we note that there is noting to prove if $d(y) \ge d_0$ since then $\mathcal{S}$ span the full tangent space. Therefore, it suffices to prove (11.12) when $d(y) \le d_0$ and with $\mathcal{S}$ replaced by $\mathcal{S}_0$ and $\mathcal{R}$ replaced by

$\mathcal{R}_0$. Then (11.12) follows from (11.4) if $r=1$ and assuming that its true for r replaced by $r-1$ we will prove that it holds for r. If we apply (11.4) to $\mathcal{L}_U^J W$, where $|J|=r-1$, we get

$$|\hat{\mathcal{L}}_U \mathcal{L}_U^J W| \leq K_1\big(\,|\operatorname{curl} \underline{\hat{\mathcal{L}}_U^J W}\,| + |\operatorname{div} \mathcal{L}_U^J W| + \sum_{S\in\mathcal{S}} |\mathcal{L}_S \mathcal{L}_U^J W| + [g]_1 |\mathcal{L}_U^J W|\big). \tag{11.13}$$

If $\mathcal{L}_U^J$ consist of all tangential derivatives then it follows that $|\mathcal{L}_U \hat{\mathcal{L}}_U^J W|$ is bounded by the right hand side of (11.12). If $\mathcal{L}_U^J$ does not consist of only tangential derivatives then, since $[\mathcal{L}_R, \mathcal{L}_S] = \mathcal{L}_{[R,S]} = 0$, if $S \in \mathcal{S}_0$, we can write $\mathcal{L}_S \hat{\mathcal{L}}_U^J W = \hat{\mathcal{L}}_U^K \mathcal{L}_{S'} W$, for some $S' \in \mathcal{S}_0$. If we now apply (11.12) with r replaced by $r-1$ to $\mathcal{L}_{S'} W$, (11.12) follows also for r.

In (11.8) we have $\mathcal{L}_U^I \operatorname{curl} \underline{W} = \operatorname{curl} \mathcal{L}_U^I \underline{W}$ which however is different from curl $\underline{\mathcal{L}_U^I W}$. We have:

$$\mathcal{L}_U^J \underline{W}_a = \mathcal{L}_U^J (g_{ab} W^b) = -g_{ab} \mathcal{L}_U^J W^b + \tilde{c}_{J_1 J_2}^J g_{ab}^{J_1} \mathcal{L}_U^{J_2} W^b, \qquad \text{where} \quad g_{ab}^J = \mathcal{L}_U^J g_{ab} \tag{11.14}$$

where the sum is over all $J_1 + J_2 = J$ and $\tilde{c}_{J_1 J_2}^J = 1$ for $|J_2| < |J|$ $\tilde{c}_{J_1 J_2}^J = 0$ if $J_2 = J$. It follows that

$$|\operatorname{curl} \underline{\mathcal{L}_U^J W} - \operatorname{curl} \mathcal{L}_U^J \underline{W}\,| \leq 2\tilde{c}_{J_1 J_2}^J \big(|\partial g^{J_1}||\mathcal{L}_U^{J_2} W| + |g^{J_1}||\partial \mathcal{L}_U^{J_2} W|\big), \qquad |J_2| < |J|, \tag{11.15}$$

where the partial derivative can be estimated by Lie derivatives. (11.9) follows by induction from (11.8). Finally, (11.10) follows from (11.12) and (11.15). In fact, applying (11.12) to W replaced by $\mathcal{L}_{D_t}^k W$ we see that (11.12) holds also for $\mathcal{R}$ replaced by $\mathcal{U}$ and $\mathcal{S}$ replaced by $\mathcal{T}$ and (11.15) also holds for $U \in \mathcal{U}$.

12. The estimates for the curl and the normal derivatives.

Note that in section 10 we only had bounds for the derivatives that are tangential at the boundary, as well as all derivatives in the interior since S span the full tangent space in the interior. We will now use estimates for the curl together with the estimates for the tangential derivatives to get estimates also for normal derivatives close to the boundary. Let

$$\dot{w}_a = g_{ab} \dot{W}^b, \quad \text{and} \quad \operatorname{curl} w_{ab} = \partial_a w_b - \partial_b w_a. \tag{12.1}$$

Then we have

$$D_t\big(g_{ab} \dot{W}^b\big) - \partial_a\big((\partial_c p) W^c\big) = -\partial_a q + \omega_{ab} \dot{W}^b + \underline{F}_a \tag{12.2}$$

Note that (12.2) can also be formulated as

$$D_t \dot{w} + \underline{A} W - \underline{C} \dot{W} = \underline{F} \tag{12.3}$$

where the underline as before means that we lowered the indices so the result is a one form. Note here that $\dot{w}$ is not equal $D_t w$ so the notation is slightly confusing. But what we mean is that we think of W as a vector field and take the time derivative as a vector field which results in $\dot{W}$ and then $\dot{w}$ is the corresponding one form obtained by lowering the indices. We obtain

$$D_t \operatorname{curl} \dot{w}_{ab} = (\partial_c\, \omega_{ab}) \dot{W}^c - \omega_{cb} \partial_a \dot{W}^c + \omega_{ca} \partial_b \dot{W}^c + \operatorname{curl} \underline{F}_{ab} \tag{12.4}$$

Since $D_t w_a = \dot{g}_{ab} W^b + g_{ab} \dot{W}^b$ and $\partial_a \dot{g}_{bc} - \partial_b \dot{g}_{ac} = \partial_c \omega_{ab}$ we also obtain

$$D_t \operatorname{curl} w_{ab} = \operatorname{curl} \dot{w}_{ab} + (\partial_c\, \omega_{ab}) W^c + \dot{g}_{bc} \partial_a W^c - \dot{g}_{ac} \partial_b W^c. \tag{12.5}$$

Since $\operatorname{div} W = \operatorname{div} \dot{W} = 0$ it follows from Lemma 11.2 and (12.4)-(12.5) that

$$|D_t \operatorname{curl} \dot{w}| \le K_1 |\omega| \big(|\operatorname{curl} \dot{w}| + \textstyle\sum_{S\in\mathcal{S}} |\mathcal{L}_S \dot{W}| + [g]_1 |\dot{W}| \big) + |\partial\omega||\dot{W}| + |\operatorname{curl} \underline{F}| \tag{12.6}$$

$$|D_t \operatorname{curl} w| \le |\operatorname{curl} \dot{w}| + K_1 |\dot{g}| \big(|\operatorname{curl} w| + \textstyle\sum_{S\in\mathcal{S}} |\mathcal{L}_S W| + [g]_1 |W| \big) + |\partial\omega||W| \tag{12.7}$$

Since we already have control of the tangential derivatives S by section 11 this obviously gives us control of $\operatorname{curl} \dot{w}$ and $\operatorname{curl} w$ as well and once we have control of these we in fact control all components by Lemma 11.3 again. The norms will be measured in L^2 since we have control of the L^2 norms of the tangential components. We will now derive higher order versions of the inequalities (12.6)-(12.7) using the higher order version of Lemma 11.2, i.e. (11.9) in Lemma 11.3.

We must now get equations for the curl of higher derivatives as well. Applying $\mathcal{L}_U^J$ to (12.4)-(12.5) gives, since the Lie derivative commutes with the curl,

$$D_t \operatorname{curl} \mathcal{L}_U^J \dot{w}_{ab} = c_{J_1 J_2} \Big((\partial_c\, \omega_{ab}^{J_1}) \mathcal{L}_U^{J_2} \dot{W}^c - \omega_{cb}^{J_1} \partial_a \mathcal{L}_U^{J_2} \dot{W}^c + \omega_{ca}^{J_1} \partial_b \mathcal{L}_U^{J_2} \dot{W}^c \Big) + (\operatorname{curl} \mathcal{L}_U^J \underline{F})_{ab}, \tag{12.8}$$

where $\omega^J = \mathcal{L}_U^J \omega$ and

$$D_t \operatorname{curl} \mathcal{L}_U^J w_{ab} = \operatorname{curl} \mathcal{L}_U^J \dot{w}_{ab} + c_{J_1 J_2} (\partial_c\, \omega_{ab}^{J_1}) \mathcal{L}_U^{J_2} W^c + c_{J_1 J_2} \Big(\dot{g}_{bc}^{J_1} \partial_a \mathcal{L}_U^{J_2} W^c - \dot{g}_{ac}^{J_1} \partial_b \mathcal{L}_U^{J_2} W^c \Big) \tag{12.9}$$

where $\dot{g}_{ab}^J = \mathcal{L}_U^J D_t g_{ab}$. Let us make a definition:

Definition 12.1. Let β be a two form. With notation as in Definition 11.2 we set

$$([g]|\beta|)_u^{\mathcal{V}} = \sum_{s+r\le u} [g]_s^{\mathcal{V}} |\beta|_r^{\mathcal{V}}. \tag{12.10}$$

Using Lemma 11.3 and Lemma 11.2 it follows that:

Lemma 12.1. *With notation as in Definition 11.1 and Definition 12.1 and the convention that* $|\operatorname{curl} \underline{W}|_{-1}^{\mathcal{V}} = |\operatorname{div} W|_{-1}^{\mathcal{V}} = 0$ *we have*

$$|D_t \operatorname{curl} \dot{w}|_{r-1}^{\mathcal{R}} \le K_1 \sum_{s=0}^{r} ([g]|\omega|)_{r-s}^{\mathcal{R}} \big(|\operatorname{curl} \dot{w}|_{s-1}^{\mathcal{R}} + |\operatorname{div} \dot{W}|_{s-1}^{\mathcal{R}} + |\dot{W}|_s^{\mathcal{S}} \big) + |\operatorname{curl} \underline{F}|_{r-1}^{\mathcal{R}} \tag{12.11}$$

$$|D_t \operatorname{curl} w|_{r-1}^{\mathcal{R}} \le K_1 \sum_{s=0}^{r} ([g]|\dot{g}|)_{r-s}^{\mathcal{R}} \big(|\operatorname{curl} w|_{s-1}^{\mathcal{R}} + |\operatorname{div} W|_{s-1}^{\mathcal{R}} + |W|_s^{\mathcal{S}} \big) + |\operatorname{curl} \dot{w}|_{r-1}^{\mathcal{R}} \tag{12.12}$$

The same inequalities hold with $\mathcal{R}$ replaced by $\mathcal{U}$ and $\mathcal{S}$ replaced by $\mathcal{T}$.

Proof of Lemma 12.1. Let us first prove (12.11). The first terms in the right hand side of (12.8) are by Lemma 11.3 bounded by a constant times

$$\sum_{u=0}^{r} |\omega|_{r-u}^{\mathcal{R}} |\dot{W}|_u^{\mathcal{R}} \le K_1 \sum_{u=0}^{r} \sum_{s=0}^{u} |\omega|_{r-u}^{\mathcal{R}} [g]_{u-s}^{\mathcal{R}} \big(|\operatorname{curl} \dot{w}|_{s-1}^{\mathcal{R}} + |\operatorname{div} \dot{W}|_{s-1}^{\mathcal{R}} + |\dot{W}|_s^{\mathcal{S}} \big) \tag{12.13}$$

The proof of (12.12) uses the same argument and that that $\partial_c \omega_{ab} = \partial_a \dot{g}_{bc} - \partial_b \dot{g}_{ac}$. $\square$

Let us now introduce some new norms and some new notation:

Definition 12.2. For $\mathcal{V}$ any of our families of vector fields let

$$\|W\|_r^{\mathcal{V}} = \|W(t)\|_{\mathcal{V}^r(\Omega)} = \sum_{|I|\le r, I\in\mathcal{V}} \Big(\int_\Omega |\mathcal{L}_U^I W(t,y)|^2\,\kappa dy\Big)^{1/2}, \tag{12.14}$$

and

$$C_r^{\mathcal{V}} = \sum_{|J|=\le r-1,\, J\in\mathcal{V}} \Big(\int_\Omega |\operatorname{curl}\mathcal{L}_U^J \dot w|^2 + |\operatorname{curl}\mathcal{L}_U^J w|^2\,\kappa dy\Big)^{1/2}, \qquad C_0^{\mathcal{V}} = 0. \tag{12.15}$$

Note that $\|W(t)\|_{\mathcal{R}^r(\Omega)}$ is equivalent to the usual Sobolev norm in the Lagrangian coordinates.

Definition 12.3. For $\mathcal{V}$ any of our families of vector fields and for β a function, a 1-form, a 2-form or a vector field let $|\beta|_s^{\mathcal{V}}$ be as in Definition 11.1 and set

$$\|\beta\|_{s,\infty}^{\mathcal{V}} = \|\,|\beta|_s^{\mathcal{V}}\|_{L^\infty(\Omega)}, \qquad [[g]]_{s,\infty}^{\mathcal{V}} = \sum_{s_1+\ldots+s_k\le s,\, s_i\ge 1} \|g\|_{s_1,\infty}^{\mathcal{V}}\cdots\|g\|_{s_k,\infty}^{\mathcal{V}}, \qquad [[g]]_{0,\infty}^{\mathcal{V}} = 1 \tag{12.16}$$

where the sum is over all combinations with $s_i \ge 1$. Furthermore, let

$$m_r^{\mathcal{V}} = [[g]]_{r,\infty}^{\mathcal{V}}, \qquad \dot m_r^{\mathcal{V}} = \sum_{s+u\le r} [[g]]_{s,\infty}^{\mathcal{V}}\big(||\dot g||_{u,\infty}^{\mathcal{V}} + ||\omega||_{u,\infty}^{\mathcal{V}}\big). \tag{12.17}$$

Let $F_r^{\mathcal{U}} = \|\operatorname{curl}\underline{F}\|_{\mathcal{U}^{r-1}(\Omega)}$. It now follows from Lemma 12.1 that

$$\Big|\frac{dC_r^{\mathcal{U}}}{dt}\Big| \le K_1\sum_{s=0}^r \dot m_{r-s}^{\mathcal{U}}\big(C_s^{\mathcal{U}} + E_s^{\mathcal{T}}\big) + F_r^{\mathcal{U}} \tag{12.18}$$

where $E_s^{\mathcal{T}}$ is the energy of the tangential derivatives defined in section 10. Hence

$$C_r^{\mathcal{U}} \le K_1 e^{\int_0^t K_1\,\dot m_0^{\mathcal{U}}\,d\tau}\int_0^t\Big(\sum_{s=1}^{r-1}\dot m_{r-s}^{\mathcal{U}} C_s^{\mathcal{U}} + \sum_{s=0}^r \dot m_{r-s}^{\mathcal{U}} E_s^{\mathcal{T}} + F_r^{\mathcal{U}}\Big)\,d\tau \tag{12.19}$$

Since we already proved a bound for $E_s^{\mathcal{T}}$ in Lemma 10.1 it inductively follows that $C_r^{\mathcal{U}}$ is bounded. Note that, if $r=1$ the interpretation of (12.19) is that the first sum is not there. By Lemma 11.3:

$$\|W(t)\|_{\mathcal{U}^r(\Omega)} + \|\dot W(t)\|_{\mathcal{U}^r(\Omega)} \le K_1\sum_{s=0}^r m_{r-s}^{\mathcal{U}}\big(C_s^{\mathcal{U}} + E_s^{\mathcal{T}}\big) \tag{12.20}$$

Hence we have:

Lemma 12.2. *Suppose that $x,p\in C^{r+2}([0,T]\times\Omega)$, $p\big|_{\partial\Omega}=0$, $\nabla_N p\big|_{\partial\Omega}\le -c_0<0$ and $\operatorname{div}V=0$, where $V=D_t x$. Then there is a constant $C=C(x,p)$ depending only on the norm of (x,p), a lower bound for c_0 and an upper bound for T, such that if $E_s^{\mathcal{T}}(0)=C_s^{\mathcal{U}}(0)=0$, for $s\le r$, then*

$$\|W\|_r^{\mathcal{U}} + \|\dot W\|_r^{\mathcal{U}} + E_r^{\mathcal{T}} \le C\int_0^t \|F\|_r^{\mathcal{U}}\,d\tau, \qquad \text{for}\quad 0\le t\le T. \tag{12.21}$$

13. The smoothed out normal operator.

In order to prove existence we first have to replace the normal operator A by a sequence A^ε of bounded symmetric and positive operators that convergence to A, as $\varepsilon \to 0$. The boundedness is needed for the existence and the symmetry and positivity is needed to get a positive term in the energy. Furthermore the commutators with Lie derivatives with respect to tangential vector fields as well as the curl have to be well behaved. Let $\rho = \rho(d)$ be a smooth function of $d = d(y) = \text{dist}\,(y, \partial\Omega)$, such that

$$\rho' \geq 0, \qquad \rho(d) = d \quad \text{for } d \leq 1/4 \quad \text{and} \quad \rho(d) = 1/2 \quad \text{for } d \geq 3/4. \tag{13.1}$$

Let $\chi(\rho)$ be a smooth function such that

$$\chi'(\rho) \geq 0, \qquad \chi(\rho) = 0, \quad \text{when} \quad \rho \leq 1/4, \quad \text{and} \quad \chi(\rho) = 1, \quad \text{when} \quad \rho \geq 3/4 \tag{13.2}$$

For a function f vanishing on the boundary we define

$$A_f^\varepsilon W^a = P\big(- g^{ab} \chi_\varepsilon(\rho) \partial_b \big(f \rho^{-1} (\partial_c \rho) W^c \big) \big) \tag{13.3}$$

where $\chi_\varepsilon(\rho) = \chi(\rho/\varepsilon)$. Then if we integrate by parts we get

$$\langle U, A_f^\varepsilon W \rangle = \int_\Omega f \rho^{-1} \chi_\varepsilon'(\rho) (U^a \partial_a \rho)(W^b \partial_b \rho) dy \tag{13.4}$$

from which it follows that A_f^ε is symmetric and

$$A^\varepsilon = A_p^\varepsilon \geq 0, \qquad \text{i.e.} \qquad \langle W, A^\varepsilon W \rangle \geq 0, \qquad \text{if} \quad p \geq 0 \tag{13.5}$$

It also follows that another expression for A_f^ε is

$$A_f^\varepsilon W^a = P\big(g^{ab} \chi_\varepsilon'(\rho) (\partial_b \rho) f \rho^{-1} (\partial_c \rho) W^c \big) \tag{13.6}$$

A^ε is now for each $\varepsilon > 0$ a bounded operator

$$\| A^\varepsilon W \| \leq C \| \nabla_N p \|_{L^\infty} \varepsilon^{-1} \| W \| \tag{13.7}$$

since $\chi_\varepsilon' \leq C/\varepsilon$ and $p\,\rho^{-1} |\partial \rho| \leq C \| \nabla_N p \|_{L^\infty(\partial\Omega)}$. In general, since the projection is continuous on $H^r(\Omega)$, see (3.6) and (3.8), if the metric and pressure are sufficiently regular we get

$$\sum_{j=0}^k \| D_t^j A^\varepsilon W \|_{H^r(\Omega)} \leq C_{\varepsilon,r,k} \sum_{j=0}^k \| D_t^j W \|_{H^r(\Omega)}. \tag{13.8}$$

Moreover

$$A^\varepsilon U \to AU, \quad \text{in} \quad L^2(\Omega), \quad \text{if} \quad U \in H^1(\Omega) \tag{13.9}$$

In fact, the projection is continuous in the norm and $\chi_\varepsilon F \to F$ in L^2 if $F \in L^2$. It follows that

$$P\big(g^{ab} \chi_\varepsilon(\rho) \partial_b \big(p\,\rho^{-1} (\partial_c \rho) U^c \big) \big) \to P\big(g^{ab} \partial_b \big(p\,\rho^{-1} (\partial_c \rho) U^c \big) \big) = P\big(g^{ab} \partial_b \big((\partial_c p) U^c \big) \big) \tag{13.10}$$

since $p\,\rho^{-1} \partial_c \rho = \partial_c p$ on the boundary.

We will now calculate the commutators with the Lie derivative $\mathcal{L}_T$ with respect to tangential vector fields T. As before the inequality

$$(13.11)\qquad |\langle U, A^\varepsilon_{fp} W\rangle| \le \langle U, A^\varepsilon_{|f|p} U\rangle^{1/2}\langle W, A^\varepsilon_{|f|p} W\rangle^{1/2} \le \|f\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)}\langle U, A^\varepsilon U\rangle^{1/2}\langle W, A^\varepsilon W\rangle^{1/2}$$

hold, where $\Omega^\varepsilon = \{y \in \Omega;\, d(y) > \varepsilon\}$. In fact, it suffices to take the supremum over the set where $d(y) \le \varepsilon$ since $\chi_\varepsilon' = 0$, when $d(y) \ge \varepsilon$. The only difference with (3.13) is that now the supremum over a small neighborhood of the boundary instead of on the boundary. The positivity properties (13.5) and (13.11) for A^ε will play the role that (3.12) and (3.13) did for A. In particular, since p vanishes on the boundary, $p > 0$ in the interior and $\nabla_N p \le -c_0 < 0$ on the boundary it follows that $\dot p = D_t\, p$ vanishes on the boundary and $\dot p / p$ is a smooth function. Therefore

$$(13.12)\qquad \dot A^\varepsilon = A^\varepsilon_{\dot p} \qquad \text{satisfies} \qquad |\langle W, \dot A^\varepsilon W\rangle| \le \|\dot p/p\|_{L^\infty(\Omega\setminus\Omega^\varepsilon)}\langle W, A^\varepsilon W\rangle$$

Here $\dot A^\varepsilon$ is the time derivative of the operator A^ε, considered as an operator with values in the one forms. It will show up in the energy estimate for the ε smoothed out equation in the next section.

The commutators between A^ε_f and Lie derivatives with respect to tangential vector fields are basically the same as for A. Note that

$$(13.13)\qquad Td = 0, \quad \text{if} \quad T \in \mathcal{T}_0 = \mathcal{S}_0 \cup \{D_t\}$$

where $\mathcal{S}_0$ are the rotations. Hence if T is any of these vector fields we have

$$(13.14)\qquad P\big(g^{ca}\mathcal{L}_T(g_{ab}A^\varepsilon_f W^b)\big) = A^\varepsilon_f \mathcal{L}_T W^c + A^\varepsilon_{Tf} W^c,$$

However, in order to get additional regularity in the interior we include the vector fields $\mathcal{S}_1$ that span the tangent space in the interior. The vector fields in $\mathcal{S}_1$ satisfy

$$(13.15)\qquad S\rho = \mathcal{L}_S \rho = 0, \quad \text{when} \quad d \le d_0/2$$

Since $\chi_\varepsilon'(\rho) = 0$ when $d \ge \varepsilon$ the commutator relation (13.14) above is true for these as well if we assume that $\varepsilon \le d_0/2$.

It remains to estimate the curl of A^ε. Whereas, the curl of A vanishes this is not the case for the curl of A^ε. It will however vanish away from the boundary. With $\underline{A}^\varepsilon W_a = g_{ab}A^\varepsilon W^b$ we have

$$(13.16)\qquad \underline{A}^\varepsilon W_a = -\chi_\varepsilon(\rho)\partial_a\big(p\,\rho^{-1}(\partial_c\rho)W^c\big) - \partial_a q_1$$

for some function q_1 vanishing on the boundary and determined so the divergence vanishes. Since the curl of the gradient vanishes and $\chi_\varepsilon'(\rho) = 0$ when $d \ge \varepsilon$ we have

$$(13.17)\qquad \operatorname{curl} \underline{A}^\varepsilon W_{ab} = 0, \qquad \text{when} \quad d(y) \ge \varepsilon$$

14. The smoothed out equation and existence of weak solutions.

The ε smoothed out linear equation:

$$(14.1)\qquad \ddot W^a_\varepsilon + A^\varepsilon W^a_\varepsilon + \dot G \dot W_\varepsilon - C\dot W^a_\varepsilon = F^a, \qquad \dot W_\varepsilon\Big|_{t=0} = W_\varepsilon\Big|_{t=0} = 0$$

is just an ordinary differential equation for $(W_\varepsilon, \dot{W}_\varepsilon)$ on the space of divergence free vector fields in $L^2(\Omega)$ since all operator are bounded so existence follows in $L^2(\Omega)$. In fact its an ordinary differential equation in the Sobolev spaces $H^r(\Omega)$ by (13.8). To get additional regularity in time as well we apply more time derivatives using (13.8) and (3.8) and that the initial conditions for these vanishes as well since we constructed F in (14.1) so it vanishes to any given order. If initial data, encoded in F, are smooth, we hence have a smooth solution of the ε approximate linear equation.

Now we want to use the existence and estimates for the ε smoothed out linear equation and pass to the limit as $\varepsilon \to 0$ to get existence for the linearized equation. Will show that $W_\varepsilon \to W$ weakly in L^2, where $W \in H^r(\Omega)$ for some large r. From the weak convergence it will follow that W is a weak solution and then from the additional regularity of W it will follow that in fact its a classical solution and hence that the *a priori* bounds in the earlier section hold.

Of course the norm of A^ε tends to infinity as $\varepsilon \to 0$ but since it is a positive operator it can be included in the energy. The energy will be the same as before with A replaced by A^ε, so (4.4) becomes

$$E^\varepsilon = \langle \dot{W}_\varepsilon, \dot{W}_\varepsilon \rangle + \langle W_\varepsilon, (A^\varepsilon + I) W_\varepsilon \rangle \tag{14.2}$$

The time derivative of the first term is the same as (4.5) with W replaced by W_ε. Since $D_t\, d = 0$ it follows from taking the time derivative of (13.4), with $f = p$, that

$$\frac{d}{dt} \langle W_\varepsilon, A^\varepsilon W_\varepsilon \rangle = 2 \langle \dot{W}_\varepsilon, A^\varepsilon W_\varepsilon \rangle + \langle W_\varepsilon, A^\varepsilon_{\dot{p}} W_\varepsilon \rangle, \tag{14.3}$$

where the last term is bounded by (13.12). Hence by (4.7)-(4.9):

$$|\dot{E}^\varepsilon| \le \Big(1 + \|\dot{g}\|_{L^\infty(\Omega)} + \|(D_t p)/p\|_{L^\infty(\Omega)} \Big) E^\varepsilon + 2\sqrt{E^\varepsilon} \|F\| \tag{14.4}$$

from which we get a uniform bound for $0 \le t \le T$ independent of ε: $E^\varepsilon(t) \le C$.

Since $\|W_\varepsilon\| \le C$ we can now choose a subsequence $W_{\varepsilon_n} \to W$ weakly in the inner product. We will show below that the limit W is a weak solution if the equation. Multiplying the ε smoothed out equation by a smooth divergence free vector field U that vanishes for $t \ge T$ and integrating by parts we get

$$\int_0^T \int_\Omega g_{ab} \Big(\ddot{U}^a + A^\varepsilon U^a + \dot{G} \dot{U}^a - C \dot{U}^a - \dot{C} U^a \Big) W_\varepsilon^b \, dy dt = \int_0^T \int_\Omega g_{ab} U^a F^b \, dy dt \tag{14.5}$$

where $\dot{C} W^c = P\big(g^{ac} \dot{\omega}_{cb} W^b \big)$, since A^ε and $D_t^2 + B D_t$ are symmetric and the adjoint of $C D_t$ is $C D_t - \dot{C}$. We proved in the previous section that $A^\varepsilon U$ converges to AU strongly in the norm if U is in H^1. Since $W_{\varepsilon_n} \to W$ weakly this proves that we have a weak solution W of the equation:

$$\int_0^T \int_\Omega g_{ab} \Big(\ddot{U}^a + A U^a + \dot{G} \dot{U}^a - C \dot{U}^a - \dot{C} U^a \Big) W^b \, dy dt = \int_0^T \int_\Omega g_{ab} U^a F^b \, dy dt \tag{14.6}$$

for any divergence free smooth vector field U that vanishes for $t \ge T$. Furthermore since W_ε is divergence free, we have

$$\int_0^T \int_\Omega (\partial_a q) W_\varepsilon^a \, dy dt = 0 \tag{14.7}$$

for any smooth q that vanishes on the boundary and hence

$$\int_0^T \int_\Omega (\partial_a q) W^a \, dy dt = 0 \tag{14.8}$$

so W is weakly divergence free.

Now that we have existence of a weak solution we will prove that we have additional regularity and in fact that, $W, \dot{W} \in H^r(\Omega)$ for any $r \geq 0$. It then follows that we can integrate by parts again in the above integrals and conclude that

$$\int_0^T \int_\Omega q\, \partial_a W^a \, dydt = 0 \tag{15.1}$$

for any smooth function q that vanishes on the boundary. Hence W is divergence free. Furthermore

$$\int_0^T \int_\Omega g_{ab} U^a \Big(\ddot{W}^b + A W^b + \dot{G} \dot{W}^b - C \dot{W}^b \Big) \, dydt = \int_0^T \int_\Omega g_{ab} U^a F^b \, dydt \tag{15.2}$$

for any smooth divergence free vector field U that vanishes for $t \geq T$. But in fact since W is divergence free it follows that $\ddot{W}^b + AW^b + \dot{G}\dot{W}^b - C\dot{W}^b$ is divergence and since by construction F is divergence free as well it follows that (15.2) holds for any smooth vector field U that vanishes for $t \geq T$. We then conclude that

$$\ddot{W}^b + AW^b + \dot{G}\dot{W}^b - C\dot{W}^b = F^b, \qquad \operatorname{div} W = 0 \tag{15.3}$$

It therefore only remains to show that $W \in H^r(\Omega)$. We must show that we have uniform bounds for the ε smooth out equation similar to the *a priori* bounds for the linearized equation.

The uniform tangential bounds for the ε smoothed out equation follows the proof of the *a priori* tangential bounds in section 10. The proof is just a change of notation. Let

$$E_I^\varepsilon = \langle \dot{W}_{\varepsilon I}, \dot{W}_{\varepsilon I} \rangle + \langle W_{\varepsilon I}, (A+I) W_{\varepsilon I} \rangle, \qquad W_{\varepsilon I} = \mathcal{L}_T^I W_\varepsilon. \tag{15.4}$$

If $\varepsilon < d_0$ then the commutator relation for A^ε, (13.14), is exactly the same as for A, (8.5). Furthermore the positivity property for A_f^ε only differs from the one for A_f by that the supremum over the boundary in (3.13) is replaced by the supremum over a neighborhood of the boundary where $d(y) < \varepsilon$ in (13.11). Hence all the calculations and inequalities in sections 10 and 12 hold with A replaced by A^ε, if we replace the supremum of $\nabla_N q / \nabla_N p$ over the boundary in (10.9) by the supremum of q/p over the domain $\Omega \setminus \Omega^\varepsilon$, where Ω^ε is given by (15.6). Therefore we will arrive at the energy bound (10.21) for $E_r^{\mathcal{T}}$ replaced by

$$E_r^{\mathcal{T},\varepsilon} = \sum_{|I| \leq r,\, I \in \mathcal{T}} \sqrt{E_I^\varepsilon}, \tag{15.5}$$

i.e. Lemma 10.1 hold for $E_r^{\mathcal{T}}$ replaced by $E_r^{\mathcal{T},\varepsilon}$ with a constant independent of ε. Note that, this is where we need to have vanishing initial conditions and an inhomogeneous term that vanishes to high order when $t = 0$ so that also the higher order time derivatives of the solution of (14.1) vanished when $t = 0$. If the initial conditions for higher order time derivatives were to be obtained from the ε smoothed out equation, then they would depend on ε and so we would not have been able to get a uniform bound for the energy, $E_r^{\mathcal{T},\,\varepsilon}$.

The bound for curl is very simple since by (13.17) the curl of A_ε vanishes in

$$\Omega^\varepsilon = \{ y;\, \operatorname{dist}(y, \partial\Omega) > \varepsilon \}, \tag{15.6}$$

it follows that all the formulas in section 12 hold when $d(y) \geq \varepsilon$. This follows from replacing $\underline{A}$ in (12.3) by $\underline{A}^\varepsilon$ and using that the curl of this vanishes for $d(y) \geq \varepsilon$. Since all the estimates used from section 11 are point wise estimates we conclude that (12.11)-(12.12) hold for W replaced by W^ε when $d(y) \geq \varepsilon$. Let

$$C_r^{\mathcal{U},\,\varepsilon} = \sum_{|J|\leq r-1,\, J\in\mathcal{U}} \Big(\int_{\Omega^\varepsilon} |\operatorname{curl} \mathcal{L}_U^J w_\varepsilon|^2 + |\operatorname{curl} \mathcal{L}_U^J \dot{w}_\varepsilon|^2 \, dy \Big)^{1/2} \tag{15.7}$$

With $C_r^{\mathcal{U}}$ replaced by $C_r^{\mathcal{U},\,\varepsilon}$ and $E_s^{\mathcal{T}}$ replaced by $E_s^{\mathcal{T},\,\varepsilon}$ we get exactly the same inequalities as before (12.18)-(12.19), since these were derived from the point wise bounds in section 11. Furthermore, the inequality (12.20) hold as well if we replace the norms by

$$\|W(t)\|_{\mathcal{U}^r(\Omega^\varepsilon)} = \sum_{|I|\leq r,\, I\in\mathcal{U}} \Big(\int_{\Omega^\varepsilon} |\mathcal{L}_U^I W(t,y)|^2 \, \kappa dy \Big)^{1/2}, \tag{15.8}$$

Therefore we conclude that the inequality in Lemma 12.2 hold with a constant C independent of ε if we replace the norms by (15.8):

Lemma 15.1. *Suppose that $x, p \in C^{r+2}([0,T]\times\Omega)$, $p\big|_{\partial\Omega} = 0$, $\nabla_N p\big|_{\partial\Omega} \leq -c_0 < 0$ and $\operatorname{div} V = 0$, where $V = D_t x$. Suppose that W_ε is a solution of (14.1) where F is divergence free and vanishing to order r as $t \to 0$. Let $E_s^{\mathcal{T},\,\varepsilon}$ be defined by (15.5). Then there is a constant C depending only on the norm of (x,p), a lower bound for c_0 and an upper bound for T, but independent of ε, such that if $E_s^{\mathcal{T},\varepsilon}(0) = C_s^{\mathcal{U},\varepsilon}(0) = 0$, for $s \leq r$, then*

$$\|W_\varepsilon(t)\|_{\mathcal{U}^r(\Omega^\varepsilon)} + \|\dot{W}_\varepsilon(t)\|_{\mathcal{U}^r(\Omega^\varepsilon)} + E_r^{\mathcal{T},\,\varepsilon}(t) \leq C\int_0^t \|F\|_r^{\mathcal{U}}\, d\tau, \qquad \textit{for} \quad 0 \leq t \leq T. \tag{15.9}$$

It therefore follows that the limit W satisfies the same bound with Ω^ε replaced by Ω, and so the weak solution in section 14 is in fact a smooth solution.

16. The energy estimate revisited and the proof of the theorem.

In section 10 we estimated the energies of the tangential derivatives without using the estimate of the normal derivatives coming from the curl. This was necessary to get uniform bounds for the ε smoothed out equation since in that case we could not estimate the curl close to the boundary. The drawback was that instead we had to include all time derivatives as well in the energy. However, now that we have existence we can obtain other bounds for the linearized equation directly. In section 9 we calculated the commutator between the linearized operator, considered as an operator from the divergence free vector fields to the one forms, and Lie derivatives with respect to tangential vector fields, and then projected the result back onto the divergence free vector fields. This was needed because the commutator between Lie derivatives and the operator A considered as an operator with values in the one forms is better behaved. However, the drawback is that the commutator with the second time derivative, considered as an operator with values in the one forms, involves second time derivatives, which is why we had to include all the time derivatives. Now we will instead commute through directly with the operator from the divergence free vector fields to the divergence free vector fields. Let us then also consider the original setting with non vanishing initial conditions and an inhomogeneous term:

$$\ddot{W}^a - g^{ab}\partial_b\big((\partial_c p)W^c - q_1\big) = -g^{ab}\big((\dot{g}_{cb} - \omega_{cb})\dot{W}^c - \partial_b q_2\big) + F^a \tag{16.1}$$

where q_1 and q_2 vanishes on the boundary and are chosen so that each term is divergence free. The second term on the left is AW^a and the term in the right is $-\dot{G}\dot{W}^a + C\dot{W}^a$. Let us now first calculate the commutators with A and tangential vector fields.

$$(16.2) \quad \mathcal{L}_S\Big(g^{ab}\partial_b\big((\partial_c p)W^c - q_1\big)\Big)$$
$$= (\mathcal{L}_S g^{ab})\partial_b\big((\partial_c p)W^c - q_1\big) + g^{ab}\partial_b\big((\partial_c Sp)W^c + (\partial_c p)(\mathcal{L}_S W)^c - Sq_1\big)$$

where $\mathcal{L}_S g^{ab} = -g^{ac}g^{bd}g^S_{cd}$, $g^S_{cd} = \mathcal{L}_S g_{cd}$. Projecting each term onto divergence free vector fields:

$$\mathcal{L}_S AW = -G_S AW + A_S W + A\mathcal{L}_S W, \tag{16.3}$$

where $A_S = A_{Sp}$ and $G_S = M_{g^S}$ is the operator $G_S W^a = P\big(g^{ac}g^S_{cb}W^b\big)$. Expressed differently

$$[\mathcal{L}_S, A]W = (A_S - G_S A)W \tag{16.4}$$

Although G_S is a bounded operator, all the positivity properties of A are lost and the best we can say is that $G_S A$ is an operator of order 1. The operator A_S is also of order 1 but in section 10 we used the positivity property to estimate it in terms of A which we controlled by the energy. It remains to calculate the commutator with G_S and C, which basically are the same.

$$(16.5) \quad \mathcal{L}_T G_S W^i = \mathcal{L}_T\Big(g^{ab}\big(g^S_{bc}W^c - \partial_b q\big)\Big)$$
$$= (\mathcal{L}_T g^{ab})\big(g^S_{bc}W^k - \partial_b q\big) + g^{ab}(\mathcal{L}_T g^S_{bc})W^c + g^{ab}g^S_{bc}\mathcal{L}_T W^c - g^{ab}\partial_b Tq$$

Projecting each term onto the divergence free vector fields we arrive at

$$[\mathcal{L}_T, G_S]W = (G_{TS} - G_T G_S)W, \tag{16.6}$$

where $G_{TS}W^a = P\big(g^{ab}g^{TS}_{bc}W^c\big)$ and $g^{TS}_{bc} = \mathcal{L}_T\mathcal{L}_S g_{bc}$.

In general using (16.4) and (16.6) to commute through we get for some constants $\tilde{d}_I^{I_1\dots I_k}$

$$\mathcal{L}_S^I AW - A\mathcal{L}_S^I W = \tilde{d}_I^{I_1 I_k} G_{I_1}\cdots G_{I_{k-2}} A_{I_{k-1}} W_{I_k} \tag{16.7}$$

where the sum is over all combinations with $I_1 + \dots + I_k = I$, with $k \geq 2$, and $|I_k| < |I|$. Here $G_J W^a = M_{g^J}W^a = P(g^{ac}g^J_{cb}W^b)$, where $g^J_{ac} = \mathcal{L}^J_S g_{ac}$, $A_J = A_{S^J p}$ and $W_J = \mathcal{L}^J_S W$. Similarly we get the commutators with $\dot{G}$ and C

$$\mathcal{L}_S^I \dot{G}W - \dot{G}\mathcal{L}_S^I W = \tilde{e}_I^{I_1 I_k} G_{I_1}\cdots G_{I_{k-2}} \dot{G}_{I_{k-1}} W_{I_k} \tag{16.8}$$

$$\mathcal{L}_S^I CW - C\mathcal{L}_S^I W = \tilde{e}_I^{I_1 I_k} G_{I_1}\cdots G_{I_{k-2}} C_{I_{k-1}} W_{I_k} \tag{16.9}$$

The only thing that matters is that these are bounded operators, and in fact they are lower order since $|I_k| < |I|$. Hence we obtain

$$L_1 W = \ddot{W}_I + AW_I + \dot{G}\dot{W}_I - C\dot{W}_I = H_I \tag{16.10}$$

where

$$H_I = F_I + \tilde d_I^{I_1 I_k} G_{I_1}\cdots G_{I_{k-2}} A_{I_{k-1}} W_{I_k} + \tilde e_I^{I_1 I_k} G_{I_1}\cdots G_{I_{k-2}} \dot G_{I_{k-1}} \dot W_{I_k} + \tilde e_I^{I_1 I_k} G_{I_1}\cdots G_{I_{k-2}} C_{I_{k-1}} \dot W_{I_k} \tag{16.11}$$

where $|I_k| < |I|$ in the right hand side. Here $F_I = \mathcal{L}_T^I F$. As before, let

$$E_I = \langle \dot W_I, \dot W_I\rangle + \langle W_I, (A+I) W_I\rangle \tag{16.12}$$

where we now only consider $W_I = \mathcal{L}_S^I W$ with $S \in \mathcal{S}$. The energy estimate is like before and we only have to be able to estimate the L^2 norm of the right hand side of (16.10). The terms on the second row of (16.11) are obviously bounded by E_J for some $|J| \le |I|$. In fact they are even lower order since we have strict inequality. Therefore it only remains to estimate the term on the right in the first row. $|I_k| < |I|$ but A_{I_k} is order one and it contains derivatives in any direction so that term has to be estimated by the $\|\partial W_{I_k}\|_{L^2(\Omega)}$, and so it does not directly help to have an estimate for $\|\mathcal{L}_S W_{I_k}\|_{L^2(\Omega)}$ for all tangential derivatives S. However the estimate of the tangential derivatives together with the estimates for curl in Lemma 12.1 gives the required estimate.

Let $C_r^{\mathcal{R}}$ be defined (12.15), let $E_s^{\mathcal{S}}$ be defined by (10.13) and let $m_s^{\mathcal{R}}$ and $\dot m_s^{\mathcal{R}}$ be as in Definition 12.3. Then by Lemma 11.3 we get the inequality corresponding to (12.20):

$$\|W\|_r + \|\dot W\|_r \le K_1 \sum_{s=0}^{r} m_{r-s}^{\mathcal{R}} \big(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}\big), \qquad \text{where} \quad \|W\|_r = \|W(t)\|_{\mathcal{R}^r(\Omega)} \tag{16.13}$$

Since the projection has norm 1, $\|G_J W\| \le \|g^J\|_\infty \|W\|$. It follows that

$$\|G_{I_1}\cdots G_{I_{k-2}} C_{I_{k-1}} \dot W_{I_k}\| \le \|g^{I_1}\|_\infty \cdots \|g^{I_{k-2}}\|_\infty \, \|\omega^{I_{k-1}}\|_\infty \, \|\dot W_{I_k}\| \le \dot m_{r-s}^{\mathcal{R}} \|\dot W\|_s \tag{16.14}$$

$$\|G_{I_1}\cdots G_{I_{k-2}} \dot G_{I_{k-1}} \dot W_{I_k}\| \le \|g^{I_1}\|_\infty \cdots \|g^{I_{k-2}}\|_\infty \, \|\dot g^{I_{k-1}}\|_\infty \, \|\dot W_{I_k}\| \le \dot m_{r-s}^{\mathcal{R}} \|\dot W\|_s \tag{16.15}$$

where $s = |I_k| < r$ and $r = |I|$. Let

$$p_r^{\mathcal{R}} = \sum_{s=0}^{r} [[g]]_{r-s,\infty}^{\mathcal{R}} \sum_{|J|\le s+1,\, J\in\mathcal{S}} \|\partial S^J p\|_{L^\infty(\partial\Omega)} \tag{16.16}$$

Since $A_J = A_{S^J p}$ it follows form (3.15) that

$$\|G_{I_1}\cdots G_{I_{k-2}} A_{I_{k-1}} W_{I_k}\| \le \|g^{I_1}\|_\infty \cdots \|g^{I_{k-2}}\|_\infty \, \|A_{I_{k-1}} W_{I_k}\| \le p_{r-s}^{\mathcal{R}} \|W\|_s + p_{r-s-1}^{\mathcal{R}} \|W\|_{s+1} \tag{16.17}$$

By (4.9) applied to (16.10) in place of (4.3):

$$|\dot E_I| \le \big(1 + \|\dot g\|_\infty + \|\partial \dot p\|_\infty / c_0\big) E_I + 2\sqrt{E_I}\, \|H_I\| \tag{16.18}$$

where c_0 is the constant in (1.6). By (16.14)-(16.17) we have

$$\|H_I\| \le C \sum_{s=0}^{r-1} \big(\dot m_{r-s}^{\mathcal{R}} \|\dot W\|_s + p_{r-s}^{\mathcal{R}} \|W\|_s\big) + p_0^{\mathcal{R}} \|W\|_r + \|F\|_r \tag{16.19}$$

and using (16.13)

$$\|H_I\| \le K_1 \sum_{s=0}^{r-1} \big(\dot m_{r-s}^{\mathcal{R}} + p_{r-s}^{\mathcal{R}}\big)\big(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}\big) + K_1 p_0^{\mathcal{R}}\big(C_r^{\mathcal{R}} + E_r^{\mathcal{S}}\big) + \|F\|_r \tag{16.20}$$

Summing (16.18) over all $I \in \mathcal{S}$ with $|I| = r$ and using (16.20) we get

$$\Big|\frac{dE_r^{\mathcal{S}}}{dt}\Big| \le K_1\big(1 + \|\dot g\|_\infty + \|\partial \dot p\|_\infty / c_0 + \sum_{S\in\mathcal{S}} \|\partial S p\|_\infty\big)\big(C_r^{\mathcal{R}} + E_r^{\mathcal{S}}\big) + K_1 \sum_{s=0}^{r-1} \big(\dot m_{r-s}^{\mathcal{R}} + p_{r-s}^{\mathcal{R}}\big)\big(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}\big) + \|F\|_r \tag{16.21}$$

Furthermore, by Lemma 12.1, (12.18) hold with $\mathcal{U}$ replaced by $\mathcal{R}$ and $\mathcal{T}$ replaced by $\mathcal{S}$:

$$\Big|\frac{dC_r^{\mathcal{R}}}{dt}\Big| \le K_1 \dot m_0^{\mathcal{R}}\big(C_r^{\mathcal{R}} + E_r^{\mathcal{S}}\big) + K_1 \sum_{s=1}^{r-1} \dot m_{r-s}^{\mathcal{R}}\big(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}\big) + \|F\|_r \tag{16.22}$$

(16.21) together with (16.22) gives us a bound for $C_r^{\mathcal{R}} + E_r^{\mathcal{S}}$ in terms of $C_s^{\mathcal{R}} + E_s^{\mathcal{S}}$ for $s < r$:

$$C_r^{\mathcal{R}}(t) + E_r^{\mathcal{S}}(t) \le K_1 e^{K_1 \int_0^t n\, d\tau}\big(C_r^{\mathcal{R}}(0) + E_r^{\mathcal{S}}(0)\big) + K_1 e^{K_1 \int_0^t n\, d\tau} \int_0^t \Big(\sum_{s=1}^{r-1} \big(\dot m_{r-s}^{\mathcal{R}} + p_{r-s}^{\mathcal{R}}\big)\big(C_s^{\mathcal{R}} + E_s^{\mathcal{S}}\big) + \|F\|_r\Big)\, d\tau \tag{16.23}$$

where $n = 1 + \|\dot g\|_\infty + \|\partial \dot p\|_\infty / c_0 + \sum_{S\in\mathcal{S}} \|\partial S p\|_\infty + \|\omega\|_\infty$. Since we already have proven the bound for $E_0^{\mathcal{S}} = E_0$ in section 4, (16.23) inductively gives a bound for $C_r^{\mathcal{R}} + E_r^{\mathcal{S}}$. Hence by (16.13) we obtain:

Lemma 16.1. *Suppose that $x, p \in C^{r+2}([0,T]\times\Omega)$, $p\,\big|_{\partial\Omega} = 0$, $\nabla_N p\,\big|_{\partial\Omega} \le -c_0 < 0$ and $\operatorname{div} V = 0$, where $V = D_t x$. Let W be the solution of (16.1) where F is divergence free. Then there is a constant C depending only on the norm of (x,p), a lower bound for the constant c_0 and an upper bound for T, such that, for $0 \le t \le T$, we have*

$$\|\dot W(t)\|_r + \|W(t)\|_r + \langle W(t)\rangle_{A,r} \le C\Big(\|\dot W(0)\|_r + \|W(0)\|_r + \langle W(0)\rangle_{A,r} + \int_0^t \|F\|_r\, d\tau\Big) \tag{16.24}$$

where

$$\|W(t)\|_r = \sum_{|I|\le r,\, I\in\mathcal{R}} \|\mathcal{L}_U^I W(t)\|_{L^2(\Omega)}, \qquad \langle W(t)\rangle_{A,r} = \sum_{|I|\le r,\, I\in\mathcal{S}} \langle \mathcal{L}_S^I W(t), A\mathcal{L}_S^I W(t)\rangle^{1/2} \tag{16.25}$$

Note that the $\|W(t)\|_r$ is equivalent to the usual time independent Sobolev norm. Since there are compactly supported divergence free vector fields $\langle W(t)\rangle_{A,r}$ is only a semi-norm on divergence free vector fields, see (3.10). Furthermore, since $0 < c_0 \le -\nabla_N p \le C$ it follows from (3.11) that $\langle W(t)\rangle_{A,r}$ is equivalent to a time independent semi-norm given by (3.11) with f the distance function $d(y)$, see (6.2). Since we only apply tangential vector fields, it also follows from (3.11) that, up to lower order terms that can be bounded by $\|W(t)\|_r$, it is equivalent to that the normal component of the vector field $W_N = N_a W^a$ is in $H^r(\partial\Omega)$.

Definition 16.1. With notation as in (16.25) define $H^r(\Omega)$ to be the completion of $C^\infty(\Omega)$ in the norm $\|W(t)\|_r$ and define $N^r(\Omega)$ to be the completion of the divergence free $C^\infty(\Omega)$ vector fields in the norm $\|W\|_{N^r} = \|W(t)\|_r + \langle W(t)\rangle_{A,r}$.

Since the projection onto divergence free vector fields is continuous in the H^r norm it follows that H^r is also the completion of the divergence free C^∞ vector fields in the H^r norm.

Theorem 16.2. *Suppose that $x, p \in C^{r+2}([0,T]\times\Omega)$, $p\big|_{\partial\Omega} = 0$, $\nabla_N p\big|_{\partial\Omega} \le -c_0 < 0$ and $\operatorname{div} D_t x = 0$. Then if initial data and the inhomogeneous term in (2.29) are divergence free and satisfy*

$$(W_0, W_1) \in N^r(\Omega)\times H^r(\Omega), \qquad F \in L^1\big([0,T], H^r(\Omega)\big) \tag{16.26}$$

the linearized equations (2.29) have a solution

$$(W, \dot W) \in C\big([0,T], N^r(\Omega)\times H^r(\Omega)\big) \tag{16.27}$$

Proof. The existence of a solution in (16.27) follows from section 15 if initial data and the inhomogeneous term are divergence free and C^∞ and the inhomogeneous term is supported in $t > 0$. By approximating the initial data and the inhomogeneous term in (16.26) with C^∞ divergence free vector fields and applying the estimate (16.24) to the differences we get a convergent sequence in (16.27) so the limit must also be in this space. □

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References

[BG]. M.S. Baouendi and C. Goulaouic, *Remarks on the abstract form of nonlinear Cauchy-Kovalevsky theorems*, Comm. Part. Diff. Eq. **2** (1977), 1151-1162.

[BHL]. T. Beale, T. Hou, J. Lowengrub, *Growth Rates for the Linearized Motion of Fluid Interfaces away from Equilibrium*, CPAM **XLVI(no 9)** (1993), 1269–1301.

[C1]. D. Christodoulou, *Self-Gravitating Relativistic Fluids: A Two-Phase Model*, Arch. Rational Mech. Anal. **130** (1995), 343-400.

[C2]. D. Christodoulou, *Oral Communication* (August 1995).

[CK]. D. Christodoulou and S. Klainerman, *The Nonlinear Stability of the Minkowski space-time*, Princeton Univ. Press, 1993.

[CL]. D. Christodoulou and H. Lindblad, *On the motion of the free surface of a liquid.*, Comm. Pure Appl. Math. **53** (2000), 1536-1602.

[Cr]. W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-deVries scaling limits*, Comm. in P. D. E. **10** (1985), 787-1003.

[DM]. B. Dacorogna and J. Moser, *On a partial differential equation involving the Jacobian determinant.*, Ann. Inst. H. Poincare Anal. Non. Lineaire **7** (1990), 1-26.

[E1]. D. Ebin, *The equations of motion of a perfect fluid with free boundary are not well posed.*, Comm. Part. Diff. Eq. **10** (1987), 1175–1201.

[E2]. D. Ebin, *Oral communication* (November 1997).

[L1]. H. Lindblad, *Well posedness for the linearized motion of the free surface of a liquid*, preprint (Jan 2001).

[L2]. ______, *The motion of the free surface of a liquid*, Seminaire Equations aux Derivees Partielles du Centre de Mathematiques de l'Ecole Polytechnique **VI-1-8** (2001).

[L3]. ______, *Well posedness for the motion of the free surface of a liquid*, in preparation.

[Na]. V.I. Nalimov, *The Cauchy-Poisson Problem (in Russian),*, Dynamika Splosh. Sredy 18 (1974,), 104-210.

[Ni]. T. Nishida, *A note on a theorem of Nirenberg*, J. Diff. Geometry **12** (1977), 629-633.

[W1]. S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Invent. Math. **130** (1997), 39-72.

[W2]. S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12** (1999), 445-495.

[Y]. H. Yosihara, *Gravity Waves on the Free Surface of an Incompressible Perfect Fluid* **18** (1982), Publ. RIMS Kyoto Univ., 49-96.

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