

ON THE MOTION OF THE FREE SURFACE OF A LIQUID

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INTRODUCTION

We consider Euler's equations

$$(1.1) \quad (\partial_t + v^k \partial_k) v_j = -\partial_j p, \quad j = 1, \dots, n \quad \text{in } \mathcal{D}, \quad \text{where } \partial_i = \partial / \partial x^i$$

describing the motion of an perfect incompressible fluid in vacuum:

$$(1.2) \quad \operatorname{div} v = \partial_k v^k = 0 \quad \text{in } \mathcal{D}$$

where $v = (v_1, \dots, v_n)$ and $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ are to be determined. Here $v^k = \delta^{ki} v_i = v_k$ and we have used the summation convention that repeated upper and lower indices are summed over. Given a simply connected bounded domain $\mathcal{D}_0 \subset \mathbb{R}^n$ and initial data v_0 , satisfying the constraint $\operatorname{div} v_0 = 0$, we want to find a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$ and a vector field v solving (1.1)-(1.2) and satisfying the initial conditions

$$(1.3) \quad \begin{cases} \{x; (0, x) \in \mathcal{D}\} = \mathcal{D}_0 \\ v = v_0, \quad \text{on } \{0\} \times \mathcal{D}_0 \end{cases}$$

Let $\mathcal{D}_t = \{x \in \mathbb{R}^n; (t, x) \in \mathcal{D}\}$. We also require the boundary conditions on the free boundary $\partial \mathcal{D}_t$:

$$(1.4) \quad \begin{cases} p = 0, \quad \text{on } \partial \mathcal{D}_t \\ v_{\mathcal{N}} = \kappa, \quad \text{on } \partial \mathcal{D}_t \end{cases}$$

for each t , where \mathcal{N} is the exterior unit normal to $\partial \mathcal{D}_t$, $v_{\mathcal{N}} = \mathcal{N}^i v_i$ and κ is the normal velocity of $\partial \mathcal{D}_t$. The second condition can also be expressed as $(\partial_t + v^k \partial_k)|_{\partial \mathcal{D}} \in T(\partial \mathcal{D})$. We will prove *a priori* bounds for the initial value problem (1.1)-(1.4), in Sobolev spaces under the assumption

$$(1.5) \quad \nabla_{\mathcal{N}} p \leq -\varepsilon < 0, \quad \text{on } \partial \mathcal{D}_t, \quad \text{where } \nabla_{\mathcal{N}} = \mathcal{N}^i \partial_{x^i}.$$

(1.5) is a natural *physical condition* since the pressure p has to be positive in the interior of the fluid. It is essential for the well posedness in Sobolev spaces. Taking the divergence of (1.1):

$$(1.6) \quad -\Delta p = (\partial_j v^k) \partial_k v^j, \quad \text{in } \mathcal{D}_t, \quad p = 0, \quad \text{on } \partial \mathcal{D}_t$$

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In the irrotational case (1.5) always hold, as shown in [CL,W1,W2]. Then $(\text{curl } v)_{ij} = \partial_i v^j - \partial_j v^i = 0$ so $\Delta p < 0$ and hence $p > 0$ and (1.5) hold by the strong maximum principle (see [GT]).

The incompressible perfect fluid is to be thought of as an idealization of a liquid. For small bodies like water drops surface tension should help holding it together and for larger denser bodies like stars its own gravity should play a role. Here we neglect the influence of such forces. Instead it is the incompressibility condition that prevents the body from expanding and it is the fact that the pressure is positive that prevents the body from breaking up in the interior. Let us also point out that, from a physical point of view one can alternatively think of the pressure as being a small positive constant on the boundary instead of vanishing. The aim of this paper is to show that we have *a priori* bounds in Sobolev spaces for the free boundary problem (1.1)-(1.5) in any number of space dimensions. What makes this problem difficult is that the regularity of the boundary enters to highest order. Roughly speaking, the velocity tells the boundary where to move and the boundary is the zero set of the pressure that determines the acceleration.

It is generally possible to prove local existence for analytic data for a free interface between two fluids with the same normal component of the velocity, see [BG] and [Ni] for the irrotational case. However, this type of problem might be subject to instability in Sobolev norms. The classical examples are Rayleigh-Taylor instability which occurs in a local linear analysis when a heavier fluid lies above a lighter fluid in a gravitational field and Kelvin-Helmholtz instability which occurs when the tangential velocities of the two fluids along the interface are different, see e.g. [BCS]. In our case its the first kind of instability that we must exclude. No gravitational fields are present in our problem, however a uniform exterior gravitational field would not make a difference because it can be transformed away by going to an accelerated frame. It is condition (1.5) which excludes the possibility of this kind of instability. In fact, without taking into account the sign condition (1.5) the problem is actually ill-posed in Sobolev spaces, see [Eb1],

Some existence results in Sobolev spaces are known in the irrotational case, for the closely related water wave problem which describes the motion of the surface of the ocean under the influence of earth's gravity. In that problem, the gravitational field can be considered as uniform, and as we remarked above, this problem reduces to our problem by going to an accelerated frame. The domain \mathcal{D}_t is unbounded for the water wave problem coinciding with a half-space in the case of still water. Nalimov[Na] and Yosihara[Y] proved local existence in Sobolev spaces in two space dimensions for initial conditions sufficiently close to still water. Beale, Hou and Lowengrab[BHL] have given an argument to show that problem is linearly well posed in a weak sense in Sobolev spaces, assuming a condition, which can be shown to be equivalent to (1.5). The condition (1.5) prevents the Rayleigh-Taylor instability from occurring when the water wave turns over. Recently Wu[W1,2] proved local existence in general in two and three dimensions for the water wave problem. Wu showed that (1.5) holds for an unbounded domain in the irrotational case. More importantly Wu[W2] is the first existence result in three space dimensions in Sobolev spaces; going from two to three dimensions required introduction of new techniques.

The method of proof in the above papers relies heavily on the assumption that the velocity is curl-free, hence satisfies Laplace's equation in the interior. This makes possible the reduction of the problem to a problem involving the boundary alone. In this reduction the Dirichlet to Neumann map enters and it is estimated in fractional Sobolev spaces on the boundary. In the general case, with non vanishing curl, no existence results in Sobolev spaces are known. However, recently Ebin[Eb2] announced a local existence result for the same equations but with the boundary condition containing surface tension, which makes the problem more regular.

We prove *a priori* bounds in the case of non vanishing vorticity in any number of space dimensions. We also show that the Sobolev norms remain bounded essentially as long as (1.5) hold, the second fundamental form of the surface is bounded and the first order derivatives of the velocity are bounded.

The proof works with lower regularity assumptions on initial data. This is partly due to the fact that our result is in terms of norms in the Eulerian space coordinates and the second fundamental form of the free surface. The norms are hence independent of a parametrization of the boundary so we do not have to be concerned with the possibility of a parametrization becoming singular. On the other hand it is more difficult to put up an iteration in this approach. However existence will follow from analogous estimates and existence in the presence of surface tension, reducing to the estimates presented here in the limit of vanishing surface tension. Let us also point out that an existence result even for infinitely differentiable data together with the *a priori* bounds here imply existence and continuation for low regularity data. This is in particular true in the irrotational case where existence is known.

Our approach is quite elementary and geometric in nature. We use a new type of energy that controls the geometry of the free surface. The energy has a boundary part and an interior part, which allows us to avoid the use of fractional Sobolev spaces on the boundary. The boundary part controls the norms of the second fundamental form of the free surface, whereas the interior part controls the norms of the velocity and hence the pressure. We show that the time derivative of the energy is controlled by the energy. A crucial point is that the time derivative of the interior part will, after integrating by parts, contribute with a boundary term that exactly cancels the leading order term in the time derivative of the boundary integral. The equations look ill-posed at first sight, but if one differentiates them one gets a well-posed system for higher order derivatives of the velocity and the pressure. Our energy contains the components of this higher order system. In the interior it contains most components and on the boundary only the tangential components. Due to the fact that the pressure vanishes on the boundary the tangential components of this higher order system are more regular. Another crucial point is then to estimate the projection of a tensor to the tangent space of the boundary, which involves the second fundamental form.

Let us first introduce *Lagrangian coordinates*. In these coordinates the boundary is fixed. Let Ω be a domain in \mathbf{R}^n and let $f_0 : \Omega \rightarrow \mathcal{D}_0$ be a diffeomorphism that is volume preserving; $\det(\partial f_0 / \partial y) = 1$. Assume that $v(t, x)$ and $p(t, x)$, $(t, x) \in \mathcal{D}$ are given satisfying (1.1)-(1.4). The Lagrangian coordinates $x = x(t, y) = f_t(y)$ are given by solving

$$(1.7) \quad \frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad y \in \Omega$$

Then $f_t : \Omega \rightarrow \mathcal{D}_t$ is a volume preserving diffeomorphism, since $\operatorname{div} v = 0$, and the boundary becomes fixed in the new y coordinates. Let us introduce the notation

$$(1.8) \quad D_t = \frac{\partial}{\partial t} \Big|_{y=\text{constant}} = \frac{\partial}{\partial t} \Big|_{x=\text{constant}} + v^k \frac{\partial}{\partial x^k},$$

for the material derivative and

$$(1.9) \quad \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}.$$

Sometimes it is convenient to work in the Eulerian coordinates (t, x) and sometimes it is easier to work in the Lagrangian coordinates (t, y) . In the Lagrangian picture the partial derivative with respect to the time coordinate has more direct significance than the partial derivative with respect to the time coordinate in the Eulerian picture. However this is not true for the partial derivatives with respect to the space coordinates. The notion of space derivative which plays a more significant role in the Lagrangian picture is that of covariant differentiation with respect to the metric $g_{ab}(t, y) = \delta_{ij} \partial x^i / \partial y^a \partial x^j / \partial y^b$, the pull back by f_t of the Eulerian metric δ_{ij} on $\mathcal{D}_t \subset \mathbf{R}^n$. The covariant space derivatives of the Lagrangian picture are simply and directly related to the partial derivatives with respect to the Cartesian space

coordinates of the Eulerian picture. We will work mostly in the Lagrangian coordinates in the paper. However, our statements are coordinate independent and to simplify the exposition we will present the results in the Eulerian picture in the introduction.

In the notation of (1.8)-(1.9) Euler's equations (1.1) become

$$(1.10) \quad D_t v_i = -\partial_i p$$

Note that the commutator satisfies

$$(1.11) \quad [D_t, \partial_i] = -(\partial_i v^k) \partial_k$$

By (1.11) we obtain the second order equation for the velocity

$$(1.12) \quad D_t^2 v_i - (\partial_k p) \partial_i v^k = -\partial_i D_t p$$

Our estimates make use of (1.12) restricted to the boundary together with the boundary condition:

$$(1.13) \quad p = 0, \quad \text{on } \partial \mathcal{D}_t \quad \implies \quad D_t p = 0, \quad \text{on } \partial \mathcal{D}_t$$

In the interior we will make use of the equation obtained by taking the curl of (1.10), using (1.11),

$$(1.14) \quad D_t (\text{curl } v)_{ij} = -(\partial_i v^k) (\text{curl } v)_{kj} + (\partial_j v^k) (\text{curl } v)_{ki}$$

together with

$$(1.15) \quad \text{div } v = 0, \quad \text{in } \mathcal{D}_t$$

Taking the divergence of (1.10) respectively (1.12) using (1.11) and (1.15) gives elliptic equations:

$$(1.16) \quad \Delta p = -(\partial_i v^\ell) \partial_\ell v^i, \quad \text{in } \mathcal{D}_t, \quad p = 0 \quad \text{on } \partial \mathcal{D}_t$$

$$(1.17) \quad \Delta D_t p = (\partial_k p) \Delta v^k + G(\partial v, \partial^2 p), \quad \text{in } \mathcal{D}_t, \quad D_t p = 0 \quad \text{on } \partial \mathcal{D}_t$$

where $G(\partial v, \partial^2 p) = 4\delta^{ij}(\partial_i v^k) \partial_j \partial_k p + 2(\partial_i v^j)(\partial_j v^k) \partial_k v^i$. Equation (1.16) gain regularity; neglecting the problem with the boundary regularity, one derivative of v in the interior gives two derivatives of p , which gives a gain of one time derivative of v in (1.10). If $\text{curl } v = 0$ then $\Delta v = 0$ so then the equation for $D_t p$ is as good as the equation for p .

To see the importance of the condition $\nabla_{\mathcal{N}} p \leq -\varepsilon < 0$ let us look at a simplified linear *model problem*, [CL]: Since $p = D_t p = 0$ on $\partial \mathcal{D}_t$ it follows that $\partial_i p = N_i \nabla_{\mathcal{N}} p$ and $\partial_i D_t p = N_i \nabla_{\mathcal{N}} D_t p$ there so by (1.12)

$$(1.18) \quad D_t^2 v_i - (\nabla_{\mathcal{N}} p) N^k \partial_i v_k = -(\nabla_{\mathcal{N}} D_t p) N_i, \quad \text{on } \partial \mathcal{D}_t$$

We linearize by taking $\mathcal{D}_t = \Omega$ and $x(t, y) = y$, independent of t . In the irrotational case $\mathcal{N}^k \partial_i v_k = \mathcal{N}^k \partial_k v_i = \nabla_{\mathcal{N}} v_i$ and $\Delta v_i = \delta^{jk} \partial_j \partial_k v_i = \delta^{jk} \partial_i \partial_j v_k = \partial_i \text{div } v = 0$. Let us therefore consider the equations

$$(1.19) \quad D_t^2 v_i + \nu^{-1} \nabla_{\mathcal{N}} v_i = F_i, \quad \text{on } \partial \Omega, \quad \Delta v_i = 0, \quad \text{in } \Omega$$

for a vector field v on Ω , depending on t , where ν and F_i are given functions on Ω and $D_t = \partial_t$. To simplify further let us assume that $\nu^{-1} = \varepsilon$ is constant, $F = 0$, and Ω is the unit disc in \mathbf{R}^2 . Then the solutions of $\Delta v = 0$ are given in polar coordinates by $v(t, r, \theta) = \sum c_k(t) r^{|k|} e^{ik\theta}$. The boundary condition

in (1.19) imply that $c_k''(t) + \varepsilon|k|c_k(t) = 0$, with solutions $c_k(t) = c_k^+ e^{t\lambda_k} + c_k^- e^{-t\lambda_k}$, $\lambda_k = \sqrt{-\varepsilon|k|}$, so the high frequencies remain bounded for $t > 0$ if $\varepsilon > 0$ but they are exponentially increasing if $\varepsilon < 0$. Note that if data are analytic, i.e. $c_k^\pm = o(e^{-\delta|k|})$, $\delta > 0$, then the solution exists independently of the sign condition. The model problem is related to Enbin's counterexample. By linearizing around a rigid rotation $v = (x_2, -x_1)$ he gets an equation for the variation similar to (1.19) with $\nu^{-1} = -\nabla_N p = -1$. (1.19) is also up to terms of lower order the equation Wu[W2] uses. Furthermore, a similar model problem shows up in [CL] when one studies the equation for the derivatives of the velocity (1.24)-(1.25).

The model problem also suggests a candidate for an energy:

$$(1.20) \quad E(t) = \int_{\Omega} |\partial v|^2 dx + \int_{\partial\Omega} |D_t v|^2 \nu dS, \quad \nu > 0$$

If we differentiate below the integral sign and integrate by parts we get a bound for the energy:

$$(1.21) \quad \begin{aligned} \frac{dE}{dt} &= 2 \int_{\Omega} \partial v \partial D_t v dx + 2 \int_{\partial\Omega} D_t v D_t^2 v \nu dS + \int_{\partial\Omega} |D_t v|^2 D_t \nu dS \\ &= -2 \int_{\Omega} \Delta v D_t v dx + 2 \int_{\partial\Omega} D_t v (D_t^2 v + \nu^{-1} \nabla_N v) \nu dS + \int_{\partial\Omega} |D_t v|^2 D_t \nu dS \\ &\leq 2 \|F\|_{L^2(\partial\Omega, \nu dS)} E^{1/2} + \|\nu^{-1} D_t \nu\|_{L^\infty(\partial\Omega)} E \end{aligned}$$

An easy modification gives (1.21) with an extra term $2 \|D_t \omega\|_{L^2(\Omega)} E^{1/2}$ also for a divergence free vector field, $\operatorname{div} v = 0$, with $\operatorname{curl} v = \omega$, satisfying $D_t^2 v_i + \nu^{-1} N^k \partial_i v_k = F_i$ on the boundary. This estimate is however, by itself not good enough to deal with (1.12) since we can not expect a bound for $\|\partial D_t p\|_{L^2(\partial\Omega)}$ from a bound for $\|\partial v\|_{L^2(\Omega)}$ due to the loss of regularity in (1.17) in the irrotational case. One derivative of v in the interior only gives one derivative of $D_t p$ in the interior and restricting to the boundary we loose half a derivative.

An additional idea is required which has to do with exploiting our special boundary conditions $D_t p = 0$. If we modify our energy so it only contains tangential components on and close to the boundary, then only the projection onto the tangential components of (1.12) on the boundary will occur in the energy estimate and the tangential components of $\partial D_t p$ vanishes. The components we loose control over in the energy can then be gotten back by elliptic estimates. Although the pressure and the regularity of the boundary did not enter in the above simplified model it will enter once we go to higher-order energies which is needed to close the argument. We will now develop these higher-order energies.

One can think of (1.10) and (1.12) as a system of equations for v and $\dot{v} = D_t v = -\partial p$:

$$(1.22) \quad D_t v_i = -\partial_i p$$

$$(1.23) \quad D_t \partial_i p + (\partial_k p) \partial_i v^k = \partial_i D_t p,$$

To see better what goes on let us differentiate once more with respect to the spatial coordinates

$$(1.24) \quad D_t \partial_i v_j = -\partial_i \partial_j p - (\partial_i v^k) \partial_k v_j$$

$$(1.25) \quad D_t \partial_i \partial_j p + (\partial_k p) \partial_i \partial_j v^k = \partial_i \partial_j D_t p - (\partial_i v^k) \partial_k \partial_j p - (\partial_j v^k) \partial_k \partial_i p$$

where we used (1.11).

We want to project (1.25) to the tangent space of the boundary. The orthogonal *projection* Π to the tangent space of the boundary, of a $(0, r)$ tensor α is defined to be the projection of each component along the normal:

$$(1.26) \quad (\Pi\alpha)_{i_1 \dots i_r} = \Pi_{i_1}^{j_1} \cdots \Pi_{i_r}^{j_r} \alpha_{j_1 \dots j_r}, \quad \text{where } \Pi_i^j = \delta_i^j - \mathcal{N}_i \mathcal{N}^j,$$

Let $\bar{\partial}_i = \Pi_i^j \partial_j$ be a tangential derivative. If $q = 0$ on $\partial\mathcal{D}_t$ it follows that $\bar{\partial}_i q = 0$ there and

$$(1.27) \quad (\Pi \partial^2 q)_{ij} = \theta_{ij} \nabla_{\mathcal{N}} q, \quad \text{where } \theta_{ij} = \bar{\partial}_i N_j$$

is the *second fundamental form* of $\partial\mathcal{D}_t$. In fact,

$$0 = \bar{\partial}_i \bar{\partial}_j q = \Pi_i^{i'} \partial_{i'} \Pi_j^{j'} \partial_{j'} q = \Pi_i^{i'} \Pi_j^{j'} \partial_{i'} \partial_{j'} q - (\bar{\partial}_i N_j) \mathcal{N}^k \partial_k q - N_j (\bar{\partial}_i \mathcal{N}^k) \partial_k q = (\Pi \partial^2 q)_{ij} - \theta_{ij} \nabla_{\mathcal{N}} q$$

since $N_k \bar{\partial}_i \mathcal{N}^k = \bar{\partial}_i (N_k \mathcal{N}^k) / 2 = 0$.

Our energy for the second order equation (1.25) will be a modification of (1.20) that contains only the tangential components $\Pi \partial D_t v = -\Pi \partial^2 p$ on the boundary and $(\tilde{\Pi} \partial^2) v$ in the interior, where $\tilde{\Pi}$ is an extension of the projection to the interior. Taking the time derivative of this energy and integrating by parts as in (1.21) we will get a boundary term that involves the projection of (1.25). Because $\Pi \partial^2 D_t p = \theta \nabla_{\mathcal{N}} D_t p$, this can be controlled by one less derivative $\partial D_t p$. The energy together with elliptic estimates controls two derivatives of v in the interior so (1.17) gives us two derivatives of $D_t p$ in the interior and hence one derivative on the boundary. In our discussion so far we have neglected the problem of boundary regularity, which comes in to highest order. However, our energy also controls the second fundamental form. By (1.27) and $|\nabla_{\mathcal{N}} p| \geq \varepsilon > 0$ the boundary part of the energy $|\Pi \partial^2 p|^2 \geq |\theta|^2 |\nabla_{\mathcal{N}} p|^2 \geq |\theta|^2 \varepsilon^2$, gives an estimate for the second fundamental form θ .

The *energies* we propose are of the form

$$(1.28) \quad E_r(t) = \int_{\mathcal{D}_t} \delta^{mn} Q(\partial^r v_m, \partial^r v_n) dx + \int_{\mathcal{D}_t} |\partial^{r-1} \text{curl } v|^2 dx + \int_{\partial\mathcal{D}_t} Q(\partial^r p, \partial^r p) \nu dS$$

where $\nu = (-\nabla_{\mathcal{N}} p)^{-1}$. Here Q is a positive definite quadratic form which restricted to the boundary is the inner product of the tangential components: $Q(\alpha, \beta) = \langle \Pi \alpha, \Pi \beta \rangle$ and in the interior $Q(\alpha, \alpha)$ increases to the norm $|\alpha|^2$. To be more specific, we define

$$(1.29) \quad Q(\alpha, \beta) = q^{i_1 j_1} \dots q^{i_r j_r} \alpha_{i_1 \dots i_r} \beta_{j_1 \dots j_r}$$

$$(1.30) \quad \text{where } q^{ij} = \delta^{ij} - \eta(d)^2 \mathcal{N}^i \mathcal{N}^j, \quad d(x) = \text{dist}(x, \partial\mathcal{D}_t), \quad \mathcal{N}^i = -\delta^{ij} \partial_j d.$$

Here η is a smooth cut off function satisfying $0 \leq \eta(d) \leq 1$, $\eta(d) = 1$, when $d < d_0/4$ and $\eta(d) = 0$, when $d > d_0/2$ and d_0 is a fixed number which is smaller than the *injectivity radius of the normal exponential map* ι_0 , defined to be the largest number ι_0 such that the map

$$(1.31) \quad \partial\mathcal{D}_t \times (-\iota_0, \iota_0) \rightarrow \{x \in \mathbb{R}^n : \text{dist}(x, \partial\mathcal{D}_t) < \iota_0\}, \quad \text{given by } (\bar{x}, \iota) \rightarrow x = \bar{x} + \iota \mathcal{N}(\bar{x}),$$

is an injection. These energies in fact control all components of $\partial^r v$, $\partial^r p$ and $\partial^{r-2} \theta$, see (1.41)-(1.42).

We prove an *energy estimate* implying that the energies are bounded as long as certain *a priori* assumptions are true. More specifically; we prove that there are continuous functions C_r such that

$$(1.32) \quad \left| \frac{dE_r(t)}{dt} \right| \leq C_r \left(K, 1/\varepsilon, L, M, \text{Vol } \mathcal{D}_t, E_{r-1}^*(t) \right) E_r^*(t), \quad \text{where } E_r^*(t) = \sum_{s=0}^r E_s(t)$$

if $0 \leq r \leq 4$ or $r \geq n/2 + 3/2$, provided that

$$(1.33) \quad |\theta| \leq K, \quad 1/\iota_0 \leq K \quad \text{on } \partial\mathcal{D}_t$$

$$(1.34) \quad -\nabla_{\mathcal{N}} p \geq \varepsilon > 0, \quad \text{on } \partial\mathcal{D}_t$$

$$(1.35) \quad |\partial^2 p| + |\nabla_{\mathcal{N}} D_t p| \leq L, \quad \text{on } \partial\mathcal{D}_t.$$

$$(1.36) \quad |\partial v| + |\partial p| \leq M \quad \text{in } \mathcal{D}_t,$$

The bounds (1.33) gives us control of the geometry of the free surface $\partial\mathcal{D}$. A bound for the *second fundamental form* θ gives a bound for the curvature of $\partial\mathcal{D}_t$ and a lower bound for the *injectivity radius of the normal exponential map* ι_0 measures how far off the surface is from self-intersecting.

Now, the lowest order energy and the volume are in fact conserved;

$$(1.37) \quad E_0(t) = \int_{\mathcal{D}_t} \delta^{mn} v_m v_n dx = E_0(0), \quad \text{Vol}\mathcal{D}_t = \int_{\mathcal{D}_t} dx = \text{Vol}\mathcal{D}_0.$$

Recursively it follows from (1.32) and (1.37):

Theorem 1.1. *Let $n \leq 7$. Then there are continuous functions \mathcal{F}_r , $r = 0, 1, \dots$, with $\mathcal{F}_r|_{t=0} = 1$, such that any smooth solution of the free boundary problem for Euler's equations (1.1)-(1.5), for $0 \leq t \leq T$, that satisfy the a priori assumptions (1.33)-(1.36) also satisfy the energy bound:*

$$(1.38) \quad E_r^*(t) \leq \mathcal{F}_r(t, K, 1/\varepsilon, L, M, E_{r-1}^*(0), \text{Vol}\mathcal{D}_0) E_r^*(0), \quad 0 \leq t \leq T,$$

Most of the *a priori* bounds (1.33)-(1.36) can be obtained from the energy through (1.41) and (1.42) below using Sobolev's lemma if $r > (n-1)/2 + 2$. However, the lower bounds for ε and ι_0 can not be obtained in this way but instead one has to try to get evolution equations for these.

Let $K(0)$ and $\varepsilon(0)$ be the minimum respectively maximum values such that (1.33) and (1.34) hold when $t = 0$.

Theorem 1.2. *Let r_0 be the smallest integer such that $r_0 > n/2 + 3/2$. Then there are continuous functions $\mathcal{T}_r > 0$, $r = r_0, r_0 + 1, \dots$, such that if*

$$(1.39) \quad T \leq \mathcal{T}_r(K(0), 1/\varepsilon(0), E_{r_0}^*(0), \text{Vol}\mathcal{D}_0)$$

then any smooth solution of the free boundary problem for Euler's equations (1.1)-(1.5), for $0 \leq t \leq T$ satisfies

$$(1.40) \quad E_r^*(t) \leq 2E_r^*(0), \quad 0 \leq t \leq T$$

Remarks. The restriction $n \leq 7$ in Theorem 1. 1, i.e. the restriction for (1.32) to hold is just a matter of that the proof becomes simpler in this case. The assumption that $\text{Vol}\mathcal{D}_0 < \infty$ is just used to get an L^2 estimate for p so it could be omitted if we add $\int p^2 dx$ to the energy. We only need a lower bound for the interior radius of injectivity of the normal exponential map in (1.31) for the energy estimates to hold. The bound for the exterior one is to prevent the surface from self intersecting.

Let us first point out that since $\text{div } v = 0$ and $-\Delta p = (\partial_i v^k) \partial_k v^i$ one can use *elliptic estimates* to control all components of $\partial^r v$ and $\partial^r p$ from the tangential components $\Pi \partial^r p$ in the energy:

$$(1.41) \quad \|\partial^r v\|_{L^2(\mathcal{D}_t)}^2 + \|\partial^{r-1} v\|_{L^2(\partial\mathcal{D}_t)}^2 + \|\partial^r p\|_{L^2(\partial\mathcal{D}_t)}^2 + \|\partial^r p\|_{L^2(\mathcal{D}_t)}^2 \leq C(K, M, \text{Vol}\mathcal{D}_0) E_r^*$$

A bound for the energy also implies a *bound for the second fundamental form* of the free boundary:

$$(1.42) \quad \|\bar{\partial}^{r-2} \theta\|_{L^2(\partial\mathcal{D}_t)}^2 \leq C(K, L, M, 1/\varepsilon, E_{r-1}^*, \text{Vol}\mathcal{D}_t) E_r^*$$

that control the regularity. In fact, we prove higher order versions of the *projection formula* (1.27):

$$(1.43) \quad \Pi \partial^r q = (\bar{\partial}^{r-2} \theta) \nabla_{\mathcal{N}} q + O(\partial^{r-1} q) + O(\bar{\partial}^{r-3} \theta), \quad \text{if } q = 0, \quad \text{on } \partial\mathcal{D}_t$$

Since $|\nabla_{\mathcal{N}} p| \geq \varepsilon > 0$ it follows from (1.43) that $|\bar{\partial}^{r-2}\theta| \leq C|\Pi\partial^r p| + O(\partial^{r-1}p) + O(\bar{\partial}^{r-3}\theta)$ where the lower order terms can be bounded using (1.41) and (1.42) for smaller r so (1.42) follows inductively.

Once we have the bound (1.42) for the second fundamental form we can get estimates for any solution of the Dirichlet problem. In particular since $D_t p$ satisfies the elliptic equation (1.17) we get

$$(1.44) \quad \|\Pi\partial^r D_t p\|_{L^2(\partial\mathcal{D}_t)}^2 + \|\partial^{r-1} D_t p\|_{L^2(\partial\mathcal{D}_t)}^2 \leq C(K, L, M, 1/\varepsilon, E_{r-1}^*, \text{Vol } \mathcal{D}_t) E_r^*$$

This follows from the elliptic estimates, used to prove (1.41), and (1.43) applied to $D_t p$, where now $\bar{\partial}^{r-2}\theta$ is bounded by (1.42) and $\partial^{r-1} D_t p$ is lower order. $\Pi\partial^r D_t p$ shows up in the energy estimate when we take the time derivative of the boundary part of the energy $\Pi\partial^r p$. Although a bound for the energy implies bounds for all components of $\partial^r p$ we can not bound the time derivative of the non-tangential components on the boundary in the case of non vanishing curl since the elliptic estimates only gives control of the tangential components $\Pi\partial^r D_t p$ in (1.44) because of the term with Δv in (1.17).

Let us now outline the proof of Theorem 1.1 and Theorem 1.2. First, we explain the proof of the *energy estimate* (1.32) which uses integration by parts as in the model problem. Then we give the main *elliptic estimates* and the *projection formula* used in proving (1.41)-(1.44). Finally, we discuss how to control the *geometry of the free surface* and the *a priori* bounds (1.33)-(1.36); the time evolution of ι_0 , ε and other geometric quantities that control the Sobolev constants, that is needed for Theorem 1.2.

Energy estimates. (Sections 7,5) We will now outline the proof of the energy estimate (1.32). In order to take the time derivative of the energy (1.28) we make use of the fact that if f is an arbitrary function on $\bar{\mathcal{D}}_t$, depending on t , then

$$\frac{d}{dt} \int_{\mathcal{D}_t} f dx = \int_{\mathcal{D}_t} D_t f dx \quad \text{and} \quad \frac{d}{dt} \int_{\partial\mathcal{D}_t} f dS = \int_{\partial\mathcal{D}_t} (D_t f - (\nabla_{\mathcal{N}} v_{\mathcal{N}}) f) dS$$

since $\text{div } v = 0$ (this can be seen, e.g. using the Lagrangian coordinates.) We have

$$(1.45) \quad \frac{dE_r}{dt} = \int_{\mathcal{D}_t} D_t (\delta^{mn} Q(\partial^r v_m, \partial^r v_n) + |\partial^{r-1} \text{curl } v|^2) dx \\ + \int_{\partial\mathcal{D}_t} D_t (Q(\partial^r p, \partial^r p) \nu) - Q(\partial^r p, \partial^r p) \nu \nabla_{\mathcal{N}} v_{\mathcal{N}} dS$$

The derivatives of the coefficients of Q and the measures can be bounded by the constants in (1.33)-(1.36):

$$(1.46) \quad |D_t q^{ij}| \leq CM, \quad |\partial q^{ij}| \leq CK, \quad |\nabla_{\mathcal{N}} v_{\mathcal{N}}| \leq CM,$$

see section 3. The time derivative of the higher order tensors $\partial^r v$ and $\partial^r p$ can be obtained from (1.22)-(1.23) by repeated use of (1.11)

$$(1.47) \quad D_t \partial^r v_n = -\partial^r \partial_n p + \sum_{0 \leq s \leq r-1} c_{sr} (\partial^{s+1} v) \cdot \partial^{r-s} v_n$$

$$(1.48) \quad D_t \partial^r p + (\partial_k p) \partial^r v^k = \partial^r D_t p + \sum_{0 \leq s \leq r-2} d_{sr} (\partial^{s+1} v) \cdot \partial^{r-s} p$$

where the symmetrized dot-product is defined in Lemma 2.4. Now

$$(1.49) \quad \|(\partial^{s+1} v) \cdot \partial^{r-s} v\|_{L^2(\mathcal{D}_t)} \leq C(K) \|\partial v\|_{L^\infty(\mathcal{D}_t)} \sum_{s \leq r} \|\partial^s v\|_{L^2(\mathcal{D}_t)}, \quad 0 \leq s \leq r-1.$$

This is clear for $s = 0, r-1$ and follows in general by interpolation. Hence by (1.45)-(1.48) and (1.41)

$$(1.50) \quad \frac{dE_r}{dt} = -2 \int_{\mathcal{D}_t} \delta^{mn} Q(\partial^r v_m, \partial_n \partial^r p) dx + 2 \int_{\partial \mathcal{D}_t} Q(\partial^r p, D_t \partial^r p) \nu dS + \text{Lower Order}$$

where 'Lower Order' means something that is controlled by the energy E_r^* and $K, L, M, 1/\varepsilon$ so it can be bounded by the right hand side of (1.32). If we integrate by parts in the first term we get

$$(1.51) \quad \frac{dE_r}{dt} = 2 \int_{\mathcal{D}_t} \delta^{mn} Q(\partial^r \partial_n v_m, \partial^r p) dx + 2 \int_{\partial \mathcal{D}_t} Q(\partial^r p, D_t \partial^r p - \nu^{-1} \mathcal{N}_m \partial^r v^m) \nu dS + \text{L.O.}$$

The first term vanishes since $\text{div} v = 0$. Since $-\nu^{-1} \mathcal{N}_m = \partial_m p$ the second is the inner product of $\Pi \partial^r p$ and

$$(1.52) \quad \Pi(D_t \partial^r p + (\partial_m p) \partial^r v^m) = \Pi(\partial^r D_t p) + \sum_{0 \leq s \leq r-2} d_{sr} \Pi((\partial^{s+1} v) \cdot \partial^{r-s} p)$$

by (1.48). Here $\Pi \partial^r D_t p$ is under control by (1.44) and we really need to use the projection since in the case of non-vanishing curl we can not control all components of $\partial^r D_t p$ on the boundary. The other terms in (1.52) are bounded by the *a priori* assumptions times (1.41). This is clear for $s = 0, r-2$ but dealing with the intermediate terms is the most involved part of the manuscript. This is because the interpolation has to be done on the boundary and the expression involves non tangential components. Note that if $0 \leq r \leq 2$ then the boundary terms simplify and the lower order terms are easily bounded by (1.32). The boundary terms vanish if $r = 0, 1$ and if $r = 2$ then $Q(\partial^2 p, \partial^2 p) = |\Pi \partial^2 p|^2 = |\theta|^2 |\nabla_N p|^2$, where $|\nabla_N p| \geq \varepsilon > 0$ and $Q(\partial^2 D_t p, \partial^2 D_t p) = |\theta|^2 |\nabla_N D_t p|^2$.

Elliptic estimates using the energy bound. (section 5) The bounds (1.41) follows from:

$$(1.53) \quad |\partial^r v|^2 \leq C(\delta^{mn} Q(\partial^r v_n, \partial^r v_m) + |\partial^{r-1} \text{div} v|^2 + |\partial^{r-1} \text{curl} v|^2)$$

$$(1.54) \quad \|\partial^r p\|_{L^2(\partial \mathcal{D}_t)}^2 + \|\partial^r p\|_{L^2(\mathcal{D}_t)}^2 \leq C(K, \text{Vol } \mathcal{D}_t) \sum_{s \leq r} \left(\|\Pi \partial^s p\|_{L^2(\partial \mathcal{D}_t)}^2 + \|\partial^{s-1} \Delta p\|_{L^2(\mathcal{D}_t)}^2 \right)$$

In fact, using that the measure in the boundary part of the energy $\geq \|\nabla_N p\|_{L^\infty}^{-1} dS$, respectively (1.16) and (1.49) we get

$$(1.55) \quad \|\Pi \partial^r p\|_{L^2(\partial \mathcal{D}_t)}^2 \leq \|\partial p\|_{L^\infty(\partial \mathcal{D}_t)} E_r \quad \text{and} \quad \|\partial^{r-1} \Delta p\|_{L^2(\mathcal{D}_t)}^2 \leq C \|\partial v\|_{L^\infty(\mathcal{D}_t)}^2 E_r$$

(1.53) follows as $\text{curl} v$ is the antisymmetric part of ∂v so only the symmetric part of $\partial^r v$ needs to be estimated and, moreover, the first term in the right contains one normal component while, since $\mathcal{N}^m \mathcal{N}^n \partial_m v_n = -q^{mn} \partial_m v_n + \delta^{mn} \partial_m v_n$, two normal components can be expressed in terms of tangential components and the divergence. (1.54) follows inductively from the following inequalities

$$(1.56) \quad \|\partial^r p\|_{L^2(\partial \mathcal{D}_t)}^2 \leq C \|\Pi \partial^r p\|_{L^2(\partial \mathcal{D}_t)}^2 + C(\|\partial^{r-1} \Delta p\|_{L^2(\mathcal{D}_t)} + K \|\partial^r p\|_{L^2(\mathcal{D}_t)}) \|\partial^r p\|_{L^2(\mathcal{D}_t)}$$

$$(1.57) \quad \|\partial^r p\|_{L^2(\mathcal{D}_t)}^2 \leq \|\partial^r p\|_{L^2(\partial \mathcal{D}_t)} \|\partial^{r-1} p\|_{L^2(\partial \mathcal{D}_t)} + \|\partial^{r-2} \Delta p\|_{L^2(\mathcal{D}_t)}^2$$

$$(1.58) \quad \|p\|_{L^2(\mathcal{D}_t)} \leq C(\text{Vol } \mathcal{D}_t)^{1/n} \|\Delta p\|_{L^2(\mathcal{D}_t)}, \quad \text{if } p = 0 \text{ on } \partial \mathcal{D}_t$$

The estimate (1.56) follows from repeated use of the fact that the square of the normal derivative minus the square of the tangential one behaves better on the boundary: Let \tilde{Q} be any quadratic form acting on $(0, r)$ tensors, constructed from δ^{ij} and q^{ij} and let $\tilde{\mathcal{N}} = \eta(d)\mathcal{N}$ be an extension of the normal to the interior, see (1.30). Let $T_{ij} = 2\tilde{Q}(\partial_i \alpha, \partial_j \alpha) - \delta_{ij} \delta^{mn} \tilde{Q}(\partial_m \alpha, \partial_n \alpha)$. Then $\partial_i T_j^i = 2\tilde{Q}(\Delta \alpha, \partial_j \alpha) + 2\delta^{im} (\partial_i \tilde{Q})(\partial_m \alpha, \partial_j \alpha) - \delta^{mn} (\partial_j \tilde{Q})(\partial_m \alpha, \partial_n \alpha)$ so

$$\left| \int_{\partial \mathcal{D}_t} (\mathcal{N}^i \mathcal{N}^j - q^{ij}) \tilde{Q}(\partial_i \alpha, \partial_j \alpha) dS \right| = \left| \int_{\partial \mathcal{D}_t} \mathcal{N}^i \mathcal{N}^j T_{ij} dS \right| = \left| \int_{\mathcal{D}_t} \partial_i (\tilde{\mathcal{N}}^j T_j^i) dx \right| \leq \int_{\mathcal{D}_t} 2|\Delta \alpha| |\partial \alpha| + CK |\partial \alpha|^2 dx$$

by the divergence theorem. (1.57) is integration by parts twice. (1.58) is Faber-Krahn's theorem, see [SY].

The projection formula and estimate for the second fundamental form. (section 4) We prove an *estimate for the projection*: If $q = 0$ on $\partial\mathcal{D}_t$ then for $m = 0, 1$ and $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$:

$$(1.59) \quad \|\Pi\partial^r q - (\nabla_{\mathcal{N}}q)\bar{\partial}^{r-2}\theta\|_{L^2} \leq \varepsilon\|\nabla_{\mathcal{N}}q\|_{L^\infty}\|\bar{\partial}^{r-2}\theta\|_{L^2} + C_\varepsilon\|\theta\|_{L^\infty}\|\partial^{r-1}q\|_{L^2} \\ + C(\|\theta\|_{L^\infty})\left(\|\theta\|_{L^\infty} + \sum_{s \leq r-2-m} \|\bar{\partial}^s\theta\|_{L^2}\right) \sum_{s \leq r-2+m} \|\partial^s q\|_{L^2}$$

for any $\varepsilon > 0$, where $L^p = L^p(\partial\mathcal{D}_t)$ and θ is the *second fundamental form*. The bound (1.42) for the second fundamental form θ follows from (1.41) and (1.59) using the *a priori* bound $|\nabla_{\mathcal{N}}p| \geq \varepsilon\|\nabla_{\mathcal{N}}p\|_{L^\infty}/2$.

Let us now briefly discuss the proof of (1.59). In section 4 we derive a *formula for the projection*:

$$(1.60) \quad \Pi\partial^r q = \bar{\partial}^r q + \nabla_{\mathcal{N}}q\bar{\partial}^{r-2}\theta + \sum_{\ell=1}^{r-2} \binom{r}{\ell} (\bar{\partial}^{r-2-\ell}\theta) \tilde{\otimes} (\bar{\partial}^\ell \nabla_{\mathcal{N}}q) \\ + \sum_{\substack{r_0+r_1+\dots+r_k+\ell=r-k \\ k-\ell=m=0 \pmod{2}, k \geq \ell \geq 0, k \geq 2}} a_{r_0\dots r_k \ell m} C^m (\bar{\partial}^{r_1}\theta \tilde{\otimes} \dots \tilde{\otimes} \bar{\partial}^{r_k}\theta \tilde{\otimes} \bar{\partial}^{r_0}\nabla_{\mathcal{N}}^\ell q)$$

where $\theta = \partial\mathcal{N}$ is the second fundamental form, $\tilde{\otimes}$ stands for some partial symmetrization of the tensor product and C^m stands for contraction over m pairs of indices, see section 4. Note that in (1.60) the total number of derivatives decreases by one as the number of factors of θ increases by one. Therefore, since we have assume that we have control of $\|\theta\|_{L^\infty}$, the terms on the second row will be lower order. (1.60) follows by expressing tangential derivatives of normal derivatives as projections onto tangential and normal components. The general form of the terms in (1.60) follows from the fact that the projections are defined in terms of the normal and each time a derivative falls on the normal we get a factor of θ and at the same time the total number of derivatives decreases by one. One way to obtain the leading order terms is to expand q in the distance to the boundary $d(x) = \text{dist}(x, \partial\mathcal{D}_t)$. To highest order $\Pi\partial^r q \sim \bar{\partial}^r q$. To calculate the next terms let us assume that $q = 0$ on $\partial\mathcal{D}_t$. Then $q/d = \nabla_{\mathcal{N}}q$ on $\partial\mathcal{D}_t$ and since $d = \Pi d = 0$ and $\theta = \nabla d$ on $\partial\mathcal{D}_t$ we have

$$(1.61) \quad \Pi\partial^r q = \Pi\partial^r(d\frac{q}{d}) = \sum_{\ell=0}^{r-2} \binom{r}{\ell} \Pi(\partial^{r-2-\ell}\theta) \tilde{\otimes} \Pi\partial^\ell(\frac{q}{d}) = \sum_{\ell=0}^{r-2} \binom{r}{\ell} (\bar{\partial}^{r-2-\ell}\theta) \tilde{\otimes} (\bar{\partial}^\ell \nabla_{\mathcal{N}}q) + \text{L.O.}$$

where here ‘‘L.O.’’ means terms that contain at least one more factor of θ . In section 8 we give interpolation inequalities to deal with the products on the first row of (1.60)

$$(1.62) \quad \|\|\bar{\partial}^\ell \nabla_{\mathcal{N}}q\|\bar{\partial}^{r-2-\ell}\theta\|_{L^2(\partial\mathcal{D}_t)} \leq \varepsilon\|\nabla_{\mathcal{N}}q\|_{L^\infty(\partial\mathcal{D}_t)}\|\bar{\partial}^{r-2}\theta\|_{L^2(\partial\mathcal{D}_t)} + C_\varepsilon\|\theta\|_{L^\infty(\partial\mathcal{D}_t)}\|\bar{\partial}^{r-2}\nabla_{\mathcal{N}}q\|_{L^2(\partial\mathcal{D}_t)}$$

The lower order terms on the second row of (1.60) are estimated by interpolation and Sobolev’s lemma.

Elliptic estimates using the bound for the second fundamental form. (section 5) If $q = 0$ on $\partial\mathcal{D}_t$ and $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$ then we obtain the following estimate from (1.59) and (1.54)

$$(1.63) \quad \|\partial^{r-1}q\|_{L^2(\partial\mathcal{D}_t)} \leq C(K, \text{Vol } \mathcal{D}_t, \|\theta\|_{L^2(\partial\mathcal{D}_t)}, \dots, \|\bar{\partial}^{r-3}\theta\|_{L^2(\partial\mathcal{D}_t)}) \left(\|\nabla_{\mathcal{N}}q\|_{L^\infty(\partial\mathcal{D}_t)} + \sum_{s \leq r-2} \|\nabla^s \Delta q\|_{L^2(\mathcal{D}_t)} \right)$$

If in addition $r > (n-1)/2 + 2$ then it follows from (1.59), (1.54) and Sobolev’s lemma:

$$(1.64) \quad \|\partial^{r-1}q\|_{L^2(\partial\mathcal{D}_t)} + \|\partial q\|_{L^\infty(\partial\mathcal{D}_t)} \leq C(K, \text{Vol } \mathcal{D}_t, \|\theta\|_{L^2(\partial\mathcal{D}_t)}, \dots, \|\bar{\partial}^{r-3}\theta\|_{L^2(\partial\mathcal{D}_t)}) \sum_{s \leq r-2} \|\nabla^s \Delta q\|_{L^2(\mathcal{D}_t)}$$

(1.63) together with (1.42) now gives a bound for $\|\partial^{s-1}D_t p\|_{L^2(\partial\mathcal{D}_t)}$, for $s \leq r$ since, by (1.17), $\|\partial^{s-2}\Delta D_t p\|_{L^2(\mathcal{D}_t)} = \|O(\partial^s p) + O(\partial^s v)\|_{L^2(\mathcal{D}_t)}$ is bounded by (1.41) for $s \leq r$ and since $\|\nabla_{\mathcal{N}}D_t p\|_{L^\infty(\partial\mathcal{D}_t)}$ is bounded by the *a priori* assumptions. The bound for $\|\partial^{s-1}D_t p\|_{L^2(\partial\mathcal{D}_t)}$ for $s \leq r$ together with (1.59) and (1.42) gives (1.44). This suffices to prove the energy estimate. However, in order to prove Theorem 1.2 we also need to get back bounds for the *a priori* assumptions which is where (1.64) will be used.

Bounds for the geometry and the *a priori* assumptions. (sections 3,7) We need to control the Sobolev constants for the surface and the derivatives of the coefficients of the quadratic form Q . These are easily controlled by an upper bound for the second fundamental form θ and a lower bound for the injectivity radius of the normal exponential map ι_0 . This proves Theorem 1.1. To prove Theorem 1.2 we also need to control the time evolution of the *a priori* assumptions (1.33)-(1.36). However, there is a difficulty with (1.33) because we do not have an evolution equation for ι_0 and the evolution equation for θ loses regularity, so we have to control these in an indirect way. It turns out that in order to control the Sobolev constants, for the interior as well as for the boundary (see Lemma 8.4 respectively Lemma 8.2), the constant in the elliptic estimate (1.41) and in the interpolation inequality (1.49) it suffices to have an upper bound $1/\iota_1 \leq K_1$ instead of (1.33), where $\iota_1 = \iota_1(\varepsilon_1)$ is defined to be the largest number such that

$$(1.65) \quad |\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1, \quad \text{whenever} \quad |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t$$

for some fixed number $0 < \varepsilon_1 < 2$. To prove this one makes a partition of unity into neighborhoods where (1.65) hold. An upper bound for θ and a lower bound for ι_1 then implies a lower bound for ι_0 :

$$(1.66) \quad \iota_0 \geq \min(\iota_1/2, 1/\|\theta\|_{L^\infty})$$

In fact, suppose that $x^* = \bar{x} - \iota_0 N(\bar{x})$, $\bar{x} \in \partial\mathcal{D}_t$ is a point in \mathcal{D}_t such that the interior normal exponential map of $\partial\mathcal{D}_t$ fails to be injective just beyond x^* along the normal line $\lambda \rightarrow \bar{x} - \lambda N(\bar{x})$, while $\text{dist}(x^*, \partial\mathcal{D}_t) = \iota_0$; the injectivity radius. Then either x^* is a focal point, i.e. θ has an eigenvalue $1/\iota_0$, or the line $\lambda \rightarrow \bar{x} - \lambda N(\bar{x})$, is contained in \mathcal{D}_t for all $\lambda \in (0, 2\iota_0)$ and intersects $\partial\mathcal{D}_t$ normally at $\lambda = 2\iota_0$, in which case (1.65) can not be true for the two endpoints. Since a similar argument holds for the exterior normal exponential map (1.66) follows.

The bounds (1.35)-(1.36) are easily controlled by the energy using (1.41), where K can be replaced by $K_1 \geq 1/\iota_1$, and Sobolev's lemma if $r \geq r_0 > n/2 + 3/2$: By Sobolev's lemma(Lemma 8.4) and (1.53)

$$(1.67) \quad \|v\|_{L^\infty(\mathcal{D}_t)}^2 + \|\partial v\|_{L^\infty(\mathcal{D}_t)}^2 \leq C(K_1) \sum_{s=0}^{r_0} \|\partial^s v\|_{L^2(\mathcal{D}_t)}^2 \leq C(K_1) E_{r_0}^*$$

The proof of that we can replace K by K_1 in (1.54) however requires some work, see Lemma 5.5. By (1.54), (1.55) (note that p enters quadratically in the left and linearly in the right), (1.67) and Sobolev's lemma(Lemma 8.4 respectively Lemma 8.2)

$$(1.68) \quad \|\partial p\|_{L^\infty(\mathcal{D}_t)}^2 + \|\partial^2 p\|_{L^\infty(\partial\mathcal{D}_t)}^2 \leq C(K_1, \text{Vol } \mathcal{D}_0, E_{r_0}^*)$$

Since the evolution equation for θ loses regularity and since the L^2 estimate for θ depends on the L^∞ estimate we will control it in an indirect way. By (1.27) and (1.68)

$$(1.69) \quad \|\theta\|_{L^\infty} \leq \mathcal{E} \|\Pi \partial^2 p\|_{L^\infty(\partial\mathcal{D}_t)} \leq \mathcal{E} \|\partial^2 p\|_{L^\infty(\partial\mathcal{D}_t)} \leq C(K_1, \text{Vol } \mathcal{D}_0, \mathcal{E}, E_{r_0}^*),$$

$$(1.70) \quad \text{where} \quad \mathcal{E}(t) = \|(\nabla_{\mathcal{N}} p(t, \cdot))^{-1}\|_{L^\infty(\partial\mathcal{D}_t)}$$

The estimate for $\|\nabla_{\mathcal{N}} D_t p\|_{L^\infty(\partial\mathcal{D}_t)}$ follows from (1.64).

It remains to control the evolution of K_1 and \mathcal{E} . The bound for K_1 follows since we can control the time evolution of the boundary in the Lagrangian coordinates: $x(t, y)$ and of the normal $\mathcal{N}(x(t, y))$

$$(1.71) \quad D_t x = v, \quad \text{and} \quad D_t \mathcal{N}_i = -(\bar{\partial}_i v_k) \mathcal{N}^k.$$

where the right hand sides are bounded by (1.67). We also have evolution equations for \mathcal{E} and E_r

$$(1.72) \quad |d\mathcal{E}/dt| \leq \|\nabla_{\mathcal{N}} D_t p\|_{L^\infty} \mathcal{E}^2 \leq C(K_1, \mathcal{E}, E_{r_0}^*, \text{Vol } \mathcal{D}_0)$$

$$(1.73) \quad |dE_r/dt| \leq C(K_1, \mathcal{E}, E_{\max(r_0, r-1)}^*, \text{Vol } \mathcal{D}_0) E_r^*$$

Assuming (1.65), the energy bound (1.40) and the bound $\mathcal{E}(t) \leq 2\mathcal{E}(0)$, integration of (1.71)-(1.73) gives back slightly better bounds if $t \leq \mathcal{T}(K_1(0), \mathcal{E}(0), E_{r_0}^*(0), \text{Vol}(\mathcal{D}_0))$ is sufficiently small, so Theorem 1.2 follows. In fact, integrating (1.71) using (1.67) we see that the change in \mathcal{N} and x are under control if $t \leq \mathcal{T}$ is small. Hence we get back the bound (1.65) if it is true with $\varepsilon_1/2$ and $2t_1$ initially.

2. TRANSFORMATION OF THE FREE BOUNDARY TO A FIXED BOUNDARY. LAGRANGIAN COORDINATES, THE METRIC AND COVARIANT DIFFERENTIATION IN THE INTERIOR.

Assume that we are given a velocity vector field $v(t, x)$ defined in a set $\mathcal{D} \subset [0, T] \times \mathbb{R}^n$, such that the boundary of $\mathcal{D}_t = \{x; (t, x) \in \mathcal{D}\}$ moves with the velocity, i.e. $(1, v) \in T(\partial\mathcal{D})$. We will now introduce Lagrangian or comoving coordinates, i.e. coordinates which are constant along the integral curves of the velocity vector field so that the boundary becomes fixed in these coordinates. Let $x = f_t(y)$ be the change of variables given by:

$$(2.1) \quad \frac{dx}{dt} = v(t, x(t, y)), \quad x(0, y) = f_0(y), \quad \text{if } (t, y) \in [0, T] \times \Omega$$

Initially, when $t = 0$, we can either start with the Euclidean coordinates in $\Omega = \mathcal{D}_0$ or we can start with some other coordinates $f_0 : \Omega \rightarrow \mathcal{D}_0$, where f_0 is a diffeomorphism, in which the domain Ω becomes simple. For each t we will then have a change of coordinates $f_t : \Omega \rightarrow \mathcal{D}_t = \{x; (t, x) \in \mathcal{D}\}$, taking $y \rightarrow x(t, y)$. The Euclidean metric δ_{ij} in \mathcal{D}_t then induces a metric

$$(2.2) \quad g_{ab}(t, y) = \delta_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$$

in Ω for each fixed t . We will use covariant differentiation in Ω , with respect to the metric $g_{ab}(t, y)$, since it corresponds to differentiation in \mathcal{D}_t under the change of coordinates $\Omega \ni y \rightarrow x(t, y) \in \mathcal{D}_t$ and we will work in both coordinate systems. This also avoids possible singularities in the change of coordinates. We will denote covariant differentiation in the y_a coordinates by ∇_a , $a = 0, \dots, n$, and differentiation in the x_i coordinates by ∂_i , $i = 1, \dots, n$. Covariant differentiation

The covariant differentiation of a $(0, r)$ tensor $k(t, y)$ is the $(0, r + 1)$ tensor given by

$$(2.3) \quad \nabla_a k_{a_1 \dots a_r} = \frac{\partial k_{a_1 \dots a_r}}{\partial y^a} - \Gamma_{aa_1}^d k_{d \dots a_r} - \dots - \Gamma_{aa_r}^d k_{a_1 \dots d}$$

where the Christoffel symbols Γ_{ab}^d are given by

$$(2.4) \quad \Gamma_{ab}^c = \frac{g^{cd}}{2} \left(\frac{\partial g_{bd}}{\partial y^a} + \frac{\partial g_{ad}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^d} \right) = \frac{\partial y^c}{\partial x^i} \frac{\partial^2 x^i}{\partial y^a \partial y^b}$$

where g^{cd} is the inverse of g_{ab} . If $w(t, x)$ is $(0, r)$ tensor expressed in the x coordinates then the the same tensor $k(t, y)$ expressed in the y coordinates is given by

$$(2.5) \quad k_{a_1 \dots a_r}(t, y) = \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} w_{i_1 \dots i_r}(t, x), \quad x = x(t, y)$$

and by the transformation properties for tensors

$$(2.6) \quad \nabla_a k_{a_1 \dots a_r} = \frac{\partial x^i}{\partial y^a} \frac{\partial x^{i_1}}{\partial y^{a_1}} \dots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial w_{i_1 \dots i_r}}{\partial x^i},$$

Covariant differentiation is constructed so the norms of tensors are invariant under changes of coordinates:

$$(2.7) \quad g^{a_1 b_1} \dots g^{a_r b_r} k_{a_1 \dots a_r} k_{b_1 \dots b_r} = \delta^{i_1 j_1} \dots \delta^{i_r j_r} w_{i_1 \dots i_r} w_{j_1 \dots j_r}$$

Furthermore, expressed in the y coordinates

$$(2.8) \quad \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^a}{\partial x^i} \frac{\partial}{\partial y^a}$$

Since the curvature vanishes in the x coordinates it must do so in the y coordinates and hence

$$(2.9) \quad [\nabla_a, \nabla_b] = 0$$

Let us introduce the notation $k_{a\dots b\dots c} = g^{bd} k_{a\dots d\dots c}$ and recall that covariant differentiation commutes with lowering and rising indices; $g^{ce} \nabla_a k_{b\dots e\dots d} = \nabla_a g^{ce} k_{b\dots e\dots d}$. Let us also introduce a notation for the material derivative;

$$(2.10) \quad D_t = \frac{\partial}{\partial t} \Big|_{y=\text{constant}} = \frac{\partial}{\partial t} \Big|_{x=\text{constant}} + v^k \frac{\partial}{\partial x^k}$$

In this section, indices a, b, c, \dots will refer to quantities in the y coordinates and indices i, j, k, \dots will refer to quantities in the x coordinates.

It is now important to be able to compute time derivatives of the change of coordinates and commutators between time derivatives and space derivatives.

Lemma 2.1. *Let $x = f_t(y)$ be the change of variables given by (2.1) and let g_{ab} be the metric given by (2.2). Let $v_i = \delta_{ij} v^j = v^i$ and set*

$$(2.11) \quad u_a(t, y) = v_i(t, x) \partial x^i / \partial y^a, \quad u^a = g^{ab} u_b, \quad h_{ab} = D_t g_{ab}, \quad h^{ab} = g^{ac} g^{bd} h_{cd}$$

Then

$$(2.12) \quad D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial x^k}{\partial y^a} \frac{\partial v_i}{\partial x^k}, \quad D_t \frac{\partial y^a}{\partial x^i} = -\frac{\partial y^a}{\partial x^k} \frac{\partial v_k}{\partial x^i},$$

$$(2.13) \quad D_t g_{ab} = \nabla_a u_b + \nabla_b u_a, \quad D_t g^{ab} = -h^{ab}, \quad D_t d\mu_g = g^{ab} h_{ab} d\mu_g / 2$$

$$(2.14) \quad D_t \Gamma_{ab}^c = \nabla_a \nabla_b u^c,$$

where $d\mu_g$ is the Riemannian volume element on Ω in the metric g .

Proof. We have

$$D_t \frac{\partial x^i}{\partial y^a} = \frac{\partial D_t x^i}{\partial y^a} = \frac{\partial v_i}{\partial y^a} = \frac{\partial x^k}{\partial y^a} \frac{\partial v_i}{\partial x^k}$$

which proves the first part of (2.12). Furthermore

$$0 = D_t \left(\frac{\partial y^b}{\partial x^i} \frac{\partial x^j}{\partial y^b} \right) = \left(D_t \frac{\partial y^b}{\partial x^i} \right) \frac{\partial x^j}{\partial y^b} + \frac{\partial y^b}{\partial x^i} D_t \frac{\partial x^j}{\partial y^b}$$

Multiplying by $\partial y^a / \partial x^j$ and using the first part of (2.12) now gives the second part. To prove the first part of (2.13) we note that that by (2.2) $D_t g_{ab}$ is the sum over i of

$$D_t \left(\frac{\partial x^i}{\partial y^a} \frac{\partial x^i}{\partial y^b} \right) = \left(D_t \frac{\partial x^i}{\partial y^a} \right) \frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a} \left(D_t \frac{\partial x^i}{\partial y^b} \right) = \frac{\partial x^k}{\partial y^a} \frac{\partial v_i}{\partial x^k} \frac{\partial x^i}{\partial y^b} + \frac{\partial x^i}{\partial y^a} \frac{\partial x^k}{\partial y^b} \frac{\partial v_i}{\partial x^k} = \nabla_a u_b + \nabla_b u_a$$

by (2.6). The second part of (2.13) follows from the first since $0 = D_t(g^{ab} g_{bc}) = D_t(g^{ab}) g_{bc} + g^{ab} D_t(g_{bc})$ so $D_t g^{ab} = -g^{ck} g^{ab} D_t g_{bc}$. The last part of (2.13) follows since in local coordinates $d\mu_g = \sqrt{\det g} dy$ and $D_t \det g = \det g g^{ab} D_t g_{ab}$. It follows from (2.4) and (2.13) that

$$D_t \Gamma_{ab}^c = \frac{g^{cd}}{2} (\nabla_a D_t g_{bd} + \nabla_b D_t g_{ad} - \nabla_d D_t g_{ab}) = g^{cd} \nabla_a \nabla_b u_d \quad \square$$

Lemma 2.2. Let $w_{i_1 \dots i_r}(t, x)$ be an arbitrary $(0, r)$ tensor and let

$$(2.15) \quad k_{a_1 \dots a_r}(t, y) = w_{i_1 \dots i_r}(t, x) \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}}, \quad \text{where } x = f(t, y).$$

Let $D_t = \partial_t|_{y=\text{constant}}$ and $v^\ell(t, x) = \partial_t f^\ell(t, y)$. Then

$$(2.16) \quad \begin{aligned} D_t k_{a_1 \dots a_r} &= \left(D_t w_{i_1 \dots i_r} + w_{\ell \dots i_r} \frac{\partial v^\ell}{\partial x^{i_1}} + \dots + w_{i_1 \dots \ell} \frac{\partial v^\ell}{\partial x^{i_r}} \right) \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \\ &= \left(\partial_t \Big|_{x=\text{const}} w_{i_1 \dots i_r} + (\mathcal{L}_v w)_{i_1 \dots i_r} \right) \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \end{aligned}$$

and \mathcal{L}_v is the Lie derivative.

Proof. Note that if the tensor and the velocity only depends of t through x then this would just be the definition of the Lie derivative. Now

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{y=\text{const}} w_{i_1 \dots i_r}(t, x) \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} &= \left(\partial_t \Big|_{x=\text{const}} w_{i_1 \dots i_r}(t, x) + (\partial_\ell w_{i_1 \dots i_r})(t, x) \frac{\partial x^\ell}{\partial t} \right) \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \\ &\quad + w_{i_1 \dots i_r}(t, x) \frac{\partial^2 x^{i_1}}{\partial t \partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} + \dots + w_{i_1 \dots i_r}(t, x) \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial^2 x^{i_r}}{\partial t \partial y^{a_r}} \end{aligned}$$

Since $v^\ell(t, x) = \partial x^\ell / \partial t$ we see that

$$w_{i_1 \dots i_r}(t, x) \frac{\partial^2 x^{i_1}}{\partial t \partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} = w_{i_1 \dots i_r}(t, x) \frac{\partial v^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} = w_{\ell \dots a_r}(t, x) \frac{\partial v^\ell}{\partial x^{i_1}} \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}},$$

and similarly for the other terms. This proves (2.16) since by definition

$$(\mathcal{L}_v w)_{i_1 \dots i_r} = v^\ell (\partial_\ell w_{i_1 \dots i_r}) + w_{\ell \dots i_r} \frac{\partial v^\ell}{\partial x^{i_1}} + \dots + w_{i_1 \dots \ell} \frac{\partial v^\ell}{\partial x^{i_r}} \quad \square$$

We will now calculate commutators between the material derivative D_t and space derivatives ∂_i , in Lemma 2.3, and covariant derivatives ∇_a , in Lemma 2.4. In order to calculate commutators between D_t and higher order derivatives $\partial_{i_1} \cdots \partial_{i_r}$ or $\nabla_{a_1} \cdots \nabla_{a_r}$ we will introduce some notation incorporating that these commutators are symmetric under permutations of the indices (i_1, \dots, i_r) respectively (a_1, \dots, a_r) . Let $(\partial^r)_{i_1 \dots i_r} = \partial_{i_1 \dots i_r}^r = \partial_{i_1} \cdots \partial_{i_r}$ and $(\nabla^r)_{a_1 \dots a_r} = \nabla_{a_1 \dots a_r}^r = \nabla_{a_1} \cdots \nabla_{a_r}$. In particular, it is convenient to introduce the symmetric dot product in (2.19) and (2.24):

Lemma 2.3. Let ∂_i be given by (2.8). Then

$$(2.17) \quad [D_t, \partial_i] = -(\partial_i v^k) \partial_k$$

Furthermore

$$(2.18) \quad [D_t, \partial^r] = \sum_{s=0}^{r-1} -\binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s},$$

where, the symmetric dot product is defined to be in components

$$(2.19) \quad ((\partial^{1+s} v) \cdot \partial^s)_{i_1 \dots i_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \left(\partial_{i_{\sigma_1} \dots i_{\sigma_{1+s}}}^{1+s} v^k \right) \partial_{k i_{\sigma_{s+2}} \dots i_{\sigma_r}}^s$$

Proof. The proof of (2.17) follows from (2.8) and (2.12). In the notation of (2.18) we can write (2.17) as

$$[D_t, \partial] = -(\partial v) \cdot \partial$$

Using this repeatedly we obtain

$$[D_t, \partial^r] = \sum_{\ell=0}^r \partial^\ell [D_t, \partial] \partial^{r-\ell-1} = - \sum_{\ell=0}^{r-1} \partial^\ell (\partial v) \cdot \partial^{r-\ell} = - \sum_{\ell=0}^{r-1} \sum_{s=0}^{\ell} \binom{\ell}{s} (\partial^{1+s} v) \cdot \partial^{r-s}$$

Since $\sum_{\ell=s}^{r-1} \binom{\ell}{s} = \binom{r}{s+1}$ this proves (2.18). \square

Lemma 2.4. *Let $T_{a_1 \dots a_r}$ is a $(0, r)$ tensor. We have*

$$(2.20) \quad [D_t, \nabla_a] T_{a_1 \dots a_r} = -(\nabla_{a_1} \nabla_a u^d) T_{da_2 \dots a_r} - \dots - (\nabla_{a_r} \nabla_a u^d) T_{a_1 \dots a_{r-1} d}$$

If $\Delta = g^{cd} \nabla_c \nabla_d$ and q is a function we have

$$(2.21) \quad [D_t, g^{ab} \nabla_a] T_b = -h^{ab} \nabla_a T_b - (\Delta u^e) T_e$$

$$(2.22) \quad [D_t, \Delta] q = -h^{ab} \nabla_a \nabla_b q - (\Delta u^e) \nabla_e q$$

Furthermore

$$(2.23) \quad [D_t, \nabla^r] q = \sum_{s=1}^{r-1} -\binom{r}{s+1} (\nabla^{s+1} u) \cdot \nabla^{r-s} q,$$

where the symmetric dot product is defined to be in components

$$(2.24) \quad ((\nabla^{s+1} u) \cdot \nabla^{r-s} q)_{a_1 \dots a_r} = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} (\nabla_{a_{\sigma_1} \dots a_{\sigma_{s+1}}}^{s+2} u^d) \nabla_{d a_{\sigma_{s+3}} \dots a_{\sigma_r}}^{r-s} q$$

Proof. (2.20) is a consequence of (2.13) since in components the covariant derivative is given by $\nabla_a T_{a_1 \dots a_r} = \partial T_{a_1 \dots a_r} / \partial y^a - \Gamma_{a_1 a}^d T_{da_2 \dots a_r} - \dots - \Gamma_{a_r a}^d T_{a_1 \dots a_{r-1} d}$. Now

$$[D_t, g^{ab} \nabla_a] T_b = (D_t g^{ab}) \nabla_a T_b + g^{ab} [D_t, \nabla_a] T_b$$

and (2.21) follows from (2.12) and (2.20). (2.22) follows from (2.21) applied to $T_b = \nabla_b \psi$, since $D_t \nabla_b q = \partial_t \partial q(t, y) / \partial y^b = \nabla_b D_t q$.

In the notation of (2.24) we have by (2.20)

$$(2.25) \quad [D_t, \nabla] \nabla^s q = -s(\nabla^2 u) \cdot \nabla^s q$$

using this repeatedly we get

$$[D_t, \nabla^r] q = \sum_{\ell=0}^{r-1} \nabla^\ell [D_t, \nabla] \nabla^{r-\ell-1} q = - \sum_{\ell=0}^{r-1} \nabla^\ell (r-\ell-1) (\nabla^2 u) \cdot \nabla^{r-\ell-1} q = - \sum_{\ell=0}^{r-2} \sum_{s=0}^{\ell} (r-\ell-1) \binom{\ell}{s} (\nabla^{s+2} u) \cdot \nabla^{r-s-1} q$$

Since $\sum_{\ell=s}^{r-2} (r-\ell-1) \binom{\ell}{s} = \binom{r}{s+2}$ this proves (2.23). \square

Notice that difference between (2.18) and (2.23) is that in (2.23) the term with $s = 0$ is absent, which is the advantage of going to covariant differentiation.

3. THE GEOMETRY AND REGULARITY OF THE BOUNDARY: THE SECOND
FUNDAMENTAL FORM AND EXTENSION OF THE NORMAL TO THE INTERIOR

In this section we will deal with the geometry and regularity of the boundary. The regularity is measured by the regularity of the normal, in particular by the first space derivative, i.e. the second fundamental form. We also need to control how far off the boundary is from self intersecting since we want to foliate the domain close the boundary into surfaces that do not self intersect. This can be achieved by the level sets of the distance function to the boundary. This gives an extension of the normal to the interior, which we need to prove our estimates. The size of the neighborhood in which the level sets are well defined and smooth determines the size of the derivatives of our extension of the normal to a vector field defined everywhere in the interior. We also want to control the time evolution of the boundary, which can be measured by the time derivative of the normal in the Lagrangian coordinates.

We will use both the Eulerian coordinates and the Lagrangian coordinates. When we calculate time derivatives it is of course most convenient to do so in the Lagrangian coordinates whereas the Eulerian coordinates are more convenient to use when we measure how the surface lies in space, since we want to be able to compare the normal at different points. In this section we will also define the projection of a tensor to the boundary which we will use to define covariant differentiation on the boundary. The projection will play an important role in our estimates and we will discuss it in detail in section 4.

Definition 3.1. Let N^a denote be the unit normal to $\partial\Omega$:

$$(3.1) \quad g_{ab}N^aN^b = 1, \quad g_{ab}N^aT^b = 0, \quad \text{if } T \in T(\partial\Omega)$$

and let $N_a = g_{ab}N^b$ denote the unit conormal; $g^{ab}N_aN_b = 1$. The induced metric γ on the tangent space to the boundary $T(\partial\Omega)$ extended to be 0 on the orthogonal complement in $T(\Omega)$ is then given by

$$(3.2) \quad \gamma_{ab} = g_{ab} - N_aN_b, \quad \gamma^{ab} = g^{ab} - N^aN^b$$

The orthogonal projection of a (r, s) tensor S to the boundary is given by

$$(3.3) \quad (\Pi S)^{a_1 \dots a_r}_{b_1 \dots b_s} = \gamma^{a_1}_{c_1} \dots \gamma^{a_s}_{c_s} \gamma_{b_1}^{d_1} \dots \gamma_{b_s}^{d_s} S^{c_1 \dots c_r}_{d_1 \dots d_s},$$

where

$$(3.4) \quad \gamma_a^c = \delta_a^c - N_aN^c \quad \text{and} \quad \gamma^a_c = \delta^a_c - N^aN_c.$$

Covariant differentiation on the boundary $\bar{\nabla}$ is given by

$$(3.5) \quad \bar{\nabla}S = \Pi \nabla S$$

The second fundamental form of the boundary is given by

$$(3.6) \quad \theta_{ab} = (\Pi \nabla N)_{ab} = \gamma_a^c \nabla_c N_b.$$

Note first that $\bar{\nabla}$ is invariantly defined since the projection and the covariant derivative are. Note also that $\bar{\nabla}$ indeed corresponds to the intrinsic covariant derivative $\bar{\nabla}$ of the boundary:

Lemma 3.1. *Suppose that the coordinates are chosen so that locally the boundary is given by $\partial\Omega = \{y; y^n = 0\}$ and parameterized by (y^1, \dots, y^{n-1}) . Let ∇ denote covariant differentiation on $\partial\Omega$. Then*

$$(3.7) \quad \bar{\nabla}^a T^b = \begin{cases} \nabla^a T^b & \text{for } a, b = 1, \dots, n-1 \\ 0 & \text{for } a = n \text{ or } b = n \end{cases}, \quad \text{if } T^n = 0.$$

Proof. The conormal is $N_a = \delta_{an}/\sqrt{g^{nn}}$ and the normal is $N^a = g^{ac}N_c = g^{an}/\sqrt{g^{nn}}$. The induced metric is given by $\gamma_{ab} = g_{ab}$, for $a, b = 1, \dots, n-1$ and its inverse is given by $\gamma^{ab} = g^{ab} - N^aN^b$, for $a, b = 1, \dots, n-1$. Note also that

$$\begin{cases} \gamma_a^n = \gamma^n_a = \gamma^{na} = \gamma^{an} = 0, & \text{when } a \leq n, \\ \gamma_a^b = \gamma^b_a = \delta_a^b, & \text{when } a < n \\ \gamma_n^b = \gamma^b_n = -g^{bn}/g^{nn}, & \text{when } b < n \end{cases}$$

Let us at this point use the notation $\nabla^a = g^{ab}\nabla_b$, $\bar{\nabla}^a = g^{ab}\bar{\nabla}_b$ and $\nabla^a = \gamma^{ab}\nabla_b$ where the last sum is only over $b = 1, \dots, n-1$. To prove (3.7) we first note that $\bar{\nabla}^a T^b = \gamma^a_{a'}\gamma^b_{b'}\nabla^{a'}T^{b'} = \gamma^a_{a'}\nabla^{a'}T^b = 0$ when $i = n$ or $b = n$ since $\gamma^n_{a'} = 0$. If on the other hand $1 \leq a, b \leq n-1$ then

$$\bar{\nabla}^a T^b = \gamma^a_{a'}g^{a'a''} \left(\frac{\partial T^b}{\partial y^{a''}} + g^{bb''}\Gamma_{a''b''c}T^c \right) = \gamma^{aa'}\frac{\partial T^b}{\partial y^{a'}} + \gamma^{aa'}\gamma^{bb'}\Gamma_{a'b'c}T^c$$

and if $1 \leq a, b, c \leq n-1$ then

$$\Gamma_{abc} = \frac{1}{2} \left(\frac{\partial g_{bc}}{\partial y^a} + \frac{\partial g_{ac}}{\partial y^b} - \frac{\partial g_{ab}}{\partial y^c} \right) = \frac{1}{2} \left(\frac{\partial \gamma_{bc}}{\partial y^a} + \frac{\partial \gamma_{ac}}{\partial y^b} - \frac{\partial \gamma_{ab}}{\partial y^c} \right) = \Gamma_{abc}$$

gives the intrinsic connection so (3.7) follows. \square

It follows that any invariant quantity formed from either side of (3.7) have to be equal. If the coordinates are chosen so $y^n = 0$ on $\partial\Omega$ then the curvature of $\partial\Omega$ is related to the second fundamental form by Gauss equations

$$(3.8) \quad \bar{R}_c^d{}_{ab} = \theta_{ac}\theta_b^d - \theta_{bc}\theta_a^d$$

Recall also that if T is tangential

$$(3.9) \quad [\bar{\nabla}_a, \bar{\nabla}_b]T_{a_1\dots a_r} = -\bar{R}_c^{a_1}{}_{ab}T_{c\dots a_r} - \dots - \bar{R}_c^{a_r}{}_{ab}T_{a_1\dots c}$$

We also need to extend the normal to a neighborhood of the boundary. The exact extension of the normal to the interior is not so important at this point. Basically we want to have control of the supremum norm of the time and space derivatives of the normal in the interior. One way to define an extension of the normal in the interior is to consider a foliation of Ω close to $\partial\Omega$

$$(3.10) \quad S_\lambda = \{y \in \Omega; d(t, y) = \lambda\}, \quad d > 0 \text{ in } \Omega, \quad d = 0 \text{ on } \partial\Omega$$

The unit conormal to S_λ is then given by

$$(3.11) \quad N_a = \frac{\partial_a d}{\sqrt{g^{bc}\partial_b d \partial_c d}}$$

It is natural to take $d(t, y) = \text{dist}_g(y, \partial\Omega)$ to be the geodesic distance to the boundary which is the same as the Euclidean distance in the x -variables. If d is the geodesic distance in the metric g then the conormal is $N_a = \nabla_a d$ and $\theta = \nabla N = \nabla^2 d = \Pi \nabla^2 d$ and the normal derivative of the normal vanishes $\nabla_N N = 0$. Since $\theta = \Pi \nabla^2 d = \nabla^2 d$ it follows that $\bar{\nabla} \theta = \Pi \nabla \Pi \nabla^2 d = \Pi \nabla^3 d$ is symmetric as well. $\bar{\nabla}^2 \theta$ is however not symmetric, but the antisymmetric part is lower order; by Gauss equations (3.8)-(3.9),

$$(3.12) \quad \bar{\nabla}_a \bar{\nabla}_b \theta_{cd} - \bar{\nabla}_b \bar{\nabla}_a \theta_{cd} = [\bar{\nabla}_a, \bar{\nabla}_b] \theta_{cd} = -\bar{R}_c{}^e{}_{ab} \theta_{ed} - \bar{R}_d{}^e{}_{ab} \theta_{de}.$$

Furthermore, since $N \cdot N = 1$ we get $N \cdot \nabla^2 N + (\nabla N) \cdot (\nabla N) = 0$, in other words

$$(3.13) \quad \nabla_N \theta_{ab} = -\theta_a{}^c \theta_{cb}$$

so the second fundamental forms for the surfaces S_λ for small λ are as regular as for $\partial\Omega$. We will discuss this and the regularity of the extension of the normal to the interior further in Lemma 3.6.

Let us now go on to discuss two definitions, to control the geometry and regularity of the boundary. Let us express our surface in the x variables $\partial\mathcal{D}_t \subset \mathbf{R}^n$ using the metric there.

Definition 3.2. Let $\mathcal{N}(\bar{x})$ is the outward unit normal to $\partial\mathcal{D}_t$ at $\bar{x} \in \partial\mathcal{D}_t$. Let $\text{dist}(x_1, x_2) = |x_1 - x_2|$ the denote the Euclidean distance in \mathbf{R}^n and for $\bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t$ let $\text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2)$ denote the geodesic distance on the boundary. Let $\text{dist}(x, \partial\mathcal{D}_t)$ be the Euclidean distance from x to the boundary.

Definition 3.3. Let ι_0 be the injectivity radius of the normal exponential map of $\partial\mathcal{D}_t$, i.e. the largest number such that the map

$$(3.14) \quad \partial\mathcal{D}_t \times (-\iota_0, \iota_0) \rightarrow \{x \in \mathbf{R}^n : \text{dist}(x, \partial\mathcal{D}_t) < \iota_0\} \quad \text{given by} \quad (\bar{x}, \iota) \rightarrow x = \bar{x} + \iota \mathcal{N}(\bar{x}),$$

is an injection. \square

Note that $\iota_0 \geq 1/|\theta|_{L^\infty(\partial\mathcal{D}_t)}$, for along the normal line from $\bar{x} \in \partial\mathcal{D}_t$, the first focal point is at a distance $1/|\theta(\bar{x})|$, where $|\theta(\bar{x})| = \sup_{|v|=1} |\theta(\bar{x}) \cdot v|$ is the greatest eigenvalue in magnitude. Instead of using the injectivity radius ι_0 we can use a radius ι_1 which, in conjunction with a bound for the second fundamental form, is comparable. The radius ι_1 works equally well for controlling the Sobolev constants and it is easier to control the time evolution off.

Definition 3.4. Let $0 < \varepsilon_1 < 2$ be a fixed number and let $\iota_1 = \iota_1(\varepsilon_1)$ the largest number such that

$$(3.15) \quad |\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1, \quad \text{whenever} \quad |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t.$$

Remark. Note that Definition 3.4 also says that the intersection $\partial\mathcal{D}_t \cap B(\iota_1, \bar{x}_0)$ of the surface with an open ball of radius ι_1 centered at any point $\bar{x}_0 \in \partial\mathcal{D}_t$ is connected and it can be written as a graph over the plane orthogonal to the normal $\mathcal{N}(\bar{x}_0)$ at the center \bar{x}_0 . In fact, we claim that the line segment in $B(\iota_1, \bar{x}_0)$ along the exterior normal $\mathcal{N}(\bar{x}_0)$ from any point \bar{x}_1 in the same component of $\partial\mathcal{D}_t$ as \bar{x}_0 is completely contained in the complement $\mathcal{C}\mathcal{D}_t$ (and the line segment in the opposite direction is completely contained in \mathcal{D}_t .) In fact, if not, then there would be a point $\bar{x}_2 \in \partial\mathcal{D}_t$ where it would enter the region \mathcal{D}_t again and at that point the exterior normal $\mathcal{N}(\bar{x}_2)$ would have to make an angle at least $\pi/2$ with $\mathcal{N}(\bar{x}_0)$ contradicting the condition in Definition 3.4.

Lemma 3.2. *Suppose that $|\theta| \leq K$ and let ι_0 and ι_1 be as in Definition 3.3 and Definition 3.4. Then*

$$(3.16) \quad \iota_0 \geq \min(\iota_1/2, 1/K) \quad \text{and} \quad \iota_1 \geq \min(2\iota_0, \varepsilon_1/K)$$

Proof. Let

$$\iota_3 = \min_{\text{dist}_{\partial\mathcal{D}_t}(\bar{x}, \bar{z}) \geq \pi/K} |\bar{x} - \bar{z}|$$

We claim that

$$\iota_0 = \iota_3/2 \geq \iota_1/2, \quad \text{if} \quad \min(\iota_0, \iota_3/2) \leq 1/K$$

By Definition 3.3 there are $\bar{x}_1 \neq \bar{x}_2$ on the boundary such that

$$\bar{x}_1 + a\mathcal{N}(\bar{x}_1) = \bar{x}_2 + b\mathcal{N}(\bar{x}_2) \quad \text{for some} \quad |a| \leq \iota_0, \quad |b| \leq \iota_0$$

If $\iota_0 < 1/K$ then, by Lemma 3.3, $\text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2) \geq \pi/K$ and hence

$$\iota_3 = \min_{\text{dist}_{\partial\mathcal{D}_t}(\bar{x}, \bar{z}) \geq \pi/K} |\bar{x} - \bar{z}| \leq |\bar{x}_1 - \bar{x}_2| \leq 2\iota_0 < 2/K.$$

If $\iota_3 < 2/K$ it follows from Lemma 3.3, that the minima above is attained at some, possibly different, $(\bar{x}_3, \bar{x}_4) \in \partial\mathcal{D}_t \times \partial\mathcal{D}_t$ with $\text{dist}_{\partial\mathcal{D}_t}(\bar{x}_3, \bar{x}_4) > \pi/K$. Hence $\partial\mathcal{D}_t \times \partial\mathcal{D}_t \ni (\bar{x}, \bar{z}) \rightarrow |\bar{x} - \bar{z}|$ has a local minimum at (\bar{x}_3, \bar{x}_4) so the normals $\mathcal{N}(\bar{x}_3)$ and $\mathcal{N}(\bar{x}_4)$ are parallel to the line between \bar{x}_3 and \bar{x}_4 . From this it follows that $\iota_0 \leq \iota_3/2$ and it also contradicts the condition in Definition 3.4 so we conclude that $\iota_3 = |\bar{x}_3 - \bar{x}_4| > \iota_1$. This proves the first part of (3.16) and the second part follows in a similar way; If $\text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2) \leq \pi/K$ then by Lemma 3.3

$$|\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq 2 \sin(K \text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2)/2) \leq K \text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2) \leq K\pi|\bar{x}_1 - \bar{x}_2|/2 \leq \varepsilon_1$$

if $|\bar{x}_1 - \bar{x}_2| \leq \varepsilon_1 2/K\pi$. If on the other hand $\text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2) > \pi/K$ then $|\bar{x}_1 - \bar{x}_2| \geq \iota_3$ and if $\iota_3 < 2/K$ then $\iota_3 = 2\iota_0$ so $|\bar{x}_1 - \bar{x}_2| \geq \min(2/K, 2\iota_0)$. \square

Lemma 3.3. *Suppose that $|\theta| \leq K$ and $0 < \text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2) < \pi/K$. Then*

$$(3.17) \quad \bar{x}_1 + a\mathcal{N}(\bar{x}_1) \neq \bar{x}_2 + b\mathcal{N}(\bar{x}_2) \quad \text{for} \quad |a| \leq 1/K, \quad |b| \leq 1/K$$

Furthermore, if $|\theta| \leq K$ and $\text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2) \leq \pi/K$ then

$$(3.18) \quad |\bar{x}_1 - \bar{x}_2| \geq 2 \text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2)/\pi, \quad \text{and} \quad \mathcal{N}(\bar{x}_1) \cdot \mathcal{N}(\bar{x}_2) \geq \cos(K \text{dist}_{\partial\mathcal{D}_t}(\bar{x}_1, \bar{x}_2))$$

Proof. Let $\alpha(s)$ be a geodesic in $\partial\Omega$ parameterized by arc length, $|\dot{\alpha}(s)| = 1$, with $\alpha(s_i) = \bar{x}_i$. Let $s_0 = (s_1 + s_2)/2$. To simplify notation we assume that $s_0 = 0$ and $\alpha(0) = 0$ and set $\dot{\alpha}(0) = \mathcal{T}$. Let $\mathcal{N}(s)$ be the normal to $\alpha(s)$ and $k(s) = \theta(\dot{\alpha}(s), \dot{\alpha}(s))$ be the (normal) curvature of $\alpha(s)$, i.e $\ddot{\alpha}(s) = \pm k(s)\mathcal{N}(s)$. We will show that $\mathcal{T} \cdot (\alpha(s) + a\mathcal{N}(s)) > 0$ for $|a| < K$ and that $\mathcal{T} \cdot \alpha(s) \geq \sin(Ks)/K$ provided that $0 < s < \pi/2K$. Since the same result is true in the negative direction, this would prove the lemma. Let $\phi(s)$ be the angle that $\dot{\alpha}(s)$ makes with \mathcal{T} i.e. $\dot{\alpha}(s) \cdot \mathcal{T} = \cos \phi(s)$. Then $|\dot{\phi}(s)| \leq K$ so $0 \leq \phi(s) \leq Ks$. Let $x(s) = \alpha(s) \cdot \mathcal{T}$ and $r(s) = |\alpha(s) - \mathcal{T}(\alpha(s) \cdot \mathcal{T})|$. Then $\dot{x}(s) = \cos \phi(s) \geq \cos(Ks)$ and $|\dot{r}(s)| \leq \sin \phi(s) \leq \sin(Ks)$. Hence $x(s) \geq \sin(Ks)/K$ and $r(s) \leq (1 - \cos(Ks))/K$. Furthermore $\mathcal{T} \cdot \mathcal{N}(s) \geq \cos(\phi(s) + \pi/2) = -\sin \phi(s) \geq -\sin(Ks)$ which proves the lemma.

Note that it follows from the remark after Definition 3.4 that in a neighborhood of $\bar{x}_0 \in \partial\mathcal{D}_t$ we can write the boundary as a graph. We can now make a partition of unity into coordinate neighborhoods where this is true, which will be used to control the Sobolev constants:

Lemma 3.4. Suppose that $\mathcal{D}_t \subset \mathbf{R}^n$ with boundary satisfying the condition in Definition 3.4 with $\iota_1 \geq 1/K_1$. Then there are $\chi_i \in C_0^\infty(\mathbf{R}^n)$, $i = 1, 2, \dots$ such that

$$(3.19) \quad \sum_p \chi_i = 1, \quad \sum_p |\partial^\alpha \chi_i| \leq C_\alpha K_1^{|\alpha|}, \quad \text{diam}(\text{supp}(\chi_i)) \leq 1/K_1$$

and for each $x \in \mathbf{R}^n$ there are at most 32^n i such that $\chi_i(x) \neq 0$. Furthermore, either $\text{supp}(\chi_i) \cap \partial\mathcal{D}_t$ empty or its part of a graph contained in $\partial\mathcal{D}_t$, which after a rotation is given by

$$(3.20) \quad x^n = f_i(x'), \quad (x', x^n) \in \mathbf{R}^n, \quad |x' - x'_i| \leq \iota_1, \quad |\partial f_i| \leq \varepsilon_1, \quad x_i \in \partial\mathcal{D}_t, \quad \mathcal{N}(x_i) = (0, \dots, 0, 1)$$

Proof. Let $B(r, x)$ denote the ball of radius r centered at x . Let $\rho_1 = \iota_1/16$ and let $\{B(2\rho_1, x_i)\}$ be a cover of \mathbf{R}^n such that $\{B(\rho_1, x_i)\}$ are disjoint. We define

$$\chi_i(x) = \frac{\chi(|x - x_i|/4\iota_1)}{\sum_i \chi(|x - x_i|/4\iota_1)}$$

where $\chi \in C_0^\infty$ satisfy $0 \leq \chi \leq 1$, $\chi(s) = 1$ when $s \leq 0$ and $\chi(s) = 0$ when $s \geq 2$. The number of disjoint balls of radius ρ_1 that can be contained in a ball of radius $16\rho_1$ is 16^n . Since $\text{supp}(\chi_i)$ is contained in a ball of radius $8\rho_1$ this proves that for each $x \in \mathbf{R}^n$ there are at most 16^n i such that $\chi_i(x) \neq 0$. \square

We will now estimate first order derivatives of the extension of the normal to the interior. In Lemma 3.5 we estimate the time derivatives on the boundary. It is now natural to work in the Lagrangian coordinates. In Lemma 3.6 we estimate the geodesic extension of the normal to the interior in a neighborhood of the boundary.

Lemma 3.5. Let N be the unit normal to $\partial\Omega$ and let $h_{ab} = D_t g_{ab}/2$. On $[0, T] \times \partial\Omega$ we have

$$(3.21) \quad D_t N_a = h_{NN} N_a, \quad D_t N^c = -2h^c{}_d N^d + h_{NN} N^c$$

$$(3.22) \quad D_t \gamma^{ab} = -2\gamma^{ac} \gamma^{bd} h_{cd}$$

The volume element on $\partial\Omega$ satisfy

$$(3.23) \quad D_t d\mu_\gamma = (\text{tr } h - h_{NN}) d\mu_\gamma = (\text{tr } \theta u \cdot N + \gamma^{ab} \bar{\nabla}_a \bar{u}_b) d\mu_\gamma$$

Proof. Since the right hand sides of (3.21) restricted to $[0, T] \times \partial\Omega$ is independent of the extension of the normal to the interior we may choose the foliation

$$N_a = \frac{\partial_a u}{\sqrt{g^{cd} \partial_c u \partial_d u}}, \quad \text{where } \partial\Omega = \{y; u(y) = 0\}, \quad u < 0, \quad \text{in } \Omega$$

then

$$D_t N_a = -\frac{1}{2} N_a (D_t g^{cd}) N_c N_d = h_{NN} N_a$$

and

$$D_t N^a = D_t g^{ad} N_d = (D_t g^{ad}) N_d + g^{ad} D_t N_d = -2h^{ad} N_d + h_{NN} N^a$$

which proves (3.21). (3.22) follows from

$$\begin{aligned} D_t \gamma^{ab} &= D_t (g^{ab} - N^a N^b) = D_t g^{ab} - (D_t N^a) N^b - N^a D_t N^b = -2h^{ab} + 2h^a{}_c N^c N^b \\ &\quad + 2h^b{}_d N^d N^a - 2h_{kl} N^k N^l N^a N^b = (\delta^a{}_d - N^a N_d) (\delta^b{}_d - N^b N_d) h^{cd} = -2\gamma^a{}_d \gamma^b{}_d h^{cd}. \end{aligned}$$

Introducing coordinates we have $d\mu_g = \sqrt{\det g} dy$ and $D_t \sqrt{\det g} = \sqrt{\det g} \text{tr } h$. Now $d\mu_\gamma = \sqrt{\det g} (\sum N_n^2)^{-1/2} dS$, where dS is the Euclidean surface measure, and $D_t (\sum N_n^2)^{-1/2} = -(1/2) (\sum N_n^2)^{-3/2} \sum 2N_n D_t N_n$. But $D_t N_n = h_{NN} N_n$ which proves that $D_t d\mu_\gamma = (\text{tr } h - h_{NN}) d\mu_\gamma$. Now $\text{tr } h - h_{NN} = \gamma^{ab} \bar{\nabla}_a v_b = \gamma^{ab} \bar{\nabla}_a (N_b v \cdot N) + \gamma^{ab} \bar{\nabla}_a \bar{v}_b$. \square

We will now extend the normal to a vector field defined an regular everywhere in the interior, such that when $d(t, y) \leq \iota_0/4$ its the normal to the sets $\{y; (t, y) = d_0\}$ and in the interior it drops off to 0.

Lemma 3.6. *Let ι_0 be as in Definition 3.3. and let $d(y) = \text{dist}_g(y, \partial\Omega)$ be the geodesic distance in the metric g from y to $\partial\Omega$. Then the conormal $n = \nabla d$ to the sets $S_a = \partial\{y \in \Omega; d(y) = a\}$ satisfies*

$$(3.24) \quad |\nabla n| \leq 2|\theta|_{L^\infty(\partial\Omega)}, \quad \text{and} \quad |D_t n(t, y)| \leq 6|h|_{L^\infty(\Omega)}, \quad \text{when} \quad d(y) < \iota_0/2.$$

Proof. Now since $n \cdot n = 1$ it follows that $n \cdot \nabla n = 0$ and hence $(\nabla n) \cdot \nabla n + n \cdot \nabla^2 n = 0$ and since $\theta = \nabla n$ we get $\nabla_N \theta = -\theta \cdot \theta$. It follows that $|\nabla_N |\theta|| \leq |\theta|^2$. If $d(y) = \text{dist}_g(y, \partial\Omega) < \iota_0$ then there is a unique $\bar{y} \in \partial\Omega$ such that $d(y, \bar{y}) = \text{dist}_g(y, \partial\Omega)$. Hence we can introduce d and \bar{y} as new variables so that $y = y(d, \bar{y})$ and in these coordinates $\nabla_N = \partial/\partial d$ so with $f(d) = |\theta(d, \bar{y})|$ we get the inequality $|f'(d)| \leq f(d)^2$ for each fixed \bar{y} . Its easy to see that $f(d) \leq 2f(0)$, if $2df(0) \leq 1$. and hence $|\theta(d, \bar{y})| \leq 2|\theta|_{L^\infty(\partial\Omega)}$, if $2d|\theta|_{L^\infty(\partial\Omega)} \leq 1$, which proves the first part of (3.24). We claim that

$$(3.25) \quad \nabla_N D_t d = h_{NN}, \quad \nabla_N \dot{n} + \theta \cdot \dot{n} = \theta \cdot h \cdot n, \quad \text{if} \quad \dot{n} = D_t n - h \cdot n$$

In fact since $g^{ab} N_a N_b = 1$ we have

$$0 = 2g^{ab} N_a D_t N_b + (D_t g^{ab}) N_a N_b = 2\nabla_N D_t d - 2h^{ab} N_a N_b$$

and the first equation in (3.25) follows. Since

$$\nabla_c h_{NN} = \nabla_c (N^a N^b h_{ab}) = N^a N^b \nabla_c h_{ab} + h_{ab} \nabla_c (N^a N^b) = N^a N^b \nabla_a h_{cb} + h_{ab} (N^b \theta_c^a + N^a \theta_c^b)$$

differentiating the first equation in (3.25) we get

$$\nabla_N D_t N_c + \theta_c^e D_t N_e = \nabla_c N^e \nabla_e D_t d = \nabla_c h_{NN} = \nabla_N (h_{cb} N^b) + \theta_c^e h_{eb} N^b + \theta_c^b N^a h_{ab}$$

With $\dot{n}_c = D_t N_c - h_{cb} N^b$, we get $\nabla_N \dot{n}_c + \theta_c^e \dot{n}_e = \theta_c^b N^a h_{ab}$ which proves the second part of (3.25)

$$(3.26) \quad |\nabla_N |\dot{n}|| \leq |\theta| |\dot{n}| + |\theta| |h| \leq K |\dot{n}| + K |h|, \quad \text{if} \quad K = 2|\theta|_{L^\infty(\partial\Omega)}$$

so using the coordinates $y = y(d, \bar{y})$ we get

$$|\dot{n}(t, y)| \leq e^{d(t, y)K} |\dot{n}(t, \bar{y})| + \int_0^{d(t, y)} e^{(d(t, y)-s)K} K |h| ds \leq e^{d(t, y)K} (|\dot{n}(t, \bar{y})| + K d(t, y) |h|_{L^\infty(\Omega)})$$

where $\bar{y} \in \partial\Omega$ satisfies $d(t, y) = \text{dist}_g(y, \bar{y})$. Since $K d_0 \leq 1/2$ we get $|\dot{n}(t, y)| \leq 2|\dot{n}(t, \bar{y})| + |h|_{L^\infty(\Omega)}$, when $d(t, y) \leq d_0$. Since $D_t n(t, \bar{y}) = h_{NN}(t, \bar{y}) n(t, \bar{y})$ and $\dot{n} = D_t n - h \cdot n$ we get $|D_t n(t, y)| \leq 6|h|_{L^\infty(\Omega)}$. \square

Lemma 3.7. *Let ι_0 be the reduced injectivity radius of the normal exponential map of $\partial\Omega$ and let d_0 be a fixed number such that $\iota_0/16 \leq d_0 \leq \iota_0/2$. Let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(s) = 1$ when $|s| \leq 1/2$, $\eta(s) = 0$ when $|s| \geq 3/4$, $0 \leq \eta(s) \leq 1$ and $|\eta'(s)| \leq 4$. Then the pseudo Riemannian metric γ given by*

$$(3.27) \quad \gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b, \quad \gamma^{ab} = g^{ab} - N^a N^b, \quad N^a = g^{ab} \tilde{n}_b, \quad \text{where} \quad \tilde{n}_c = \eta(d/d_0) \nabla_c d$$

satisfies

$$(3.28) \quad |\nabla \gamma|_{L^\infty(\Omega)} \leq 256(|\theta|_{L^\infty(\partial\Omega)} + 1/\iota_0) \quad \text{and} \quad |D_t \gamma(t, y)| \leq 64|h|_{L^\infty(\Omega)}$$

Proof. We have $\nabla_c \tilde{n}_a = -\eta(d/d_0) \nabla_c N_a - \eta'(d/d_0) N_a N_c / d_0$ which in view of (3.27) proves that $|\nabla \tilde{\gamma}| \leq 2|\nabla n| + 16/d_0$, so the first inequality in (3.28) follows. Since $\gamma_{ab} = g_{ab} - \tilde{n}_a \tilde{n}_b$, where $\tilde{n}_b = \eta(d/d_0) N_b$ and so $D_t \tilde{n}_b = \eta(d/d_0) D_t N_b + \eta'(d/d_0) N_b D_t d / d_0$. Integrating the first equation in (3.25) gives $|D_t d(t, y)| \leq |h_{NN}|_{L^\infty(\Omega)} d(t, y)$ and since $d/d_0 \leq 1$ in the support of $\eta(d/d_0)$ this proves the second part of (3.28). \square

Note that in a neighborhood of $\partial\Omega$, $\tilde{\gamma}$ is just the induced metric on the surfaces $S_\lambda = \{y \in \mathbb{R}^n; d(y, \partial\Omega) = \lambda\}$ and in the interior $\tilde{\gamma}$ is just the interior metric g .

4. ESTIMATES FOR THE PROJECTION OF A TENSOR TO THE TANGENT SPACE OF THE BOUNDARY

Definition 4.1. Let N be the unit normal to $\partial\Omega$ and let $\nabla_N = N^j \nabla_j$ be the normal derivative. Let $d(t, y) = \text{dist}_g(y, \partial\Omega)$ be the geodesic distance from y to $\partial\Omega$ and let $N_i = \nabla_i d$ be the geodesic extension of the normal to the interior. Let $\theta_{ij} = \nabla_i N_j = \nabla_i \nabla_j d$ be the second fundamental form of $\partial\Omega$. Let $\gamma_i^j = \delta_i^j - N_i N^j$ and if $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ are multi indices of length $|I| = r$ set $\gamma_I^J = \gamma_{i_1}^{j_1} \cdots \gamma_{i_r}^{j_r}$ and $N^I = N^{i_1} \cdots N^{i_r}$. If β is a $(0, r)$ tensor in Ω define the projection $\Pi\beta$ to a tensor on $\partial\Omega$ to be $(\Pi\beta)_I = \gamma_I^J \beta_J$. Let $\bar{\nabla}\beta = \Pi\nabla\beta$ denote the tangential covariant derivative. This is the intrinsic covariant derivative of $\partial\Omega$ if β is already tangential to $\partial\Omega$, i.e. if $\beta_{i_1 \dots i_k \dots i_r} N^k = 0$, $k = 1, \dots, r$, see Lemma 3.1. Furthermore, let ∇^r and $\bar{\nabla}^r$ be the operators which in components are given by $\nabla_I^r = \nabla_{i_1} \cdots \nabla_{i_r}$ respectively $\bar{\nabla}_I^r = \bar{\nabla}_{i_1} \cdots \bar{\nabla}_{i_r}$.

Definition 4.2. Let α is a $(0, s)$ tensor and β is a $(0, r)$ tensor. We will let $\alpha \tilde{\otimes} \beta$ denote some partial symmetrization of the tensor product $\alpha \otimes \beta$. I.e. a sum over some subset of the permutations of the indices divided by the number of permutations in that subset. In each situation there is of course a specific subset, but in our estimates it does not matter which one so to simplify the exposition we do not write out the exact permutations. Similarly, we let $\alpha \tilde{\cdot} \beta$ denote a partial symmetrization of the dot product $\alpha \cdot \beta$, which in turn is defined to be a contraction of the last index of α with the first index of β : $(\alpha \cdot \beta)_{i_1 \dots i_{r+s-2}} = g^{ij} \alpha_{i_1 \dots i_{s-1} i} \beta_{j i_s \dots i_{r+s-2}}$ \square

The simple observation that will help us is that if $q = 0$ on $\partial\Omega$ then the projection of the tensor $\nabla^2 q$ to the boundary will only contain first order derivatives of q and it will contain all components of the second fundamental form. In fact

$$(4.1) \quad \Pi\nabla^2 q = \bar{\nabla}^2 q + \theta \nabla_N q$$

where the tangential derivatives $\bar{\nabla}^2 q = 0$ on the boundary. To prove (4.1) we note that

$$(4.2) \quad \gamma_j^k \nabla_i \gamma_k^l = -\gamma_j^k \nabla_i (N_k N^l) = -\gamma_j^k \theta_{ik} N^l - \gamma_j^k N_k \theta_i^l = -\theta_{ij} N^l$$

so

$$(4.3) \quad \begin{aligned} \bar{\nabla}_i \bar{\nabla}_j q &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \gamma_{j'}^{j''} \nabla_{j''} q = \gamma_i^{i'} \gamma_j^{j'} \gamma_{j'}^{j''} \nabla_{i'} \nabla_{j''} q + \gamma_i^{i'} \gamma_j^{j'} (\nabla_{i'} \gamma_{j'}^{j''}) \nabla_{j''} q \\ &= \gamma_i^{i'} \gamma_j^{j'} \nabla_{i'} \nabla_{j'} q - \theta_{ij} \nabla_N q \end{aligned}$$

We now want to find a higher order versions of (4.1). One way to understand why there should be such a formula if $q = 0$ on $\partial\Omega$ is to expand q in a Taylor series in the geodesic distance d from the boundary. If $q = 0$ on $\partial\Omega$ then $q/d \sim \nabla_N q$ is a well defined function in a neighborhood of $\partial\Omega$ and hence we can write

$$\Pi\nabla^r q = \Pi\nabla^r \left(d \frac{q}{d} \right) = \sum_{s=0}^r \binom{r}{s} \Pi(\nabla^{r-s} d) \tilde{\otimes} \Pi\nabla^s \left(\frac{q}{d} \right)$$

Since however $d = \Pi\nabla d = 0$ on $\partial\Omega$ and $\nabla^2 d = \theta$ we obtain

$$(4.4) \quad \Pi\nabla^r q = \sum_{s=0}^{r-2} \binom{r}{s} \Pi(\nabla^{r-2-s} \theta) \tilde{\otimes} \Pi\nabla^s \left(\frac{q}{d} \right)$$

Proposition 4.1. *On $\partial\Omega$ we have*

$$(4.5) \quad \left| (\Pi\nabla^r)q - \bar{\nabla}^r q - \nabla_N q \bar{\nabla}^{r-2}\theta - \sum_{s=1}^{r-2} \binom{r}{s} (\bar{\nabla}^{r-2-s}\theta) \tilde{\otimes} (\bar{\nabla}^s \nabla_N q) \right| \\ \leq C \sum_{\substack{r_0+r_1+\dots+r_k+\ell=r-k \\ k-\ell=0 \pmod{2}, k \geq \ell \geq 0, k \geq 2}} |\bar{\nabla}^{r_1}\theta| \dots |\bar{\nabla}^{r_k}\theta| |\bar{\nabla}^{r_0} \nabla_N^\ell q|$$

and

$$(4.6) \quad |\bar{\nabla}^{r_0} \nabla_N^\ell q| \leq C \sum_{\tilde{r}_0+\tilde{r}_1+\dots+\tilde{r}_k=r_0+\ell-k} |\bar{\nabla}^{\tilde{r}_1}\theta| \dots |\bar{\nabla}^{\tilde{r}_k}\theta| |\nabla^{\tilde{r}_0} q|$$

$$(4.7) \quad |\nabla^{r_0} q| \leq C \sum_{\tilde{r}_0+\ell+\tilde{r}_1+\dots+\tilde{r}_k=r_0-k} |\bar{\nabla}^{\tilde{r}_1}\theta| \dots |\bar{\nabla}^{\tilde{r}_k}\theta| |\bar{\nabla}^{\tilde{r}_0} \nabla_N^\ell q|$$

where the sums are over all positive integers $r_i \geq 0$, $\tilde{r}_i \geq 0$ and $k, \ell \geq 0$.

Proposition 4.2. *We have*

$$(4.8) \quad (\Pi\nabla^r)_{Jq} = \sum_{r_0+r_1+\dots+r_k+\ell=r-k} c_{k\ell J I_0 \dots I_k}(g) (\bar{\nabla}^{r_1}\theta)_{I_1} \otimes \dots \otimes (\bar{\nabla}^{r_k}\theta)_{I_k} \otimes \bar{\nabla}_{I_0}^{r_0} (\nabla_N^\ell q)$$

where the sum is over positive integers $k, \ell, m \geq 0$, $k - \ell = 2m \geq 0$, $r_i \geq 0$ and all permutations (I_0, I_1, \dots, I_k) of (J, i_1, \dots, i_{2m}) . Here

$$(4.9) \quad c_{k\ell J I_0 \dots I_k}(g) = d_{k\ell m J I_0 \dots I_k} g^{i_1 i_2} \dots g^{i_{2m-1} i_{2m}}$$

denotes contractions over m indices. Furthermore.

$$(4.10) \quad (\Pi\nabla^r)q = \tilde{\nabla}^r q + \sum_{s=0}^{r-2} \binom{r}{s} (\bar{\nabla}^{r-2-s}\theta) \tilde{\otimes} (\bar{\nabla}^s \nabla_N q) + F$$

where F is of the form in the right hand side of (4.8) but with $k \geq 2$ in the sum.

Remark. Proposition 4.1 and Proposition 4.2 applies to the function q being replaced by $(0, s)$ tensor α as well if the projections, tangential and normal derivatives are correctly interpreted: only the first r indices should be projected. This will be explained later in this section, see Proposition 4.8.

The proof of Proposition 4.1 and Proposition 4.2 consists of turning projections onto the tangential and normal components into tangential derivatives of normal derivatives. The basic idea is that any derivative ∇^r of order r can be expressed as a sum of combinations of tangential derivatives $\bar{\nabla}$ and normal derivatives ∇_N of total order at most $s \leq r$ and similarly any combination of normal and tangential derivatives of total order r can be expressed as a sum of derivatives ∇^s for $s \leq r$. Since the coefficients of both the normal derivative and of the projection involved in the tangential derivative are made up out of the normal it follows that the coefficients in expressing a derivative ∇^r in terms of normal ∇_N and tangential $\bar{\nabla}$ derivatives will consists of derivatives of the normal, i.e. derivatives of the second fundamental form θ . Whenever a derivative in say (4.5)-(4.8) falls on the normal it produces a new factor θ and at the same time the total number of derivatives involved has gone down by one, so the total number of derivatives in the expressions (4.5)-(4.8) goes down by one for each new factor of θ . This simple observation will prove (4.6)-(4.7) and (4.8). The more detailed information in (4.5) and (4.10) formally follows from (4.4) and the above argument.

The key to turn tangential and normal components into tangential derivatives of normal components is Lemma 4.3 below. In Lemma 4.4 it is then expressed in a form which is more directly adapted to the situation in Proposition 4.1 and Proposition 4.2.

Lemma 4.3. *Suppose that S is a $(0, r + \ell + s)$ tensor which is symmetric with respect to the first $r + \ell$ indices. Let*

$$(4.11) \quad S_{i_1 \dots i_{r+s}}^{r, \ell} = (\Pi^{r, \ell} S)_{i_1 \dots i_{r+s}} = \gamma_{i_1}^{j_1} \dots \gamma_{i_r}^{j_r} N^{j_{r+1}} \dots N^{j_{r+\ell}} S_{j_1 \dots j_{r+\ell} i_{r+1} \dots i_{r+s}}$$

be the projection of the first indices onto r tangential and ℓ normal components. Then

$$(4.12) \quad \Pi^{r+1, 0} \nabla \Pi^{r, \ell} S = \Pi^{r+1, \ell} \nabla S - r \theta \tilde{\otimes} \Pi^{r-1, \ell+1} S + \ell \theta \cdot \Pi^{r+1, \ell-1} S$$

where

$$(4.13) \quad (\theta \tilde{\otimes} \Pi^{r-1, \ell+1} S)_{i_0 i_1 \dots i_r i_{r+1} \dots i_{r+s}} = \frac{1}{r} \sum_{p=1}^r \theta_{i_0 i_p} (\Pi^{r-1, \ell+1} S)_{I_p i_{r+1} \dots i_{r+s}}$$

$$(4.14) \quad (\theta \cdot \Pi^{r+1, \ell-1} S)_{i_0 i_1 \dots i_r i_{r+1} \dots i_{r+s}} = \theta_{i_0}^j (\Pi^{r+1, \ell-1} S)_{j i_1 \dots i_r i_{r+1} \dots i_{r+s}}$$

where $I_p = (i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_r)$.

Proof of Lemma 4.3. To simplify notation we assume that $s = 0$. Now

$$S_{i_1 \dots i_r}^{r, \ell} = \gamma_I^J N^{J'} S_{JJ'} = \gamma_{i_1}^{j_1} \dots \gamma_{i_r}^{j_r} N^{j_{r+1}} \dots N^{j_{r+\ell}} S_{j_1 \dots j_{r+\ell}}$$

where $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ be multi indices of length r and $J' = (j_{r+1}, \dots, j_{r+\ell})$ is a multi index of length ℓ . Now

$$\begin{aligned} \bar{\nabla}_{i_0} S_{i_1 \dots i_r}^{r, \ell} &= \gamma_{i_0}^{j_0} \gamma_I^L \nabla_{j_0} (\gamma_L^J N^{J'} S_{JJ'}) = \\ &= \gamma_{i_0}^{j_0} \gamma_I^J N^{J'} \nabla_{j_0} S_{JJ'} + \gamma_{i_0}^{j_0} \gamma_I^L (\nabla_{j_0} \gamma_L^J) N^{J'} S_{JJ'} + \gamma_{i_0}^{j_0} \gamma_I^J (\nabla_{j_0} N^{J'}) S_{JJ'} \end{aligned}$$

By (4.2)

$$\gamma_I^L \nabla_{i_0} \gamma_L^J = - \sum_{p=1}^r \theta_{i_0 i_p} \gamma_{I_p}^{J_p} N^{j_p}$$

where $I_p = (i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_r)$ and $J_p = (j_1, \dots, j_{p-1}, j_{p+1}, \dots, j_r)$. Furthermore

$$\nabla_{i_0} N^{J'} = \sum_{p=r+1}^{r+\ell} \theta_{i_0}^{j'_p} N^{j'_p}$$

where $J'_p = (j_{r+1}, \dots, j_{p-1}, j_{p+1}, \dots, j_{r+\ell})$. If we now assume that S is symmetric the notation simplifies a bit and we obtain the lemma. \square

Now we want to apply Lemma 4.3 to $S = \nabla^{r+\ell} q$. Since in geodesic coordinates $\nabla_N N = 0$ it follows that

$$(4.15) \quad [\nabla_N, \Pi] = 0, \quad \nabla_N^\ell = N^{i_1} \dots N^{i_\ell} \nabla_{i_1} \dots \nabla_{i_\ell}$$

Lemma 4.4. *Let*

$$(4.16) \quad S^{r,\ell} = \Pi^{r,\ell} \nabla^{r+\ell} q = \nabla_N^\ell \Pi \nabla^r q,$$

Then

$$(4.17) \quad S^{r+1,\ell} = \bar{\nabla} S^{r,\ell} + r \theta \tilde{\otimes} S^{r-1,\ell+1} - \ell \theta \cdot S^{r+1,\ell-1}$$

Furthermore

$$(4.18) \quad S^{r,\ell} - \bar{\nabla}^r S^{0,\ell} = \sum_{k=0}^{r-2} \bar{\nabla}^{r-2-k} ((k+1) \theta \tilde{\otimes} S^{k,\ell+1} - \ell \theta \cdot S^{k+2,\ell-1})$$

and

$$(4.19) \quad \begin{aligned} S^{r,\ell} - \bar{\nabla}^r S^{0,\ell} &= \sum_{m=0}^{r-2} \binom{r}{m} (\bar{\nabla}^m \theta) \tilde{\otimes} \bar{\nabla}^{r-2-m} S^{0,\ell+1} - \sum_{m=0}^{r-2} \ell \binom{r-1}{m} (\bar{\nabla}^m \theta) \tilde{\cdot} \bar{\nabla}^{r-m} S^{0,\ell-1} \\ &\quad + \sum a_{r_1 r_2 k} (\bar{\nabla}^{r_1} \theta) \tilde{\otimes} (\bar{\nabla}^{r_2} \theta) \tilde{\otimes} \bar{\nabla}^{r-4-r_1-r_2-k} S^{k,\ell+2} \\ &\quad + (\ell+1) \sum b_{r_1 r_2 k} (\bar{\nabla}^{r_1} \theta) \tilde{\otimes} (\bar{\nabla}^{r_2} \theta) \tilde{\cdot} \bar{\nabla}^{r-2-r_1-r_2-k} S^{k,\ell} \\ &\quad + \ell \sum c_{r_1 r_2 k} (\bar{\nabla}^{r_1} \theta) \tilde{\cdot} (\bar{\nabla}^{r_2} \theta) \tilde{\otimes} \bar{\nabla}^{r-2-r_1-r_2-k} S^{k,\ell} \\ &\quad + \ell(\ell-1) \sum d_{r_1 r_2 k} (\bar{\nabla}^{r_1} \theta) \tilde{\cdot} (\bar{\nabla}^{r_2} \theta) \tilde{\cdot} \bar{\nabla}^{r-r_1-r_2-k} S^{k,\ell-2} \end{aligned}$$

where the sums are over all integers $r_1, r_2, k \geq 0$ such that all exponents of differentiation also are ≥ 0 .

Proof. (4.17) follows from (4.12). Now by repeated use of (4.17)

$$\begin{aligned} S^{r,\ell} &= \bar{\nabla} S^{r-1,\ell} + (r-1) \theta \tilde{\otimes} S^{r-2,\ell+1} - \ell S^{r,\ell-1} \\ &= \bar{\nabla} (\bar{\nabla} S^{r-2,\ell} + (r-2) \theta \tilde{\otimes} S^{r-3,\ell+1} - \ell \theta \cdot S^{r-1,\ell-1}) + (r-1) \theta \tilde{\otimes} S^{r-2,\ell+1} - \ell \theta \cdot S^{r,\ell-1} \\ &= \dots = \bar{\nabla}^r S^{0,\ell} + \sum_{k=0}^{r-2} \bar{\nabla}^{r-2-k} ((k+1) \theta \tilde{\otimes} S^{k,\ell+1} - \ell \theta \cdot S^{k+2,\ell-1}) \end{aligned}$$

which proves (4.18). To proceed further we must use (4.18) twice. In the right hand side of (4.18) we use (4.18) to write $S^{k,\ell+1}$ as $\bar{\nabla}^k S^{k,\ell+1}$ plus terms involving one factor of θ and write $S^{k+2,\ell-1}$ as $\bar{\nabla}^{k+2} S^{0,\ell-1}$ plus terms involving one factor of θ . Let us first calculate the term involving one factor of θ . By Leibnitz' rule we have

$$\begin{aligned} &\sum_{k=0}^{r-2} \bar{\nabla}^{r-2-k} ((k+1) \theta \tilde{\otimes} \bar{\nabla}^k S^{0,\ell+1} - \ell \theta \cdot \bar{\nabla}^k S^{0,\ell-1}) \\ &= \sum_{k=0}^{r-2} \sum_{m=0}^{r-2-k} \binom{r-2-k}{m} (k+1) (\bar{\nabla}^m \theta) \tilde{\otimes} \bar{\nabla}^{r-2-m} S^{0,\ell+1} - \ell \sum_{k=0}^{r-2} \sum_{m=0}^{r-2-k} \binom{r-2-k}{m} (\bar{\nabla}^m \theta) \tilde{\cdot} \bar{\nabla}^{r-2-m} S^{0,\ell-1} \\ &= \sum_{m=0}^{r-2} \binom{r}{m} (\bar{\nabla}^m \theta) \tilde{\otimes} \bar{\nabla}^{r-2-m} S^{0,\ell+1} - \ell \sum_{m=0}^{r-2} \binom{r-1}{m} (\bar{\nabla}^m \theta) \tilde{\cdot} \bar{\nabla}^{r-m} S^{0,\ell-1} \end{aligned}$$

since $\sum_{k=0}^{r-2-m} (k+1) \binom{r-2-k}{m} = \binom{r}{m}$ and $\sum_{k=0}^{r-2-m} \binom{r-2-k}{m} = \binom{r-1}{m}$. This explains the terms involving one factor of θ in the first row of (4.19). Using (4.18) and Leibnitz' rule its easy to see that the term involving two factors of θ has to be of the form in (4.19). \square

Proof of Proposition 4.1 and Proposition 4.2. The proof is just an application of Lemma 4.4: (4.5) follows from (4.8). (4.8) follows by induction from (4.17), noticing that the total order of the tensor goes down by one for each new factor of θ . (4.10) follows from (4.19). (4.6) and (4.7) follows from the same argument. \square

Using (4.17)-(4.18) one can show that

$$(4.20) \quad \Pi \nabla^2 q = \bar{\nabla}^2 q + \theta \nabla_N q,$$

$$(4.21) \quad \Pi \nabla^3 q = \bar{\nabla}^3 q - 2\theta \tilde{\otimes} (\theta \cdot \bar{\nabla} q) + (\bar{\nabla} \theta) \nabla_N q + 3\theta \tilde{\otimes} \bar{\nabla} \nabla_N q$$

$$(4.22) \quad \begin{aligned} \Pi \nabla^4 q = & \bar{\nabla}^4 q - \theta \tilde{\otimes} (5(\bar{\nabla} \theta) \cdot \bar{\nabla} q + 8\theta \cdot \bar{\nabla}^2 q) - 2(\bar{\nabla} \theta) \tilde{\otimes} (\theta \cdot \bar{\nabla} q) \\ & + (\bar{\nabla}^2 \theta) \nabla_N q + 4(\bar{\nabla} \theta) \tilde{\otimes} \bar{\nabla} \nabla_N q + 6\theta \tilde{\otimes} \bar{\nabla}^2 \nabla_N q - 3\theta \tilde{\otimes} (\theta \cdot \theta) \nabla_N q + 3\theta \tilde{\otimes} \theta \nabla_N^2 q \end{aligned}$$

Since $\nabla_N = N \cdot \nabla$, $\Pi \nabla N = \nabla N = \theta$ and $\Pi \nabla^2 N = \Pi \nabla \theta = \Pi \nabla \Pi \theta = \bar{\nabla} \theta$ and $\nabla_N \theta = \Pi N \cdot \nabla^2 q = -\Pi(\nabla N) \cdot (\nabla N) = -\theta \cdot \theta$ (see (4.51)) we get

$$(4.23) \quad \bar{\nabla} q = \Pi \nabla q$$

$$(4.24) \quad \bar{\nabla} \nabla_N q = \Pi N \cdot \nabla^2 q + \theta \cdot \nabla q$$

$$(4.25) \quad \nabla_N^2 q = N \cdot (N \cdot \nabla^2 q)$$

$$(4.26) \quad \bar{\nabla}^2 q = \Pi \nabla^2 q - \theta N \cdot \nabla q$$

$$(4.27) \quad \bar{\nabla}^2 \nabla_N q = \Pi N \cdot \nabla^3 q + 2\theta \cdot \Pi \nabla^2 q + (\bar{\nabla} \theta) \cdot \Pi \nabla q - \theta \cdot \theta N \cdot \nabla q - \theta N \cdot (N \cdot \nabla^2 q)$$

where in (4.27) we used that $\bar{\nabla}^2 \nabla_N q = \Pi \nabla^2 \nabla_N q - \theta \nabla_N^2 q$.

Proposition 4.5. *Suppose that $q = 0$ on $\partial\Omega$ and $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$. Let $L^p = L^p(\partial\Omega)$ and suppose that $\iota_1 \geq 1/K_1$, where ι_1 is as in Definition 3.3. Then for $m = 0, 1$ and any $\varepsilon > 0$ we have*

$$(4.28) \quad \begin{aligned} \|\Pi \nabla^r q - (\nabla_N q) \bar{\nabla}^{r-2} \theta\|_{L^2} \leq & \varepsilon \|\nabla_N q\|_{L^\infty} \|\bar{\nabla}^{r-2} \theta\|_{L^2} + C(1/\varepsilon) \sum_{k=1}^{r-1} \|\theta\|_{L^\infty}^k \|\nabla^{r-k} q\|_{L^2} \\ & + C(K_1, 1/\varepsilon, \|\theta\|_{L^\infty}) \left(\|\theta\|_{L^\infty} + \sum_{0 \leq s \leq r-2-m} \|\bar{\nabla}^s \theta\|_{L^2} \right) \sum_{0 \leq s \leq r-2+m} \|\nabla^s q\|_{L^2} \end{aligned}$$

where the second line drops out if $r \leq 4$.

Proof of Proposition 4.5 for $r \leq 4$. We want to prove (4.28) for $r = 4$, since the proof for $r \leq 3$ is simpler and it follows in the same way. By (4.22) we have if $q = 0$ on $\partial\Omega$;

$$\Pi \nabla^4 q = (\bar{\nabla}^2 \theta) \nabla_N q + 4(\bar{\nabla} \theta) \tilde{\otimes} \bar{\nabla} \nabla_N q + 6\theta \tilde{\otimes} \bar{\nabla}^2 \nabla_N q - 3\theta \tilde{\otimes} (\theta \cdot \theta) \nabla_N q + 3\theta \tilde{\otimes} \theta \nabla_N^2 q$$

The only problematic term can be controlled by Lemma 8.1 (here $L^p = L^p(\partial\Omega)$):

$$\begin{aligned} \|\bar{\nabla} \theta \cdot \bar{\nabla} \nabla_N q\|_{L^2} & \leq \|\bar{\nabla} \theta\|_{L^4} \|\bar{\nabla} \nabla_N q\|_{L^4} \leq C \|\theta\|_{L^\infty}^{1/2} \|\bar{\nabla}^2 \theta\|_{L^2}^{1/2} \|\nabla_N q\|_{L^\infty}^{1/2} \|\bar{\nabla}^2 \nabla_N q\|_{L^2}^{1/2} \\ & \leq C 2^{-1} \varepsilon \|\nabla_N q\|_{L^\infty} \|\bar{\nabla}^2 \theta\|_{L^2} + C 2^{-1} \varepsilon^{-1} \|\theta\|_{L^\infty} \|\bar{\nabla}^2 \nabla_N q\|_{L^2}, \quad \text{for any } \varepsilon > 0. \end{aligned}$$

By (4.27), since $\Pi \nabla q = 0$ on $\partial\Omega$;

$$\|\bar{\nabla}^2 \nabla_N q\|_{L^2} \leq \|\nabla^3 q\|_{L^2} + 3\|\theta\|_{L^\infty} \|\nabla^2 q\|_{L^2} + \|\theta\|_{L^\infty}^2 \|\nabla q\|_{L^2} \quad \square$$

The basic inequalities that we will use on the boundary for the proof of Theorem 4.5 in general can be summarized in:

Lemma 4.6. *Let $L^p = L^p(\partial\Omega)$ and let $t = r - 2$. Then if $t - m \geq s$*

$$(4.29) \quad \|\bar{\nabla}^s \alpha\|_{L^{2t/(s+m)}} \leq C \|\alpha\|_{L^{2t/m}}^{1-s/(t-m)} \|\bar{\nabla}^{t-m} \alpha\|_{L^2}^{s/(t-m)}, \quad m \geq 0, \quad t - m \geq s$$

$$(4.30) \quad \|\nabla^s \alpha\|_{L^{2t/(s-m)}} \leq C(K_1) \sum_{\ell=s}^{t+m} \|\nabla^\ell \alpha\|_{L^2}, \quad \text{if } t \geq \frac{n-1}{2} \quad s - m \geq 0, \quad t + m \geq s$$

where K_1 is a constant such that $\iota_1 \geq 1/K_1$, and ι_1 is as in Definition 3.3. Furthermore

$$(4.31) \quad \|\bar{\nabla}^{r_1} \theta \cdots \bar{\nabla}^{r_k} \theta\|_{L^p} \leq C \|\theta\|_{L^\infty}^{k-1} \|\bar{\nabla}^{r_1+\dots+r_k} \theta\|_{L^p}$$

$$(4.32) \quad \|\bar{\nabla}^s \theta\|_{L^{2t/(s+m)}} \leq C \|\theta\|_{L^{2t/m}}^{1-(s+m)/t} \|\bar{\nabla}^{t-m} \theta\|_{L^2}^{(s+m)/t}, \quad m \geq 0$$

Furthermore, we have for every $\varepsilon > 0$ if $1 \leq s \leq t$

$$(4.33) \quad \|\bar{\nabla}^{t-s} \theta | \bar{\nabla}^s \nabla_N q\|_{L^2} \leq \varepsilon \|\nabla_N q\|_{L^\infty} \|\bar{\nabla}^t \theta\|_{L^2} + C \varepsilon^{-(t-s)/s} \|\theta\|_{L^\infty} \|\bar{\nabla}^t \nabla_N q\|_{L^2},$$

and if $0 \leq m \leq s \leq t - m$

$$(4.34) \quad \|\bar{\nabla}^{t-s} \theta | \nabla^s q\|_{L^2} \\ \leq \|\bar{\nabla}^{t-s} \theta\|_{L^{2t/(t-s+m)}} \|\nabla^s q\|_{L^{2t/(s-m)}} \leq C(K_1) \|\theta\|_{L^{2t/m}}^{1-(t-s+m)/t} \|\bar{\nabla}^{t-m} \theta\|_{L^2}^{(t-s+m)/t} \sum_{\ell=s}^{t+m} \|\nabla^\ell q\|_{L^2},$$

Proof of Lemma 4.6. (4.29), (4.31)-(4.33) are just the interpolation inequality (8.4) in Lemma 8.1. For the proof of (4.31) one first uses Hölder's inequality. (4.30) on the other hand is a special case of Sobolev's lemma, Lemma 8.2, which by the remark after the lemma holds with the covariant differentiation of the interior restricted to the boundary. By Hölder's inequality and (4.29) with $m = 0$:

$$\begin{aligned} \|\bar{\nabla}^{t-s} \theta | \bar{\nabla}^s \nabla_N q\|_{L^2} &\leq \|\bar{\nabla}^{t-s} \theta\|_{L^{2t/(t-s)}} \|\bar{\nabla}^s \nabla_N q\|_{L^{2t/s}} \\ &\leq C \|\theta\|_{L^\infty}^{s/t} \|\bar{\nabla}^t \theta\|_{L^2}^{1-s/t} \|\nabla_N q\|_{L^\infty}^{1-s/t} \|\bar{\nabla}^t \nabla_N q\|_{L^2}^{s/t} \\ &\leq \varepsilon \|\nabla_N q\|_{L^\infty} \|\bar{\nabla}^t \theta\|_{L^2} + C \varepsilon^{-(t-s)/s} \|\theta\|_{L^\infty} \|\bar{\nabla}^t \nabla_N q\|_{L^2}, \quad \text{for any } \varepsilon > 0, \quad \square \end{aligned}$$

which proves (4.33). (4.34) follows from Hölder's inequality and (4.30) applied to $\alpha = q$ and (4.32).

Proof of Proposition 4.5 in case $r \geq 5$. The proof is an application of Proposition 4.1 and Lemma 4.6. Since $q = 0$ the term $\bar{\nabla}^r q = 0$ in the left of (4.5) and the terms in the right with $\ell = 0$ vanishes as well so $\ell \geq 2$ in the right sum. Each term in the sum on the left of (4.5) can be estimated using (4.33). Then we can use (4.6) to estimate $\|\theta\|_{L^\infty} \|\bar{\nabla}^{r-2} \nabla_N q\|_{L^2}$ by $\|\theta\|_{L^\infty} \|\nabla^{r-1} q\|_{L^2}$ plus a sum of terms of the form

$$(4.35) \quad \|\theta\|_{L^\infty} \|\bar{\nabla}^{r_2} \theta \cdots \bar{\nabla}^{r_k} \theta | \nabla^{r_0} q\|_{L^2}, \quad r_0 + r_2 + \dots + r_k = r - k, \quad k \geq 2$$

Similarly, if we use (4.6) we can estimate the terms in the right of (4.5) (the second line of (4.5)) by

$$(4.36) \quad \|\bar{\nabla}^{r_1} \theta \cdots \bar{\nabla}^{r_k} \theta | \nabla^{r_0} q\|_{L^2}, \quad r_0 + r_1 + \dots + r_k = r - k, \quad k \geq 2$$

Now, a typical term looks like

$$\|\theta\|_{L^\infty} \|\bar{\nabla}^{r-2-s} \theta | \nabla^s q\|_{L^2}$$

which can be estimate by (4.34) with $m = 0, 1$. The general term is not much harder: Using Hölder's inequality and (4.31) we see that we must estimate

$$(4.37) \quad \|\theta\|_{L^\infty}^{k-1} \|\bar{\nabla}^{r'} \theta\|_{L^p} \|\nabla^{r_0} q\|_{L^{p'}}, \quad r_0 + r' = r - k, \quad k \geq 2$$

for some $1/p + 1/p' = 1/2$ which are to be determined. If $r' = 0$ then we can take $p = \infty$ so we may assume that $r' \geq 1$. Similarly, we may assume that $r_0 \geq 2$ since if $r_0 = 1$ we can take $p' = \infty$. We pick

$$(4.38) \quad p = 2(r-2)/(r'+m), \quad p' = 2(r-2)/(r-2-r'-m)$$

and use (4.34) with $m = 0, 1$. \square

Note that Proposition 4.1 and Proposition 4.2 applies to q being replaced by $(0, t)$ tensor α as well if the projections, tangential and normal derivatives are correctly interpreted. Only the first r indices should be projected, i.e. all indices referring to θ should be projected as well as the ones referring to differentiation of α but the ones referring to α itself should not. So we should replace $\Pi \nabla^r$ by $\Pi^{r,0} \nabla^r$ and we should replace $\bar{\nabla}^r$ when applied to α by $\bar{\bar{\nabla}}^r = \Pi^{r,0} \nabla \Pi^{r-1,0} \nabla \dots \Pi^{2,0} \nabla \Pi^{1,0} \nabla$. (One should keep the old definition of $\bar{\nabla}^r \theta$ since all these indices are projected over.) In components this means:

Definition 4.3. Let

$$(4.39) \quad (\Pi^{r,0} \nabla^r)_{i_1 \dots i_r} \alpha_{i_{r+1} \dots i_{r+t}} = \gamma_{i_1}^{j_1} \dots \gamma_{i_r}^{j_r} \nabla_{j_1} \dots \nabla_{j_r} \alpha_{i_{r+1} \dots i_{r+t}}, \quad \nabla_N \alpha_{i_1 \dots i_t} = N^k \nabla_k \alpha_{i_1 \dots i_t}$$

and

$$(4.40) \quad (\bar{\bar{\nabla}}^r)_{i_1 \dots i_r} \alpha_{i_{r+1} \dots i_{r+t}} = \gamma_{i_1}^{j_1} \dots \gamma_{i_r}^{j_r} \nabla_{j_1} \left(\gamma_{j_2}^{k_2} \dots \gamma_{j_r}^{k_r} \nabla_{k_2} \left(\dots \right. \right. \\ \left. \left. \dots \gamma_{m_{r-2}}^{N_{r-2}} \gamma_{m_{r-1}}^{N_{r-1}} \gamma_{m_r}^{N_r} \nabla_{N_{r-2}} \left(\gamma_{N_{r-1}}^{o_{r-1}} \gamma_{N_r}^{o_r} \nabla_{o_{r-1}} \left(\gamma_{o_r}^{p_r} \nabla_{p_r} \alpha_{i_{r+1} \dots i_{r+t}} \right) \right) \dots \right) \right)$$

In fact, with this modification the proofs of Lemma 4.3 and Lemma 4.4 goes through. Also the interpolation inequality in Lemma 8.1 remains true. One just has to modify the proof to work with mixed tangential and full inner products

$$(4.41) \quad \langle \alpha, \beta \rangle_{\gamma g} = \gamma^{i_1 j_1} \dots \gamma^{i_s j_s} g^{i_{s+1} j_{s+1}} \dots g^{i_{s+t} j_{s+t}} \alpha_{i_1 \dots i_s i_{s+1} \dots i_{s+t}} \beta_{j_1 \dots j_s j_{s+1} \dots j_{s+t}}$$

Hence we obtain the following version of the interpolation inequality:

Lemma 4.7. *Suppose that α is a $(0, t)$ tensor and let $\bar{\bar{\nabla}}^s$ be defined as in (4.40). Then if $s \leq r - 2$*

$$(4.42) \quad \left\| \bar{\bar{\nabla}}^s \alpha \right\|_{L^{2(r-2)/s}} \leq C \|\alpha\|_{L^\infty}^{1-s/(r-2)} \left\| \bar{\bar{\nabla}}^{r-2} \alpha \right\|_{L^2}^{s/(r-2)},$$

In order to deal with some lower order terms the following is useful:

Proposition 4.8. *Suppose that α is a $(0, \mu)$ tensor and let $\Pi^{s,0} \nabla^s$ and $\bar{\bar{\nabla}}^s$ be defined as in (4.39) and (4.40). Let $t = r - 2$. Then*

$$(4.43) \quad |(\Pi^{s,0} \nabla^s) \alpha - \bar{\bar{\nabla}}^s \alpha| \leq C \sum_{\substack{r_0+r_1+\dots+r_k=s-k \\ k \geq 1, r_0 \geq 1}} |\bar{\nabla}^{r_1} \theta| \dots |\bar{\nabla}^{r_k} \theta| |\nabla^{r_0} \alpha|$$

Here $\overline{\nabla}^r \alpha$ is defined by projecting only over the first r components as in (4.40) whereas $\overline{\nabla}^r \theta$ is defined as before by projecting over all $r + 2$ components. If $s \leq t$

(4.44)

$$\|(\Pi^{s,0} \nabla^s) \alpha\|_{L^{2t/s}} \leq C \|\alpha\|_{L^\infty}^{1-s/t} \|\overline{\nabla}^t \alpha\|_{L^2}^{s/t} + C(K_1)(1 + \|\theta\|_{L^\infty})^s (\|\theta\|_{L^\infty} + \|\overline{\nabla}^t \theta\|_{L^2})^{s/t} \sum_{\ell=0}^{t-1} \|\nabla^\ell \alpha\|_{L^2}$$

where K_1 is a constant such that $\iota_1 \geq 1/K_1$ and ι_1 is as in Definition 3.3. Furthermore

$$(4.45) \quad \|\overline{\nabla}^t \alpha\|_{L^2} \leq C \|\nabla^t \alpha\|_{L^2} + C(K_1)(1 + \|\theta\|_{L^\infty})^t (\|\theta\|_{L^\infty} + \|\overline{\nabla}^t \theta\|_{L^2}) \sum_{\ell=0}^{t-1} \|\nabla^\ell \alpha\|_{L^2}$$

and

$$(4.46) \quad \begin{aligned} & \| |(\Pi^{s,0} \nabla^s) \alpha| |(\Pi^{t-s,0} \nabla^{t-s}) \beta| \|_{L^2} \leq \|(\Pi^{s,0} \nabla^s) \alpha\|_{L^{2t/s}} \|(\Pi^{t-s,0} \nabla^{t-s}) \beta\|_{L^{2t/(t-s)}} \\ & \leq C(K_1) \left(\|\alpha\|_{L^\infty} + \sum_{\ell=0}^{t-1} \|\nabla^\ell \alpha\|_{L^2} \right) \|\nabla^t \beta\|_{L^2} + C(K_1) \left(\|\beta\|_{L^\infty} + \sum_{\ell=0}^{t-1} \|\nabla^\ell \beta\|_{L^2} \right) \|\nabla^t \alpha\|_{L^2} + \\ & + C(K_1)(1 + \|\theta\|_{L^\infty})^t (\|\theta\|_{L^\infty} + \|\overline{\nabla}^t \theta\|_{L^2}) \left(\|\alpha\|_{L^\infty} + \sum_{\ell=0}^{t-1} \|\nabla^\ell \alpha\|_{L^2} \right) \left(\|\beta\|_{L^\infty} + \sum_{\ell=0}^{t-1} \|\nabla^\ell \beta\|_{L^2} \right) \end{aligned}$$

Proof. (4.43) follows from Lemma 4.3. And if $r' = r_1 + \dots + r_k$, $r' + r_0 = s - k$ then by Hölder's inequality, (4.32) with $m = 0$ respectively (4.30) with $m = -k$

$$(4.47) \quad \begin{aligned} & \| |\overline{\nabla}^{r_1} \theta| \cdots |\overline{\nabla}^{r_k} \theta| |\nabla^{r_0} \alpha| \|_{L^{2t/s}} \leq C \|\theta\|_{L^\infty}^{k-1} \|\overline{\nabla}^{r'} \theta\|_{L^{2t/r'}} \|\nabla^{r_0} \alpha\|_{L^{2t/(r_0+k)}} \\ & \leq C(K_1) \|\theta\|_{L^\infty}^{k-r'/t} \|\overline{\nabla}^t \theta\|_{L^2}^{r'/t} \sum_{\ell=r_0}^{t-k} \|\nabla^\ell \alpha\|_{L^2} \leq C(K_1)(1 + \|\theta\|_{L^\infty})^s (\|\theta\|_{L^\infty} + \|\overline{\nabla}^t \theta\|_{L^2})^{s/t} \sum_{\ell=r_0}^{t-k} \|\nabla^\ell \alpha\|_{L^2}. \end{aligned}$$

If $s = t$ this proves (4.45). (4.44) follows from (4.43), (4.42) and (4.47). (4.46) follows from (4.44), (4.45) and our usual convexity inequality $a^{s/t} b^{1-s/t} \leq a + b$. \square

Let us now derive some properties of the projection. Since $g^{ij} = \gamma^{ij} + N^i N^j$ we have

$$(4.48) \quad \Pi(S \cdot R) = \Pi(S) \cdot \Pi(R) + \Pi(S \cdot N) \tilde{\otimes} \Pi(N \cdot R),$$

Note also that

$$(4.49) \quad [\nabla_N, \Pi]S = 0, \quad [\overline{\nabla}, \Pi]S = 0, \quad [\nabla_N, \nabla]S = -\theta \cdot \nabla S, \quad [\nabla_N, \overline{\nabla}]S = -\theta \cdot \overline{\nabla} S,$$

where we have used that $[\nabla_N, \overline{\nabla}] = [\nabla_N, \Pi \nabla \Pi] = \Pi[\nabla_N, \nabla] \Pi$. Since $N \cdot \overline{\nabla}^k \theta = 0$ we get

$$(4.50) \quad [\nabla_N, \overline{\nabla}^r]S = \sum_{\ell=0}^{r-1} \overline{\nabla}^\ell [\nabla_N, \overline{\nabla}] \overline{\nabla}^{r-1-\ell} S = - \sum_{k=0}^{r-1} \binom{r}{k+1} (\overline{\nabla}^k \theta) \cdot \overline{\nabla}^{r-k} S$$

where we used that $\sum_{\ell=0}^{r-1} \binom{\ell}{k} = \binom{r}{k+1}$ and $\overline{\nabla}((\Pi R) \cdot \Pi S) = (\overline{\nabla} \Pi R) \cdot \Pi S + (\Pi R) \cdot \overline{\nabla} \Pi S$. Furthermore, $0 = \nabla^2(N \cdot N) = 2N \cdot \nabla^2 N + 2(\nabla N) \cdot \nabla N$ so $\nabla_N \theta = -\theta \cdot \theta$ so (4.50) applied to $S = \theta$ gives

$$(4.51) \quad \nabla_N \overline{\nabla}^r \theta = - \sum_{k=0}^r \left(\binom{r}{k+1} + \binom{r}{k} \right) (\overline{\nabla}^k \theta) \cdot \overline{\nabla}^{r-k} \theta$$

5. ELLIPTIC ESTIMATES AND ENERGY ESTIMATES FOR THE BOUNDARY PROBLEM

Most of the results here will be stated in a coordinate independent way. We can however take advantage of that we have a transformation $f_t : \Omega \rightarrow \mathcal{D}_t \subset \mathbb{R}^n$ such that the metric is Euclidean in \mathcal{D}_t . Also, since we are looking for a short time existence our metric expressed in the y -coordinates in Ω $g_{ij}(t, y)$ is equivalent to the metric at $t = 0$, $g_{ij}(0, y)$ and similarly the induced metric on $\partial\Omega$ $\gamma_{ij}(t, \bar{y})$ is equivalent to $\gamma_{ij}(0, \bar{y})$. Throughout this section, ∇ will refer to covariant differentiation with respect to the metric g_{ij} in Ω and $\bar{\nabla}$ will refer to covariant differentiation on $\partial\Omega$ with respect to the induced metric γ_{ij} on $\partial\Omega$ as defined in the beginning of section 3.

We will assume that the normal N to $\partial\Omega$ is extended to a vector field in the interior of Ω satisfying $g_{ij}N^iN^j \leq 1$ there, such that in a neighborhood of $\partial\Omega$ N is the unit normal to the sets $\partial\Omega_\rho = \{y; \text{dist}_g(y, \partial\Omega) = \rho\}$ and N has the regularity described by Lemma 3.6 and Lemma 3.7. Then $\gamma_{ij} = g_{ij} - N_iN_j$, where $N_i = g_{ij}N^j$, is a positive semidefinite pseudo Riemannian metric in Ω . Using the decomposition into normal and tangential components $g^{ij} = N^iN^j + \gamma^{ij}$ we can write

$$(5.1) \quad g^{ij}g^{kl}\nabla_i\beta_k\nabla_j\beta_l = (N^iN^jg^{kl} + g^{ij}N^kN^l + \gamma^{ik}\gamma^{jl} - N^iN^kN^jN^l + \gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\nabla_i\beta_k\nabla_j\beta_l$$

$$(5.2) \quad g^{ij}g^{kl}\nabla_k\beta_i\nabla_l\beta_j = (g^{ij}\gamma^{kl} + \gamma^{ij}g^{kl} - (\gamma^{ik}\gamma^{jl} - N^iN^kN^jN^l) - (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}))\nabla_i\beta_k\nabla_j\beta_l$$

The terms $(\gamma^{ik}\gamma^{jl} - N^iN^kN^jN^l)\nabla_i\beta_k\nabla_j\beta_l$ and $(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\nabla_i\beta_k\nabla_j\beta_l$ are going to be lower order. The first one because it can be controlled by $\text{div } \beta = g^{ik}\nabla_i\beta_k$ which we expect to be lower order and the second one because the boundary term vanishes if we integrate by parts using Greens theorem. Hence, (5.1) and (5.2) says that we essentially can control $|\nabla\beta|^2 = g^{ij}g^{kl}\nabla_i\beta_k\nabla_j\beta_l$ by the normal-tangential components $\gamma^{ij}N^kN^l\nabla_i\beta_k\nabla_j\beta_l$ and either the normal-normal components $N^iN^jN^kN^l\nabla_i\beta_k\nabla_j\beta_l$ or the tangential-tangential components $\gamma^{ij}\gamma^{kl}\nabla_i\beta_k\nabla_j\beta_l$.

Definition 5.1. Let $\beta_k = \beta_{Ik} = \nabla_I^r u_k$, where $\nabla_I^r = \nabla_{i_1} \dots \nabla_{i_r}$ and u is a $(0, 1)$ tensor and $[\nabla_i, \nabla_j] = 0$. Let $\text{div } \beta = \nabla_i \beta^i = \nabla^r \text{div } u$ and let $\text{curl } \beta_{ij} = \nabla_i \beta_j - \nabla_j \beta_i = \nabla^r \text{curl } u_{ij}$.

Lemma 5.1. *Let β as in Definition 5.1 and let Q be a positive semidefinite quadratic form $Q(\nabla_i\beta_k, \nabla_j\beta_l) = q^{IJ}(\nabla_i\beta_{Ik})\nabla_j\beta_{Jl}$. Then*

$$(5.3) \quad g^{ij}g^{kl}Q(\nabla_i\beta_k, \nabla_j\beta_l) \leq (2(N^iN^jg^{kl} + g^{ij}N^kN^l) + 2g^{ik}g^{jl} + (\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl}))Q(\nabla_i\beta_k, \nabla_j\beta_l)$$

$$(5.4) \quad g^{ij}g^{kl}Q(\nabla_k\beta_i, \nabla_l\beta_j) \leq (n(g^{ij}\gamma^{kl} + \gamma^{ij}g^{kl}) + 2g^{ik}g^{jl})Q(\nabla_i\beta_k, \nabla_j\beta_l)$$

and

$$(5.5) \quad N^iN^j\gamma^{kl}Q(\nabla_i\beta_k, \nabla_j\beta_l) \leq 2N^kN^l\gamma^{ij}Q(\nabla_i\beta_k, \nabla_j\beta_l) + N^kN^l\gamma^{ij}Q(\text{curl } \beta_{ik}, \text{curl } \beta_{jl})$$

Proof. Since $g^{ik} = \gamma^{ik} + N^iN^k$ we obtain

$$(5.6) \quad \gamma^{ik}\gamma^{jl}Q(\nabla_i\beta_k, \nabla_j\beta_l) \leq (2g^{ik}g^{jl} + 2N^iN^kN^jN^l)Q(\nabla_i\beta_k, \nabla_j\beta_l)$$

$$(5.7) \quad N^iN^kN^jN^lQ(\nabla_i\beta_k, \nabla_j\beta_l) \leq (2g^{ik}g^{jl} + 2\gamma^{ik}\gamma^{jl})Q(\nabla_i\beta_k, \nabla_j\beta_l)$$

(5.3)-(5.4) follows from (5.6)-(5.7) and

$$(5.8) \quad \gamma^{ik}\gamma^{jl}Q(\alpha_{ik}, \alpha_{jl}) \leq (n-1)\gamma^{ij}\gamma^{kl}Q(\alpha_{ik}, \alpha_{jl})$$

To prove (5.8) let $\text{tr}_\gamma(\alpha) = \gamma^{ik}\alpha_{ik}$ and $\hat{\alpha}_{ik} = \alpha_{ik} - \gamma_{ik}\gamma^{pq}\alpha_{pq}/(n-1)$ is the traceless part. Then

$$\text{tr}_\gamma(\alpha)\text{tr}_\gamma(\sigma) = (n-1)(\gamma^{ij}\gamma^{kl}\alpha_{ik}\sigma_{jl} - \gamma^{ij}\gamma^{kl}\hat{\alpha}_{ik}\hat{\sigma}_{jl}) \quad \square$$

Let us recall Gauss formula for Ω and $\partial\Omega$;

$$(5.9) \quad \int_{\Omega} \nabla_m(\beta^m) d\mu_g = \int_{\partial\Omega} N_m \beta^m d\mu_{\gamma}, \quad \text{and} \quad \int_{\partial\Omega} \bar{\nabla}_i \bar{f}^i d\mu_{\gamma} = 0,$$

if \bar{f} is tangential to $\partial\Omega$ and N is the unit conormal to $\partial\Omega$. The last part of (5.9) follows since by (3.8) $\bar{\nabla}_i \bar{f}^i = \nabla_i \bar{f}^i$ is the intrinsic divergence on $\partial\Omega$ if the coordinates are chosen so $\partial\Omega$ is given by $y^n = 0$.

Lemma 5.2. *Let R^{ijklIJ} be any quadratic form q^{IJ} multiplied with $(N^k N^l g^{ij} - g^{ki} N^l N^j)$ or $(g^{kl} \gamma^{ij} - \gamma^{ik} g^{lj})$. Then*

$$(5.10) \quad \int_{\Omega} R^{ijklIJ} \nabla_k \alpha_{Ii} \nabla_j \beta_{Jl} d\mu = \int_{\partial\Omega} N^l \gamma^{ij} q^{IJ} \alpha_{Ii} \nabla_j \beta_{Jl} d\mu_{\gamma} - \int_{\Omega} (\nabla_k R^{ijklIJ}) \alpha_{Ii} \nabla_j \beta_{Jl} d\mu$$

$$(5.11) \quad \int_{\Omega} R^{ijklIJ} \nabla_k \alpha_{Ii} \nabla_j \beta_{Jl} d\mu = - \int_{\partial\Omega} N^l \gamma^{ik} q^{IJ} \nabla_k \alpha_{Ii} \beta_{Jl} d\mu_{\gamma} - \int_{\Omega} (\nabla_j R^{ijklIJ}) \nabla_k \alpha_{Ii} \beta_{Jl} d\mu$$

Moreover if R^{ijklIJ} is any quadratic form q^{IJ} multiplied with $(\gamma^{kl} \gamma^{ij} - \gamma^{ik} \gamma^{lj})$ then

$$(5.12) \quad \int_{\Omega} R^{ijklIJ} \nabla_k \alpha_{iI} \nabla_j \beta_{lJ} d\mu = - \int_{\Omega} (\nabla_k R^{ijklIJ}) \alpha_{iI} \nabla_j \beta_{lJ} d\mu$$

Proof. Note that we have the following identities

$$(5.13) \quad R^{ijklIJ} \nabla_k \alpha_{Ii} \nabla_j \beta_{Jl} = \nabla_k (R^{ijklIJ} \alpha_{Ii} \nabla_j \beta_{Jl}) - (\nabla_k R^{ijklIJ}) \alpha_{Ii} \nabla_j \beta_{Jl}$$

$$(5.14) \quad R^{ijklIJ} \nabla_k \alpha_{Ii} \nabla_j \beta_{Jl} = \nabla_j (R^{ijklIJ} \nabla_k \alpha_{Ii} \beta_{Jl}) - (\nabla_j R^{ijklIJ}) \nabla_k \alpha_{Ii} \beta_{Jl}$$

Integrating (5.13)-(5.14) over Ω using Gauss formula (5.7) we get a boundary term from the divergence. The lemma now follows from

$$(5.15) \quad \begin{aligned} N_k (N^k N^l g^{ij} - g^{ki} N^l N^j) &= N_k (g^{kl} \gamma^{ij} - \gamma^{ik} g^{lj}) = N^l \gamma^{ij} \\ N_j (N^k N^l g^{ij} - g^{ki} N^l N^j) &= N_j (g^{kl} \gamma^{ij} - \gamma^{ik} g^{lj}) = -N^l \gamma^{ik}, \quad \square \\ N_k (\gamma^{kl} \gamma^{ij} - \gamma^{ik} \gamma^{lj}) &= 0, \end{aligned}$$

Definition 5.2. If $|I| = |J| = r$ set $g^{IJ} = g^{i_1 j_1} \dots g^{i_r j_r}$ and $\gamma^{IJ} = \gamma^{i_1 j_1} \dots \gamma^{i_r j_r}$. If α and β are $(0, r)$ tensors let $\langle \alpha, \beta \rangle = g^{IJ} \alpha_I \beta_J$ and $|\alpha|^2 = \langle \alpha, \alpha \rangle$. If $(\Pi\beta)_I = \gamma_I^J \beta_J$ is the projection then $\langle \Pi\alpha, \Pi\beta \rangle = \gamma^{IJ} \alpha_I \beta_J$. Let

$$\|\beta\|_{L^2(\Omega)} = \left(\int_{\Omega} |\beta|^2 d\mu_g \right)^{1/2}, \quad \|\beta\|_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |\beta|^2 d\mu_{\gamma} \right)^{1/2}, \quad \|\Pi\beta\|_{L^2(\partial\Omega)} = \left(\int_{\partial\Omega} |\Pi\beta|^2 d\mu_{\gamma} \right)^{1/2},$$

where $d\mu_g$ is the Riemannian volume element on Ω and $d\mu_{\gamma}$ is the induced surface measure on $\partial\Omega$.

Lemma 5.3. *Let β be as in Definition 5.1 and ι_0 be as in Definition 3.3. If $|\theta| + 1/\iota_0 \leq K$ then*

$$(5.16) \quad |\nabla\beta|^2 \leq C \left(g^{ij} \gamma^{kl} \gamma^{IJ} \nabla_k \beta_{Ii} \nabla_l \beta_{Jj} + |\operatorname{div} \beta|^2 + |\operatorname{curl} \beta|^2 \right)$$

$$(5.17) \quad \int_{\Omega} |\nabla\beta|^2 d\mu \leq C \int_{\Omega} \left(N^i N^j g^{kl} \gamma^{IJ} \nabla_k \beta_{Ii} \nabla_l \beta_{Jj} + |\operatorname{div} \beta|^2 + |\operatorname{curl} \beta|^2 + K^2 |\beta|^2 \right) d\mu$$

Proof. The proof follows by induction from repeated use of Lemma 5.1. $|\beta|^2 = g^{IJ} \beta_I \beta_J$ can be written as a sum of terms of the form

$$(5.18) \quad N^{i_1} N^{j_1} \dots N^{i_s} N^{j_s} \gamma^{i_{s+1} j_{s+1}} \dots \gamma^{i_r j_r} \beta_{i_1 \dots i_r} \beta_{j_1 \dots j_r}$$

If $s = 0, 1$ then (5.18) is bounded by the right hand side of (5.16). If we inductively assume that we can bound the right hand side of (5.18) for $s \leq s_0$ then the bound for $s = s_0 + 1$ follows from (5.4)-(5.5) in Lemma 5.1. On the other hand, if we control the right hand side of (5.17) then we have a bound for the integral of (5.18) for $s = 1, 2$. However, by (5.3) in Lemma 5.1 and (5.12) in Lemma 5.2 this gives us the integral of (5.18) also for $s = 0$, but then we can use (5.16) to obtain (5.17). \square

Lemma 5.4. *Let β be as in Definition 5.1 and ι_0 be as in Definition 3.3. If $|\theta| + 1/\iota_0 \leq K$ then*

$$(5.19) \quad \|\beta\|_{L^2(\partial\Omega)}^2 \leq C(\|\nabla\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)})\|\beta\|_{L^2(\Omega)}$$

$$(5.20) \quad \|\beta\|_{L^2(\partial\Omega)}^2 \leq C\|\Pi\beta\|_{L^2(\partial\Omega)}^2 + C(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)})\|\beta\|_{L^2(\Omega)}.$$

and

$$(5.21) \quad \|\nabla\beta\|_{L^2(\Omega)}^2 \leq C\|\nabla\beta\|_{L^2(\partial\Omega)}\|\beta\|_{L^2(\partial\Omega)} + C(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)})^2$$

Furthermore;

$$(5.22) \quad \|\nabla\beta\|_{L^2(\Omega)}^2 \leq C\|\Pi\nabla\beta\|_{L^2(\partial\Omega)}\|\Pi N \cdot \beta\|_{L^2(\partial\Omega)} + C(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)})^2$$

$$(5.23) \quad \|\nabla\beta\|_{L^2(\Omega)}^2 \leq C\|\Pi N \cdot \nabla\beta\|_{L^2(\partial\Omega)}\|\Pi\beta\|_{L^2(\partial\Omega)} + C(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)})^2$$

where $N \cdot \beta_I = N^i \beta_{iI}$ and $N \cdot \nabla \beta_{kI} = N^i \nabla_k \beta_{iI}$.

Proof. Let N be the extension of the normal to the interior as in Lemma 3.6-Lemma 3.7. Then

$$\int_{\partial\Omega} |\beta|^2 d\mu_\gamma = \int_{\Omega} \nabla_k (N^k |\beta|^2) d\mu$$

and since $|\nabla N| \leq K$, by Lemma 3.6- Lemma 3.7, (5.19) follows. (5.20) follows by induction as in the proof of Lemma 5.3, from:

$$\left| \int_{\partial\Omega} q^{IJ} (N^i N^j - \gamma^{ij}) \beta_{Ii} \beta_{Jj} d\mu_\gamma \right| \leq C(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)})\|\beta\|_{L^2(\Omega)},$$

if q^{IJ} is any product of factors $q^{i_k j_k}$ of the form $g^{i_k j_k}$, $\gamma^{i_k j_k}$ or $N^{i_k} N^{j_k}$. The left hand side is

$$\begin{aligned} & \int_{\Omega} \nabla_k \left(N^k q^{IJ} (N^i N^j - \gamma^{ij}) \beta_{Ii} \beta_{Jj} \right) d\mu \\ &= 2 \int_{\Omega} N^k q^{IJ} (N^i N^j - \gamma^{ij}) \beta_{Ii} \nabla_k \beta_{Jj} d\mu + \int_{\Omega} (\nabla_k N^k) q^{IJ} (N^i N^j - \gamma^{ij}) \beta_{Ii} \beta_{Jj} d\mu \\ &= -2 \int_{\Omega} N^k q^{IJ} \gamma^{ij} \beta_{Ii} (\nabla_k \beta_{Jj} - \nabla_j \beta_{Jk}) d\mu + 2 \int_{\Omega} q^{IJ} (N^i N^j + \gamma^{ij}) (\nabla_j \beta_{Ii}) N^k \beta_{Jk} d\mu \\ & \quad + 2 \int_{\Omega} \nabla_j (q^{IJ} \gamma^{ij} N^k) \beta_{Ii} \beta_{Jk} d\mu + \int_{\Omega} (\nabla_k N^k) q^{IJ} (N^i N^j - \gamma^{ij}) \beta_{Ii} \beta_{Jj} d\mu \end{aligned}$$

(5.21) is just integration by parts twice. (5.22)-(5.23) follows from Lemma 5.3 and Lemma 5.2. \square

One can actually get away with a less regular boundary for some of the estimates:

Lemma 5.5. *Let β be as in Definition 5.1. Then there is $\varepsilon_1(r) > 0$ such that if the condition in Definition 3.4 holds with $\varepsilon_1 \leq \varepsilon_1(r)$, we have with $K_1 \geq 1/\nu_1$:*

$$(5.24) \quad \|\beta\|_{L^2(\partial\Omega)}^2 \leq C(\|\nabla\beta\|_{L^2(\Omega)} + K_1\|\beta\|_{L^2(\Omega)})\|\beta\|_{L^2(\Omega)}$$

$$(5.25) \quad \|\beta\|_{L^2(\partial\Omega)}^2 \leq C\|\Pi\beta\|_{L^2(\partial\Omega)}^2 + C(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)} + K_1\|\beta\|_{L^2(\Omega)})\|\beta\|_{L^2(\Omega)}.$$

Proof. We will prove (5.24)-(5.25) in the x coordinates $\Omega \ni y \rightarrow x(t, y) \in \mathcal{D}_t \subset \mathbf{R}^n$. Since the metric there is the induced metric from \mathbf{R}^n we can then compare the normal \mathcal{N} to $\partial\mathcal{D}_t$ at different points. Let χ_p be the partition of unity in Lemma 3.4 and let $\mathcal{N}_p = \mathcal{N}(x_p)$ be the unit normal at some fixed point $x_p \in \operatorname{supp}(\chi_p) \cap \partial\mathcal{D}_t$ and let N be the unit normal to $\partial\mathcal{D}_t$. Then

$$\int_{\partial\mathcal{D}_t} \chi_p |\beta|^2 \langle \mathcal{N}_p, N \rangle dS = \int_{\mathcal{D}_t} \mathcal{N}_p^k \partial_k (\chi_p |\beta|^2) dx$$

where N is the unit normal to $\partial\mathcal{D}_t$ and $\langle \mathcal{N}_p, N \rangle = \delta_{ij} \mathcal{N}_p^i \mathcal{N}^j \geq 1/2$. Since $|\partial\chi_p| \leq CK_1$ (5.24) follows.

To prove (5.25), we will use a similar estimate to the one in the proof of (5.20), with \mathcal{N} replaced by \mathcal{N}_p and $\gamma^{ij} = \delta^{ij} - \mathcal{N}^i \mathcal{N}^j$ replaced by $\gamma_p^{ij} = \delta^{ij} - \mathcal{N}_p^i \mathcal{N}_p^j$ and q^{IJ} replaced by q_p^{IJ} , a product of factors δ^{ij} , γ_p^{ij} and $\mathcal{N}_p^i \mathcal{N}_p^j$. We will use the identity

$$\begin{aligned} & \mathcal{N}_p^k \partial_k (\delta^{ij} q_p^{IJ} \chi_p \beta_{Ii} \beta_{Jj}) - 2\delta^{ij} \partial_i (\mathcal{N}_p^k q_p^{IJ} \chi_p \beta_{Ik} \beta_{Jj}) \\ &= -2\mathcal{N}_p^k q_p^{IJ} \chi_p \beta_{Ik} \delta^{ij} \partial_i \beta_{Jj} + 2\delta^{ij} \mathcal{N}_p^k q_p^{IJ} \chi_p (\partial_i \beta_{Ik} - \partial_k \beta_{Ii}) \beta_{Jj} \\ & \quad + \mathcal{N}_p^k (\partial_k \chi_p) (\delta^{ij} q_p^{IJ} \beta_{Ii} \beta_{Jj}) - 2\delta^{ij} (\partial_i \chi_p) (\mathcal{N}_p^k q_p^{IJ} \beta_{Ik} \beta_{Jj}) \end{aligned}$$

Integrating this over \mathcal{D}_t using Gauss theorem we get

$$\left| \int_{\partial\mathcal{D}_t} (\langle \mathcal{N}_p, \mathcal{N} \rangle \delta^{ij} - 2\mathcal{N}^j \mathcal{N}_p^i) q_p^{IJ} \chi_p \beta_{Ii} \beta_{Jj} dS \right| \leq \int_{\mathcal{D}_t} (2\chi_p (|\operatorname{div}\beta| + |\operatorname{curl}\beta|) + 3|\partial\chi_p||\beta|) |\beta| dx$$

We now assume that $|\mathcal{N} - \mathcal{N}_p| \leq \varepsilon_1$ in the support of χ_p , where $\varepsilon_1 = \varepsilon_1(r)$ is to be determined. Writing $\mathcal{N} = a\mathcal{N}_p + b\mathcal{T}_p$, where $a = \langle \mathcal{N}_p, \mathcal{N} \rangle$, $b = \sqrt{1 - a^2} \leq \varepsilon_1$ and $\langle \mathcal{T}_p, \mathcal{T}_p \rangle = 1$ and $\langle \mathcal{T}_p, \mathcal{N}_p \rangle = 0$. We get

$$\langle \mathcal{N}_p, \mathcal{N} \rangle \delta^{ij} - 2\mathcal{N}^j \mathcal{N}_p^i = a(\gamma_p^{ij} - \mathcal{N}_p^i \mathcal{N}_p^j) - 2b\mathcal{N}_p^i \mathcal{T}_p^j$$

Let $Q_p(\beta_i, \beta_j) = q_p^{IJ} \chi_p \beta_{Ii} \beta_{Jj}$ and let $R_p(\beta, \beta) = (a(\gamma_p^{ij} - \mathcal{N}_p^i \mathcal{N}_p^j) - 2b\mathcal{N}_p^i \mathcal{T}_p^j) Q_p(\beta_i, \beta_j)$. It follows that

$$\begin{aligned} \mathcal{N}_p^i \mathcal{N}_p^j Q_p(\beta_i, \beta_j) &\leq (\gamma_p^{ij} - \frac{b}{a}(\mathcal{N}_p^i \mathcal{T}_p^j + \mathcal{T}_p^i \mathcal{N}_p^j)) Q_p(\beta_i, \beta_j) + \frac{1}{a} R_p(\beta, \beta) \\ &\leq (\gamma_p^{ij} - \frac{b}{a}(\frac{1+b}{a} \mathcal{T}_p^i \mathcal{T}_p^j + \frac{a}{1+b} \mathcal{N}_p^i \mathcal{N}_p^j)) Q_p(\beta_i, \beta_j) + \frac{1}{a} R_p(\beta, \beta) \\ &\leq (\frac{1}{1-b} \gamma_p^{ij} + \frac{b}{1+b} \mathcal{N}_p^i \mathcal{N}_p^j) Q_p(\beta_i, \beta_j) + \frac{1}{a} R_p(\beta, \beta) \end{aligned}$$

since $\mathcal{T}_p^i \mathcal{T}_p^j Q_p(\beta_i, \beta_j) \leq \gamma_p^{ij} Q_p(\beta_i, \beta_j)$ and $a^2 = 1 - b^2$. Moving the term with the normal component over to the other side we obtain

$$\delta^{ij} Q_p(\beta_i, \beta_j) \leq \frac{2}{1-b} \gamma_p^{ij} Q_p(\beta_i, \beta_j) + \frac{1+b}{a} R_p(\beta, \beta)$$

Integrating this gives

$$\int_{\partial\mathcal{D}_t} \delta^{ij} q_p^{IJ} \chi_p \beta_{Ii} \beta_{Jj} dS \leq \frac{2}{1-\varepsilon_1} \int_{\partial\mathcal{D}_t} \gamma_p^{ij} q_p^{IJ} \chi_p \beta_{Ii} \beta_{Jj} dS + 4 \int_{\mathcal{D}_t} (\chi_p (|\operatorname{div}\beta| + |\operatorname{curl}\beta|) + |\partial\chi_p||\beta|) |\beta| dx$$

Repeated use of this gives

$$(5.26) \quad \int_{\partial\mathcal{D}_i} \delta^{ij} \delta^{IJ} \chi_p \beta_{Ii} \beta_{Jj} dS \leq A \int_{\partial\mathcal{D}_i} \gamma_p^{ij} \gamma_p^{IJ} \chi_p \beta_{Ii} \beta_{Jj} dS + B \int_{\mathcal{D}_i} (\chi_p (|\operatorname{div} \beta| + |\operatorname{curl} \beta|) + |\partial \chi_p| |\beta|) |\beta| dx$$

for some constants A and B that only depends on the order r of the tensor β . We now claim that if q^{IJ} is any positive definite quadratic form then

$$(5.27) \quad \gamma_p^{ij} q^{IJ} \chi_p \beta_{Ii} \beta_{Jj} \leq \gamma^{ij} q^{IJ} \chi_p \beta_{Ii} \beta_{Jj} + b \delta^{ij} q^{IJ} \chi_p \beta_{Ii} \beta_{Jj}$$

In fact if $Q(\beta_i, \beta_j) = q^{IJ} \chi_p \beta_{Ii} \beta_{Jj}$

$$\begin{aligned} \gamma_p^{ij} Q(\beta_i, \beta_j) - \gamma^{ij} Q(\beta_i, \beta_j) &= (\mathcal{N}^i \mathcal{N}^j - \mathcal{N}_p^i \mathcal{N}_p^j) Q(\beta_i, \beta_j) \\ &= (b^2 \mathcal{T}_p^i \mathcal{T}_p^j - b^2 \mathcal{N}_p^i \mathcal{N}_p^j + ab(\mathcal{N}_p^i \mathcal{T}_p^j + \mathcal{T}_p^i \mathcal{N}_p^j)) Q(\beta_i, \beta_j) \\ &\leq (b^2 \mathcal{T}_p^i \mathcal{T}_p^j - b^2 \mathcal{N}_p^i \mathcal{N}_p^j + ab(\frac{1+b}{a} \mathcal{N}_p^i \mathcal{N}_p^j + \frac{a}{1+b} \mathcal{T}_p^i \mathcal{T}_p^j)) Q(\beta_i, \beta_j) \\ &= b(\mathcal{N}_p^i \mathcal{N}_p^j + \mathcal{T}_p^i \mathcal{T}_p^j) Q(\beta_i, \beta_j) \leq b \delta^{ij} Q(\beta_i, \beta_j) \end{aligned}$$

since $a^2 = 1 - b^2$. Using (5.27) now we can replace $\gamma_p^{ij} \gamma_p^{IJ}$ by $\gamma^{ij} \gamma^{IJ}$ in (5.26) with a small error that can be absorbed into the left-hand side is $b \leq \varepsilon_1$ is sufficiently small. Finally, summing over p using that $\sum_p \chi_p = 1$ and $\sum_p |\partial \chi_p| \leq CK_1$ and Hölder's inequality gives (5.25). \square

Lemma 5.4 applied to $\beta = \nabla q$, where q is a function, gives estimates for both the Dirichlet problem and the Neumann problem. In fact if $q = 0$ on $\partial\Omega$ then $\Pi \nabla^2 q = \theta \nabla_N q$ so (5.22) and (5.20) gives

$$\|\nabla^2 q\|_{L^2(\Omega)}^2 \leq CK \|\nabla_N q\|_{L^2(\partial\Omega)}^2 + C(\|\Delta q\|_{L^2(\Omega)} + K \|\nabla q\|_{L^2(\Omega)})^2 \leq C(\|\Delta q\|_{L^2(\Omega)} + K \|\nabla q\|_{L^2(\Omega)})^2$$

Similarly, if $\nabla_N q = 0$ on $\partial\Omega$ then $N^i \bar{\nabla}_j \nabla_i q = -\theta_j^i \bar{\nabla}_i q$ and by (5.23) and (5.20)

$$\|\nabla^2 q\|_{L^2(\Omega)}^2 \leq CK \|\bar{\nabla} q\|_{L^2(\partial\Omega)}^2 + C(\|\Delta q\|_{L^2(\Omega)} + K \|\nabla q\|_{L^2(\Omega)})^2 \leq C(\|\Delta q\|_{L^2(\Omega)} + K \|\nabla q\|_{L^2(\Omega)})^2$$

Similarly we can get estimates for higher order derivatives. More generally, we have

Proposition 5.6. *Let ι_0 and ι_1 be as in Definitions 3.3-3.4 and suppose that $|\theta| + 1/\iota_0 \leq K$ and $1/\iota_1 \leq K_1$. Then with $\tilde{K} = \min(K, K_1)$ we have for any $r \geq 2$ and $\delta > 0$:*

$$(5.28) \quad \|\nabla^r q\|_{L^2(\partial\Omega)} + \|\nabla^r q\|_{L^2(\Omega)} \leq C \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} + C(\tilde{K}, \operatorname{Vol}(\Omega)) \sum_{s \leq r-1} \|\nabla^s \Delta q\|_{L^2(\Omega)}$$

$$(5.29) \quad \|\nabla^r q\|_{L^2(\Omega)} + \|\nabla^{r-1} q\|_{L^2(\partial\Omega)} \leq \delta \|\Pi \nabla^r q\|_{L^2(\partial\Omega)} + C(1/\delta, K, \operatorname{Vol}(\Omega)) \sum_{s \leq r-2} \|\nabla^s \Delta q\|_{L^2(\Omega)}$$

Proof. (5.28) with an extra lower order term $C(\tilde{K}) \|\nabla q\|_{L^2(\Omega)}$ in the right follows from (5.20) or (5.25) together with repeated use of (5.21) and (5.19) or (5.24). The lower order term can then be bounded by (5.17) in Lemma 8.5. (5.29) with the same extra lower order term follows from (5.22) together with repeated use of (5.19) and (5.21). \square

Remark. One should be able to improve the results of Proposition 5.6 and replace the sum in the right hand side of (5.28) by the sum over $s = 0, 1/2$ at least when $|\nabla_N q| > \varepsilon > 0$ on $\partial\Omega$. However, then one has to make sense of fractional derivatives.

Proposition 5.7. *Assume that $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$. Suppose that $|\theta| \leq K$ and $\iota_1 \geq 1/K_1$, where ι_1 is as in Definition 3.4. If $q = 0$ on $\partial\Omega$ then for $m = 0, 1$*

$$(5.30) \quad \|\Pi\nabla^r q\|_{L^2(\partial\Omega)} \leq 2\|\bar{\nabla}^{r-2}\theta\|_{L^2(\partial\Omega)}\|\nabla_N q\|_{L^\infty(\partial\Omega)} + C\sum_{k=1}^{r-1}\|\theta\|_{L^\infty(\partial\Omega)}^k\|\nabla^{r-k}q\|_{L^2(\partial\Omega)} \\ + C(K, K_1)\left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-2-m}\|\bar{\nabla}^k\theta\|_{L^2(\partial\Omega)}\right)\sum_{k \leq r-2+m}\|\nabla^k q\|_{L^2(\partial\Omega)}$$

and if $r > (n-1)/2 + 2$ then for any $\delta > 0$

$$(5.31) \quad \|\Pi\nabla^{r-1}q\|_{L^2(\partial\Omega)} \leq \delta\|\nabla^{r-1}q\|_{L^2(\partial\Omega)} + C_\delta(K, K_1, \|\theta\|_{L^2(\partial\Omega)}, \|\bar{\nabla}^{r-3}\theta\|_{L^2(\partial\Omega)})\sum_{k=0}^{r-2}\|\nabla^k q\|_{L^2(\partial\Omega)}$$

If in addition $|\nabla_N q| \geq \varepsilon > 0$ and $|\nabla_N q| \geq 2\varepsilon\|\nabla_N q\|_{L^\infty(\partial\Omega)}$ then

$$(5.32) \quad \|\bar{\nabla}^{r-2}\theta\|_{L^2(\partial\Omega)} \leq C(1/\varepsilon)\left(\|\Pi\nabla^r q\|_{L^2(\partial\Omega)} + \sum_{k=1}^{r-1}\|\theta\|_{L^\infty(\partial\Omega)}^k\|\nabla^{r-k}q\|_{L^2(\partial\Omega)}\right) \\ + C(K, K_1, 1/\varepsilon)\left(\|\theta\|_{L^\infty(\partial\Omega)} + \sum_{k \leq r-3}\|\bar{\nabla}^k\theta\|_{L^2(\partial\Omega)}\right)\sum_{k \leq r-1}\|\nabla^k q\|_{L^2(\partial\Omega)}$$

Furthermore if $r \leq 4$ then the second line of (5.30) and (5.32) drop out.

Proof. (5.30) and (5.32) follows from Proposition 4.5. To prove (5.30) we can take $\varepsilon = 1$ and to prove (5.32) we take $m = 1$ in Proposition 4.5. (5.31) follows from (5.30) and Sobolev's lemma; (8.8). \square

Proposition 5.8. *Assume that $0 \leq r \leq 4$ or $r \geq (n-1)/2 + 2$ and that $|\theta| + 1/\iota_0 \leq K$. If $q = 0$ on $\partial\Omega$ then*

$$(5.33) \quad \|\nabla^{r-1}q\|_{L^2(\partial\Omega)} \leq C\left(\|\bar{\nabla}^{r-3}\theta\|_{L^2(\partial\Omega)}\|\nabla_N q\|_{L^\infty(\partial\Omega)} + \|\nabla^{r-2}\Delta q\|_{L^2(\Omega)}\right) \\ + C(K, \text{Vol}(\Omega), \|\theta\|_{L^2(\partial\Omega)}, \dots, \|\bar{\nabla}^{r-4}\theta\|_{L^2(\partial\Omega)})\left(\|\nabla_N q\|_{L^\infty(\partial\Omega)} + \sum_{s \leq r-3}\|\nabla^s \Delta q\|_{L^2(\Omega)}\right)$$

If $r > (n-1)/2 + 2$ then

$$(5.34) \quad \|\nabla^{r-1}q\|_{L^2(\partial\Omega)} + \|\nabla q\|_{L^\infty(\partial\Omega)} \\ \leq C\|\nabla^{r-2}\Delta q\|_{L^2(\Omega)} + C(K, \text{Vol}(\Omega), \|\theta\|_{L^2(\partial\Omega)}, \dots, \|\bar{\nabla}^{r-3}\theta\|_{L^2(\partial\Omega)})\sum_{s \leq r-3}\|\nabla^s \Delta q\|_{L^2(\Omega)}$$

Proof. (5.33) follows from (5.28) and (5.30) with $m = 1$ and r replaced by $r-1$. The estimate for $\|\nabla^{r-1}q\|_{L^2(\partial\Omega)}$ in (5.34) follows from (5.28), with r replaced by $r-1$, and (5.31). The estimate for $\|\nabla q\|_{L^\infty(\partial\Omega)}$ in (5.34) follows from the estimate for $\|\nabla^{r-1}q\|_{L^2(\partial\Omega)}$ and Sobolev's lemma, Lemma 8.2. \square

There are two possible energies, given in Proposition 5.9 respectively Proposition 5.10 :

Proposition 5.9. *Let $Q(\alpha, \alpha) = \gamma^{IJ} \alpha_I \alpha_J$, $h_{ij} = D_t g_{ij}/2$ and set*

$$E(t) = \int_{\partial\Omega} \gamma^{ij} Q(\alpha_i, \alpha_j) \nu d\mu_\gamma + \int_{\Omega} g^{ij} N^k N^l Q(\nabla_i \beta_k, \nabla_j \beta_l) d\mu_g,$$

where $0 < \nu < \infty$. Let K be a constant such that

$$(5.35) \quad |h| \leq K \quad \text{in } [0, T] \times \Omega$$

$$(5.36) \quad |\theta| + 1/\nu_0 + |\nu_t/\nu| \leq K, \quad \text{on } [0, T] \times \partial\Omega.$$

Then

$$(5.37) \quad \frac{dE}{dt} \leq C\sqrt{E} (\|\Pi(D_t \alpha + \nu N^k \nabla \beta_k)\|_{L^2(\partial\Omega)} + \|D_t \nabla \beta - \nabla \alpha\|_{L^2(\Omega)}) + CK E \\ + C (\|\operatorname{div} \alpha\|_{L^2(\Omega)} + \|\operatorname{curl} \alpha\|_{L^2(\Omega)} + K\|\alpha\|_{L^2(\Omega)} + \|\operatorname{div} \beta\|_{L^2(\Omega)} + \|\operatorname{curl} \beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)})^2$$

Proof. Since by Lemma 3.5 $D_t d\mu_\gamma = (\operatorname{tr} h - h_{NN})d\mu_\gamma$ and $D_t d\mu = \operatorname{tr} h d\mu$ we obtain

$$(5.38) \quad \frac{dE}{dt} = 2 \int_{\partial\Omega} \gamma^{ij} Q(\alpha_i, D_t \alpha_j) \nu d\mu_\gamma + 2 \int_{\Omega} g^{ij} N^k N^l Q(\nabla_i \beta_k, D_t \nabla_j \beta_l) d\mu_g \\ + \int_{\partial\Omega} (D_t(\gamma^{ij} \gamma^{IJ}) + (\operatorname{tr} h - h_{NN} + \nu_t/\nu) \gamma^{ij} \gamma^{IJ}) \alpha_{Ii} \alpha_{Jj} \nu d\mu_\gamma \\ + \int_{\Omega} (D_t(g^{ij} N^k N^l \gamma^{IJ}) + \operatorname{tr} h g^{ij} N^k N^l \gamma^{IJ}) \nabla_i \beta_{Ik} \nabla_j \beta_{Jl} d\mu_g$$

Since $D_t \gamma^{ij} = -2\gamma^{im} \gamma^{jn} h_{mn}$ the second line is bounded by the boundary term in the energy E and the third line is bounded by $\|\nabla \beta\|_{L^2(\Omega)}^2$. By Lemma 5.2

$$\int_{\Omega} g^{ij} N^k N^l \gamma^{IJ} \nabla_k \alpha_{Ii} \nabla_j \beta_{Jl} d\mu_g = \int_{\partial\Omega} N^l \gamma^{ij} \gamma^{IJ} \alpha_{Ii} \nabla_j \beta_{Jl} d\mu_\gamma \\ + \int_{\Omega} g^{ik} N^j N^l \gamma^{IJ} \nabla_k \alpha_{Ii} \nabla_j \beta_{Jl} d\mu_g - \int_{\Omega} \nabla_k (g^{ij} N^k N^l \gamma^{IJ} - g^{ik} N^j N^l \gamma^{IJ}) \alpha_{Ii} \nabla_j \beta_{Jl} d\mu_g$$

The first term on the second line is bounded by $\|\operatorname{div} \alpha\|_{L^2(\Omega)} \|\nabla \beta\|_{L^2(\Omega)}$ and the second by $K\|\alpha\|_{L^2(\Omega)} \|\nabla \beta\|_{L^2(\Omega)}$. Recall now that by Lemma 5.3

$$\|\nabla \beta\|_{L^2(\Omega)}^2 \leq CE + C \left(\|\operatorname{div} \beta\|_{L^2(\Omega)} + \|\operatorname{curl} \beta\|_{L^2(\Omega)} + K\|\beta\|_{L^2(\Omega)} \right)^2.$$

This proves Proposition 5.9. \square

Proposition 5.10. *Let $Q(\alpha, \alpha) = \gamma^{IJ} \alpha_I \alpha_J$, $h_{ij} = D_t g_{ij}/2$ and set*

$$(5.39) \quad E(t) = \int_{\partial\Omega} \gamma^{ij} Q(\alpha_i, \alpha_j) \nu d\mu_\gamma + \int_{\Omega} g^{kl} \gamma^{ij} Q(\nabla_i \beta_k, \nabla_j \beta_l) d\mu_g,$$

where $0 < \nu < \infty$. Let K be a constant such that

$$(5.40) \quad |h| \leq K \quad \text{in } [0, T] \times \Omega$$

$$(5.41) \quad |\theta| + 1/\nu_0 + |\nu_t/\nu| \leq K, \quad \text{on } [0, T] \times \partial\Omega.$$

Then

$$(5.42) \quad \frac{dE}{dt} \leq C\sqrt{E} (\|\Pi(D_t\alpha + \nu N^k \nabla\beta_k)\|_{L^2(\partial\Omega)} + \|D_t\nabla\beta - \nabla\alpha\|_{L^2(\Omega)}) + CK E \\ + C\|\operatorname{curl}\alpha\|_{L^2(\Omega)}\sqrt{E} + C\|\alpha\|_{L^2(\Omega)}\|\nabla\operatorname{div}\beta\|_{L^2(\Omega)} + (K\|\alpha\|_{L^2(\Omega)} + \|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)})^2$$

Proof. Since by Lemma 3.5 $D_t d\mu_\gamma = (\operatorname{tr} h - h_{NN})d\mu_\gamma$ and $D_t d\mu = \operatorname{tr} h d\mu$ we obtain

$$\frac{dE}{dt} = 2 \int_{\partial\Omega} \gamma^{ij} Q(D_t\alpha_i, \alpha_j) \nu d\mu_\gamma + 2 \int_{\Omega} g^{kl} \gamma^{ij} Q(D_t\nabla_i\beta_k, \nabla_j\beta_l) d\mu_g \\ + \int_{\partial\Omega} (D_t(\gamma^{ij}\gamma^{IJ}) + (\operatorname{tr} h - h_{NN} + \nu_t/\nu)\gamma^{ij}\gamma^{IJ}) \alpha_{Ii} \alpha_{Jj} \nu d\mu_\gamma \\ + \int_{\Omega} (D_t(g^{kl}\gamma^{ij}\gamma^{IJ}) + \operatorname{tr} h g^{kl}\gamma^{ij}\gamma^{IJ}) \nabla_i\beta_{Ik} \nabla_j\beta_{Jl} d\mu_g$$

Since $D_t\gamma^{ij} = -2\gamma^{im}\gamma^{jn}h_{mn}$ the second line is bounded by the boundary term in the energy E and the third line is bounded by $\|\nabla\beta\|_{L^2(\Omega)}^2$. The second term on the first line is bounded by $\|\operatorname{curl}\alpha\|_{L^2(\Omega)}\sqrt{E}$ plus

$$\int_{\Omega} g^{kl}\gamma^{ij}\gamma^{IJ} \nabla_k\alpha_{Ii} \nabla_j\beta_{Jl} d\mu_g = \int_{\partial\Omega} N^l \gamma^{ij}\gamma^{IJ} \alpha_{Ii} \nabla_j\beta_{Jl} d\mu_\gamma \\ + \int_{\Omega} \gamma^{ik} g^{jl} \gamma^{IJ} \alpha_{Ii} \nabla_k \nabla_j \beta_{Jl} d\mu_g - \int_{\Omega} \nabla_k (g^{kl}\gamma^{ij}\gamma^{IJ}) \alpha_{Ii} \nabla_j\beta_{Jl} d\mu_g$$

where we have used Lemma 5.2. The first term on the second line is bounded by $\|\alpha\|_{L^2(\Omega)}\|\nabla\operatorname{div}\beta\|_{L^2(\Omega)}$ and the second by $K\|\alpha\|_{L^2(\Omega)}\|\nabla\beta\|_{L^2(\Omega)}$. Recall now that by Lemma 5.3

$$\|\nabla\beta\|_{L^2(\Omega)}^2 \leq CE + C\left(\|\operatorname{div}\beta\|_{L^2(\Omega)} + \|\operatorname{curl}\beta\|_{L^2(\Omega)}\right)^2.$$

This proves Proposition 5.10 \square

6. EULER'S EQUATIONS AND HIGHER ORDER DERIVED EQUATIONS.

Recall Euler's equations

$$(6.1) \quad D_t v_i + \partial_i p = 0, \quad \partial_i v^i = 0$$

where

$$(6.2) \quad D_t = \frac{d}{dt} \Big|_{y=\text{const}} = \frac{d}{dt} \Big|_{x=\text{const}} + v^k \partial_k \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial y^d}{\partial x^i} \frac{\partial}{\partial y^d}$$

We now want to get higher order versions of (6.1) in terms of higher order tensors $\partial^r v_i$. By Lemma 2.3

$$(6.3) \quad D_t \partial^r v_i + \partial^r \partial_i p = - \sum_{s=0}^{r-1} \binom{r}{s+1} (\partial^{1+s} v) \cdot \partial^{r-s} v_i$$

In particular if $r = 1$

$$(6.4) \quad D_t \partial_i v_j + \partial_i \partial_j p = -(\partial_i v^k) \partial_k v_j$$

We now want to change coordinates and calculate $D_t \nabla^r u$. By Lemma 2.2:

$$(6.5) \quad \begin{aligned} D_t \nabla_{a_1} \cdots \nabla_{a_r} u_a &= \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \partial_{i_1} \cdots \partial_{i_r} v_i \\ &= \frac{\partial x^{i_1}}{\partial y^{a_1}} \cdots \frac{\partial x^{i_r}}{\partial y^{a_r}} \frac{\partial x^i}{\partial y^a} \left(\partial_{i_1} \partial_{i_2} \cdots \partial_{i_r} v_i + \frac{\partial v^l}{\partial x^{i_1}} \partial_{i_2} \cdots \partial_{i_r} v_i + \cdots + \frac{\partial v^l}{\partial x^{i_r}} \partial_{i_1} \cdots \partial_{i_{r-1}} v_i + \frac{\partial v^l}{\partial x^i} \partial_{i_1} \cdots \partial_{i_r} v_l \right) \end{aligned}$$

It follows from (6.4)-(6.5) that

$$(6.6) \quad \begin{aligned} D_t \nabla^r u_a + \nabla^r \nabla_a p &= - \sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a + (\nabla_a u^c) \nabla^r u_c \\ &= (\nabla_a u_c - \nabla_c u_a) \nabla^r u^c - \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} u_a \end{aligned}$$

In particular if $r = 1$ we get

$$(6.7) \quad D_t \nabla_a u_b + \nabla_a \nabla_b p = (\nabla_a u^c) \nabla_b u_c$$

so

$$(6.8) \quad D_t (\nabla_a u_b - \nabla_b u_a) = 0$$

The higher order Euler's equations (6.3) or (6.6) will be used in the interior together with that

$$(6.9) \quad \operatorname{div} v = 0, \quad D_t \operatorname{curl} v = O(\nabla v)$$

On the boundary we will instead use an equation which has to do with the geometry of the boundary which only depends on Euler's equations indirectly through the change of coordinates. By Lemma 2.3:

$$(6.10) \quad \begin{aligned} D_t \partial_i p &= \partial_i D_t p - (\partial_i v^k) \partial_k p \\ D_t \partial_i \partial_j p &= \partial_i \partial_j D_t p - (\partial_i v^k) \partial_k \partial_j p - (\partial_i v^k) \partial_k \partial_j p + (\partial_i \partial_j v^k) \partial_k p \end{aligned}$$

It is however, more convenient to formulate the higher order version for $D_t \nabla^r p$. By Lemma 2.4

$$\begin{aligned}
(6.11) \quad D_t \nabla^r p &= \nabla^r D_t p - \sum_{s=1}^{r-1} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} p \\
&= \nabla^r D_t p - (\nabla^r u) \cdot \nabla p - \sum_{s=1}^{r-2} \binom{r}{s+1} (\nabla^{1+s} u) \cdot \nabla^{r-s} p
\end{aligned}$$

We also want to calculate equations for p . By (6.1)

$$\begin{aligned}
(6.12) \quad 0 &= D_t (\delta^{ij} \partial_i v_j) = \delta^{ij} \partial_i D_t v_j - \delta^{ij} (\partial_i v^k) \partial_k v_j \\
\Delta p &= -(\partial_i v^k) \partial_k v^i
\end{aligned}$$

Since Δ is invariant we also have

$$(6.13) \quad \Delta p = -(\nabla_a u^b) \nabla_b u^a = -g^{ab} g^{cd} (\nabla_a u_d) \nabla_c u_b = -\text{tr}((\nabla u)^2)$$

where we used the notation $(\nabla u)_{ab}^2 = ((\nabla u) \cdot \nabla u)_{ab} = (\nabla_a u^c) \nabla_c u_b$ and tr of a tensor is defined to be the trace over the first and last index. It follows that

$$(6.14) \quad \nabla^r \Delta p = -\nabla^r (\text{tr}(\nabla u)^2) = -\sum_{s=0}^r \binom{r}{s} (\nabla^{r-s} \nabla_a u) \cdot \nabla^{s+1} u^a$$

By Lemma 2.4

$$\begin{aligned}
\Delta D_t p &= -D_t (g^{ab} g^{cd} (\nabla_a u_d) \nabla_c u_b) + h^{ab} \nabla_a \nabla_b p + (\Delta u^e) \nabla_e p \\
&= 2g^{ab} h^{cd} (\nabla_a u_d) \nabla_c u_b + 2g^{ab} g^{cd} (\nabla_a u_d) (\nabla_c \nabla_b p - (\nabla_c u^e) \nabla_b u_e) + h^{ab} \nabla_a \nabla_b p - (\Delta u^e) \nabla_e p \\
&= 4g^{ab} g^{cd} (\nabla_a u_c) \nabla_b \nabla_d p + 2(\nabla_a u^d) (\nabla_d u^c) \nabla_c u^a - (\Delta u^e) \nabla_e p
\end{aligned}$$

since $D_t g^{ab} = -h^{ab}$, $h_{ab} = \nabla_a u_b + \nabla_b u_a$. In order to write things in a more appealing way we will use the notation $(\nabla u)_{ab}^3 = ((\nabla u) \cdot (\nabla u) \cdot \nabla u)_{ab} = (\nabla_a u^d) (\nabla_d u^c) \nabla_c u_b$ and $((\nabla u) \cdot \nabla^2 p)_{ab} = (\nabla_a u^d) \nabla_d \nabla_b p$

$$(6.15) \quad \Delta D_t p = 4 \text{tr}((\nabla u) \cdot \nabla^2 p) + 2 \text{tr}((\nabla u)^3) - (\Delta u) \cdot \nabla p$$

and hence

$$(6.16) \quad \nabla^{r-2} \Delta D_t p = \nabla^{r-2} (4 \text{tr}((\nabla u) \cdot \nabla^2 p) + 2 \text{tr}((\nabla u)^3) - (\Delta u) \cdot \nabla p)$$

The exact interpretation of what the dot product and traces means is not so important since the right hand side will be lower order and since ∇^{r-2} will be subject to Leibnitz' rule. Summing up, we have:

Lemma 6.1.

$$(6.17) \quad |D_t \nabla^r u + \nabla^{r+1} p| + |D_t \nabla^{r-1} \text{curl} u| + |\nabla^{r-1} \Delta p| \leq C \sum_{s=0}^{r-1} |\nabla^{1+s} u| |\nabla^{r-s} u|$$

$$(6.18) \quad \left| \Pi \left(D_t \nabla^r p + (\nabla^r u) \cdot \nabla p - \nabla^r D_t p \right) \right| \leq C \sum_{s=1}^{r-2} \left| \Pi \left((\nabla^{1+s} u) \cdot \nabla^{r-s} p \right) \right|$$

and

$$\begin{aligned}
(6.19) \quad |\nabla^{r-2} \Delta D_t p - (\nabla^{r-2} \Delta u) \cdot \nabla p| &\leq C \sum_{s=0}^{r-2} |\nabla^{1+s} u| |\nabla^{r-s} p| \\
&\quad + C \sum_{r_1+r_2+r_3=r-2} |\nabla^{1+r_1} u| |\nabla^{1+r_2} u| |\nabla^{1+r_3} u|
\end{aligned}$$

7. ENERGY ESTIMATES FOR EULER'S EQUATIONS

Let

$$(7.1) \quad E_r(t) = \int_{\Omega} g^{mn} \gamma^{ij} Q(\nabla^{r-1} \nabla_i u_m, \nabla^{r-1} \nabla_j u_n) d\mu + \int_{\Omega} |\nabla^{r-1} \operatorname{curl} u|^2 d\mu \\ + \int_{\partial\Omega} \gamma^{ij} Q(\nabla^{r-1} \nabla_i p, \nabla^{r-1} \nabla_j p) \nu d\mu_{\gamma}$$

where $\nu = 1/(-\nabla_N p)$. We will prove that there are continuous functions C_r such that

$$(7.2) \quad \left| \frac{dE_r(t)}{dt} \right| \leq C_r \left(K, 1/\varepsilon, L, M, \operatorname{Vol} \Omega, \sum_{s=0}^{r-1} E_s(t) \right) \sum_{s=0}^r E_s(t)$$

if $0 \leq r \leq 4$ or $r \geq n/2 + 3/2$, provided that some *a priori* assumption are true;

$$(7.3) \quad |\theta| + 1/\iota_0 \leq K, \quad \text{on } [0, T] \times \partial\Omega$$

$$(7.4) \quad -\nabla_N p \geq \varepsilon > 0, \quad \text{on } [0, T] \times \partial\Omega$$

$$(7.5) \quad |\nabla^2 p| + |\nabla_N p_t| \leq L, \quad \text{on } [0, T] \times \partial\Omega.$$

Since $h_{ab} = \nabla_a u_b + \nabla_b u_a$, the bound for $|h|$ of course follows from the bound for $|\nabla u|$. We also assume

$$(7.6) \quad |\nabla p| \leq M, \quad |\nabla u| \leq M, \quad \text{in } [0, T] \times \Omega$$

It is not clear to what extent we need the bound for $\nabla^2 p$, but it is natural to assume it, since $\Delta p = -\operatorname{tr}(\nabla u)^2$ and $\Pi \nabla^2 p = \theta \nabla_N p$. The bound for $\nabla^2 p$ together with (7.4) of course implies the bound for θ

Remark. Instead of the energy (7.1) coming from Proposition 5.10 we could alternatively have used the energy coming from Proposition 5.9. The one we use gives a better control of $\|\nabla^r u\|_{L^2(\Omega)}$ which is need for proving Theorem 7.2 below with minimal r_0 , but it only works when $\operatorname{div} u = 0$.

Since $E_0(t) = \int_{\Omega} |v|^2 d\mu = E_0(0)$ and $\operatorname{Vol} \Omega(t) = \operatorname{Vol} \Omega(0)$ it follows recursively from (7.2):

Theorem 7.1. *If $r \geq 0$ and $n \leq 7$, then there are continuous functions \mathcal{F}_r , with $\mathcal{F}_r|_{t=0} = 1$, such that for any smooth solution of Euler's equations (1.1)-(1.5) for $0 \leq t \leq T$ satisfying (7.3)-(7.6) we have*

$$(7.7) \quad \sum_{s=0}^r E_s(t) \leq \mathcal{F}_r(t, K, 1/\varepsilon, L, M, E_0(0), \dots, E_{r-1}(0), \operatorname{Vol} \Omega) \sum_{s=0}^r E_s(0), \quad 0 \leq t \leq T,$$

Let $\mathcal{K}(t)$ and $\varepsilon(t)$ be the maximum respectively minimum values such that (7.3)-(7.4) hold at time t

$$(7.8) \quad \mathcal{K}(t) = \max(\|\theta(t, \cdot)\|_{L^\infty(\partial\Omega)}, 1/\iota_0(t)), \quad \text{and} \quad \mathcal{E}(t) = \|(\nabla_N p(t, \cdot))^{-1}\|_{L^\infty(\partial\Omega)} = 1/\varepsilon(t)$$

Theorem 7.2. *Let $r \geq r_0 > n/2 + 3/2$. Then there is a continuous functions $\mathcal{T}_r > 0$ such that if*

$$(7.9) \quad T \leq \mathcal{T}_r(\mathcal{K}(0), \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \operatorname{Vol} \Omega)$$

any smooth solution of the free boundary problem for Euler's equations (1.1)-(1.5), for $0 \leq t \leq T$ satisfies

$$(7.10) \quad \sum_{s=0}^r E_s(t) \leq 2 \sum_{s=0}^r E_s(0) \quad 0 \leq t \leq T$$

Proof of Theorem 7.1. In the proof it is convenient to replace the *a priori* bound (7.3) by

$$(7.11) \quad |\theta| \leq K', \quad 1/\iota_1 \leq K_1$$

see Definition 3.3 for ι_0 and Definition 3.4 for ι_1 . However, by Lemma 3.2

$$(7.12) \quad 1/\iota_0 \leq \max(K_1/2, \|\theta\|_{L^\infty}) \quad \text{and} \quad 1/\iota_1 \leq \max(\|\theta\|_{L^\infty}/\varepsilon_1, 1/2\iota_0)$$

Now, to get the iteration started we need bounds for some low norms. For u , $E_0 = \|u\|_{L^2(\Omega)}^2$ is conserved but we can not control the low norms of p and p_t in terms of the energies only so to control these we must use the fact that the $\text{Vol } \Omega$ is conserved.

Before starting with the proof of (7.2) let us first see what a bound for the energy (7.1) implies:

Lemma 7.3. *We have*

$$(7.13) \quad \|\nabla^r u\|_{L^2(\Omega)}^2 \leq CE_r, \quad \|\Pi \nabla^r p\|_{L^2(\partial\Omega)}^2 \leq \|\nabla p\|_{L^\infty(\partial\Omega)} E_r$$

$$(7.14) \quad \|\nabla^r p\|_{L^2(\partial\Omega)}^2 + \|\nabla^r p\|_{L^2(\Omega)}^2 \leq C(K_1, \text{Vol } \Omega) (\|\nabla p\|_{L^\infty(\partial\Omega)} + \|\nabla u\|_{L^\infty(\Omega)}^2) \sum_{k=0}^r E_k$$

Proof of Lemma 7.3. That $\|\Pi \nabla^r p\|_{L^2(\partial\Omega)} \leq \|\partial p\|_{L^\infty(\partial\Omega)} E_r$ follows from the definition of the projection; $\gamma^{ij} Q(\alpha_i, \alpha_j) = |\Pi \alpha|^2$ on $\partial\Omega$, and the fact that the measure in the energy is $(-\nabla_N p)^{-1} dS$. Since $\text{div } u = 0$, the bound $\|\nabla^r u\|_{L^2(\Omega)}^2 \leq CE_r$ follows from Lemma 5.3. By Lemma 6.1 and Lemma 8.3

$$\|\nabla^{r-1} \Delta p\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^\infty(\Omega)} \sum_{k=0}^r K_1^{r-\ell} \|\nabla^k u\|_{L^2(\Omega)}$$

(7.14) follows from (5.28) in Proposition 5.6 and the second part of (8.17) in Lemma 8.5. \square

The most interesting observation is now that that the bounds in particular of the boundary term in Lemma 7.3 actually imply a bound on the second fundamental form of the boundary:

Lemma 7.4. *With $L^\infty = L^\infty(\partial\Omega)$ we have*

$$(7.15) \quad \|\bar{\nabla}^{r-2} \theta\|_{L^2}^2 \leq C \left(K_1, \|\theta\|_{L^\infty}, \|(\nabla_N p)^{-1}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty(\Omega)}, \text{Vol } \Omega, \sum_{s=0}^{r-1} E_s(t) \right) \sum_{s=0}^r E_s(t)$$

Proof. Lemma 7.4 is of course just (5.32) in Proposition 5.7 and (7.14) in Lemma 7.3, the crucial point being a lower bound $-\nabla_N p > \varepsilon > 0$. \square

Lemma 7.3 suffices to control the interior terms, as we shall see. To control the boundary terms it turns out that the crucial point is to estimate

$$\|\Pi \nabla^r D_t p\|_{L^2(\partial\Omega)}$$

which uses the bound in Lemma 7.4. to estimate $\|\nabla^{r-2} \Delta D_t p\|_{L^2(\Omega)}$. We have

Lemma 7.5. *Let $p_t = D_t p$ and $L^\infty = L^\infty(\partial\Omega)$. We have*

$$(7.16) \quad \|\Pi \nabla^r p_t\|_{L^2(\partial\Omega)}^2 + \|\nabla^{r-1} p_t\|_{L^2(\partial\Omega)}^2 + \|\nabla^r p_t\|_{L^2(\Omega)}^2 \\ \leq C \left(K_1, \|\theta\|_{L^\infty}, \|(\nabla_N p)^{-1}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla_N p_t\|_{L^\infty}, \text{Vol } \Omega, \sum_{s=0}^{r-1} E_s(t) \right) \sum_{s=0}^r E_s(t)$$

Proof. By Lemma 6.1 and Lemma 8.3

$$(7.17) \quad \|\nabla^{r-2} \Delta D_t p\|_{L^2(\Omega)} \leq C(K_1) (\|\nabla p\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)}) \left(\sum_{k=0}^r (\|\nabla^k u\|_{L^2(\Omega)} + \|\nabla^k p\|_{L^2(\Omega)}) \right) \\ + C(K_1) \|\nabla u\|_{L^\infty(\Omega)}^2 \sum_{k=0}^{r-1} \|\nabla^k u\|_{L^2(\Omega)}.$$

The bound in (7.16) for $\|\nabla^{r-1} p_t\|_{L^2(\partial\Omega)}$ is just (5.33) in Proposition 5.8 together with (7.17), Lemma 7.3 and Lemma 7.4. The bound for $\|\Pi \nabla^r p_t\|_{L^2(\partial\Omega)}$ follows (5.30) in Proposition 5.7 and bound just obtained for $\|\nabla^s p_t\|_{L^2(\partial\Omega)}$, for $s \leq r-1$. Finally, the bound for $\|\nabla^r p_t\|_{L^2(\Omega)}$ follows from (5.29) in Proposition 5.6 and the bounds for $\|\nabla^{r-1} p_t\|_{L^2(\partial\Omega)}$ and $\|\Pi \nabla^r p_t\|_{L^2(\partial\Omega)}$ just obtained. \square

After having seen what a bound for the energy implies we now want to prove (7.2). The main ingredient is Proposition 5.10 applied to $\alpha = -\nabla^r p$, $\beta = \nabla^{r-1} u$ and $\nu = 1/(-\nabla_N p)$. Then $\text{div } \beta = 0$ and $\text{curl } \alpha = 0$ so we get from Proposition 5.10 and Lemma 7.3:

$$(7.18) \quad \frac{dE_r}{dt} \leq C(K_1, \|\theta\|_{L^\infty}, \|(\nabla_N p)^{-1}\|_{L^\infty}, \|\nabla p\|_{L^\infty}, \|\nabla u\|_{L^\infty(\Omega)}) E_r + \\ C \sqrt{E_r} \left(\|\Pi(-D_t \nabla^r p + \nu N^k \nabla^r u_k)\|_{L^2(\partial\Omega)} + \|D_t \nabla^r u + \nabla^{r+1} p\|_{L^2(\Omega)} + \|D_t \nabla^{r-1} \text{curl } u\|_{L^2(\Omega)} \right)$$

Using, Lemma 6.1 and Lemma 8.3 we can directly control the interior terms in (7.18):

$$(7.19) \quad \|D_t \nabla^r u + \nabla^{r+1} p\|_{L^2(\Omega)} + \|D_t \nabla^{r-1} \text{curl } u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^\infty(\Omega)} \sum_{k=0}^r K_1^{r-k} \|\nabla^k u\|_{L^2(\Omega)}$$

Hence it only remains to control the boundary term in (7.18). By Lemma 6.1

$$(7.20) \quad \|\Pi(D_t \nabla^r p + (\nabla^r u) \cdot \nabla p)\|_{L^2(\partial\Omega)} \leq \|\Pi \nabla^r D_t p\|_{L^2(\partial\Omega)} + C \sum_{s=1}^{r-2} \|\Pi((\nabla^{1+s} u) \cdot \nabla^{r-s} p)\|_{L^2(\partial\Omega)}$$

Since the first term in the right hand side of (7.20) is controlled by Lemma 7.5 it only remains to estimate

$$(7.21) \quad \|\Pi((\nabla^{1+s} u) \cdot \nabla^{r-s} p)\|_{L^2(\partial\Omega)}, \quad \text{for } 1 \leq s \leq r-2$$

Clearly these terms are lower order so there is no problem estimating them say using Sobolev's Lemma to bound them with interior norms. However, in order to get a bound that is linear in the highest

order derivative provided the *a priori* assumptions (7.3)-(7.6) hold, we must work a bit harder. Let us therefore look at the endpoints. If $s = r - 2$ this can be estimated by

$$(7.22) \quad \|\nabla^2 p\|_{L^\infty(\partial\Omega)} \|\nabla^{r-1} u\|_{L^2(\partial\Omega)} \leq CL \left(\sum_{k=0}^r E_k \right)^{1/2}$$

where we used the *a priori* assumption (7.5) and Sobolev's Lemma (Lemma 8.2):

$$(7.23) \quad \|\nabla^{r-1} u\|_{L^2(\partial\Omega)} \leq C \|\nabla^{r-1} u\|_{L^{2(n-1)/(n-2)}(\partial\Omega)} \leq C(K_1) \sum_{k=0}^r \|\nabla^k u\|_{L^2(\Omega)}$$

and if $s = 0$ (which actually is excluded) we could estimate it with

$$(7.24) \quad \|\nabla u\|_{L^\infty(\partial\Omega)} \|\nabla^r p\|_{L^2(\partial\Omega)} \leq C(K_1) M \left(\sum_{k=0}^r E_k \right)^{1/2}$$

by Lemma 7.3. Hence, we must now somehow control the intermediate terms. If the derivatives were tangential we could do this with the interpolation inequality Lemma 8.1. But because of the projection to the tangential components in (7.21) the highest order derivatives will be mostly tangential. By (4.48)

$$(7.25) \quad \begin{aligned} & \|\Pi((\nabla^{1+s} u) \cdot \nabla^{r-s} p)\|_{L^2(\partial\Omega)} \\ & \leq \|\Pi(\nabla^{1+s} u)\|_{L^{2(r-2)/s}(\partial\Omega)} \|\Pi \nabla^{r-s} p\|_{L^2(\partial\Omega)} + \|\Pi(N^k \nabla^{1+s} u_k)\|_{L^{2(r-2)/s}(\partial\Omega)} \|\Pi N^k \nabla^{r-1-s} \nabla_k p\|_{L^2(\partial\Omega)} \\ & \leq \|\Pi(\nabla^{1+s} u)\|_{L^{2(r-2)/s}(\partial\Omega)} \|\Pi \nabla^{r-s} p\|_{L^{2(r-2)/(r-2-s)}(\partial\Omega)} \\ & \quad + \|\Pi(N^k \nabla^{1+s} u_k)\|_{L^{2(r-2)/s}(\partial\Omega)} \|\Pi N^k \nabla^{r-1-s} \nabla_k p\|_{L^{2(r-2)/(r-2-s)}(\partial\Omega)} \end{aligned}$$

These terms can now be estimated by (4.46) in Proposition 4.8 with $\alpha = \nabla u$ and $\beta = \nabla^2 p$. This concludes the proof of Theorem 7.1.

Proof of Theorem 7.2. Let us now show how Theorem 7.2 follows. We will be using Sobolev's lemma, (Lemma 8.2-8.4). But then we must first make sure that we can control the Sobolev constants. By the results in section 8 these depend on the constant $K_1 = 1/\iota_1$ in Definition 3.4. Alternatively the change of the Sobolev constants in time are controlled by a bound for the time derivative of the metric in the y coordinates, see section 8. We also need to have control of the constant $1/\varepsilon$. We have

Lemma 7.6. *Let K_1 be as in Definition 3.4, $\mathcal{E}(t)$ as in (7.8) and $r_0 > n/2 + 3/2$. Then there are continuous functions $G_{r_0}, H_{r_0}, I_{r_0}$ and J_{r_0} such that*

$$(7.26) \quad \|\nabla u\|_{L^\infty(\Omega)} \leq G_{r_0}(K_1, E_0, \dots, E_{r_0})$$

$$(7.27) \quad \|\nabla p\|_{L^\infty(\Omega)} + \|\nabla^2 p\|_{L^\infty(\partial\Omega)} \leq H_{r_0}(K_1, E_0, \dots, E_{r_0}, \text{Vol } \Omega)$$

$$(7.28) \quad \|\theta\|_{L^\infty(\partial\Omega)} \leq I_{r_0}(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \text{Vol } \Omega)$$

$$(7.29) \quad \|\nabla p_t\|_{L^\infty(\partial\Omega)} \leq J_{r_0}(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \text{Vol } \Omega)$$

Proof. By Sobolev's lemma

$$(7.30) \quad \|\nabla u\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{s \leq r} \|\nabla^s u\|_{L^2(\Omega)}, \quad r-1 > n/2$$

$$(7.31) \quad \|\nabla p\|_{L^\infty(\Omega)} \leq C(K_1) \sum_{s \leq r} \|\nabla^s p\|_{L^2(\Omega)}, \quad r-1 > n/2$$

$$(7.32) \quad \|\nabla^2 p\|_{L^\infty(\partial\Omega)} \leq C(K_1) \sum_{s \leq r} \|\nabla^s p\|_{L^2(\partial\Omega)}, \quad r-2 > (n-1)/2$$

(7.26) follows from (7.30) and (7.13) in Lemma 7.3, (7.27) follows from (7.31), (7.14) and (7.26) (Note that p enters quadratic in the left hand side of (7.14) but only linear in the right hand side.). (7.32) follows in the same way. The bounds for $\|\theta\|_{L^\infty}$ and $\|\nabla p_t\|_{L^\infty}$ can not be obtained directly by Sobolev's lemma since the right hand side of (7.15) depends on $\|\theta\|_{L^\infty}$ and the right hand side of (7.16) depends on $\|\nabla p_t\|_{L^\infty}$. However,

$$(7.33) \quad |\nabla^2 p| \geq |\Pi \nabla^2 p| = |\nabla_N p| |\theta| \geq \mathcal{E}^{-1} |\theta|$$

so (7.28) follows from (7.27). (7.29) follows from (5.34) in Proposition 5.8. \square

Lemma 7.7. *Let $K_1 \geq 1/\iota_1$ and $\varepsilon_1 = \varepsilon_1(r)$ be as in Definition 3.4 and Lemma 5.5. Then if $r_0 > n/2 + 3/2$*

$$(7.34) \quad \left| \frac{d}{dt} E_r \right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \text{Vol } \Omega) \sum_{s=0}^r E_s$$

and

$$(7.35) \quad \left| \frac{d}{dt} \mathcal{E} \right| \leq C_r(K_1, \mathcal{E}, E_0, \dots, E_{r_0}, \text{Vol } \Omega).$$

Proof. (7.34) is a consequence of Lemma 7.6 and the estimates in the proof of Theorem 7.1. (7.35) follows from

$$\left| \frac{d}{dt} \|(-\nabla_N p(t, \cdot))^{-1}\|_{L^\infty(\partial\Omega)} \right| \leq C \|(-\nabla_N p(t, \cdot))^{-1}\|_{L^\infty(\partial\Omega)}^2 \|\nabla_N p_t(t, \cdot)\|_{L^\infty(\partial\Omega)}$$

and (7.29). \square

It follows from Lemma 7.7:

Lemma 7.8. *If $r \geq r_0$, there is continuous function $\mathcal{T}_r(K_1, \mathcal{E}(0), E_0(0), \dots, E_r(0), \text{Vol } \Omega) > 0$ such that for*

$$(7.36) \quad 0 \leq t \leq \mathcal{T}_r(K_1, \mathcal{E}(0), E_0(0), \dots, E_r(0), \text{Vol } \Omega)$$

the following statements hold: We have

$$(7.37) \quad E_s(t) \leq 2E_s(0), \quad 0 \leq s \leq r, \quad \mathcal{E}(t) \leq 2\mathcal{E}(0),$$

Furthermore

$$(7.38) \quad g_{ij}(0, y) X^i X^j / 2 \leq g_{ij}(t, y) X^i X^j \leq 2g_{ij}(0, y) X^i X^j$$

and with $\varepsilon_1(r) > 0$ as in Lemma 5.5

$$(7.39) \quad |\mathcal{N}(x(t, \bar{y})) - \mathcal{N}(x(0, \bar{y}))| \leq \varepsilon_1(r)/16, \quad \bar{y} \in \partial\Omega$$

$$(7.40) \quad |x(t, y) - x(0, y)| \leq \iota_1/16, \quad y \in \Omega$$

$$(7.41) \quad |\partial x / \partial y(t, \bar{y}) - \partial x / \partial y(0, \bar{y})| \leq \varepsilon_1(r)/16, \quad \bar{y} \in \partial\Omega$$

Proof. We get (7.37) from Lemma 7.7 if $\mathcal{T}_r(K_1, \mathcal{E}(0), E_0(0), \dots, E_r(0), \text{Vol } \Omega) > 0$ is sufficiently small. We have

$$(7.42) \quad \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^\infty(\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0))$$

$$(7.43) \quad \|\nabla^2 p\|_{L^\infty(\partial\Omega)} + \|\theta\|_{L^\infty(\partial\Omega)} \leq C(K_1, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \text{Vol } \Omega)$$

$$(7.44) \quad \|\nabla p_t\|_{L^\infty(\Omega)} \leq D(K_1, \mathcal{E}(0), E_0(0), \dots, E_{r_0}(0), \text{Vol } \Omega)$$

In fact, (7.42)-(7.44) follows from (7.37) and Lemma 7.6. It follows from this that

$$(7.45) \quad \|\nabla u(t, \cdot)\|_{L^\infty(\partial\Omega)} \leq 2\|\nabla u(0, \cdot)\|_{L^\infty(\partial\Omega)}$$

$$(7.46) \quad \|\nabla p(t, \cdot)\|_{L^\infty(\Omega)} \leq 2\|\nabla p(0, \cdot)\|_{L^\infty(\Omega)}$$

$$(7.47) \quad \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq 2\|v(0, \cdot)\|_{L^\infty(\Omega)}$$

In fact, by (6.7) we have

$$(7.48) \quad |D_t \nabla u| \leq |\nabla^2 p| + |\nabla u|^2, \quad |D_t \partial v| \leq |\partial^2 p| + |\partial v|^2$$

Using (7.42)-(7.44) we get that

$$(7.49) \quad \int_0^T \|\nabla^2 p(t, \cdot)\|_{L^\infty} + \|\nabla u(t, \cdot)\|_{L^\infty}^2 dt \leq \|\nabla u(0, \cdot)\|_{L^\infty},$$

if T is sufficiently small, so (7.45) follows after possibly making $\mathcal{T} > 0$ smaller. (7.46) and (7.47) follows in a similar manner from $|D_t \nabla p| = |\nabla p_t|$ respectively $|D_t v| = |\partial p|$.

Also (7.38) follows from the same argument since

$$(7.50) \quad D_t g_{ab} = h_{ab} = \nabla_a u_b + \nabla_b u_a$$

and by (7.44)

$$(7.51) \quad 2 \int_0^T \|\nabla_a u_b\|_{L^\infty(\Omega)} dt X^a X^b \leq g_{ab} X^a X^b / 2$$

if T is sufficiently small. Now the estimate for \mathcal{N} follows from

$$(7.52) \quad D_t n_a = h_{NN} n_a$$

and the estimates for x and $\partial x / \partial y$ from

$$(7.53) \quad D_t x(t, y) = v(t, x(t, y)), \quad \text{and} \quad D_t \frac{\partial x}{\partial y} = \frac{\partial v(t, x(t, y))}{\partial y} = \frac{\partial v(t, x)}{\partial x} \frac{\partial x}{\partial y}$$

and (7.47) respectively (7.45). \square

The idea is now to use (7.38)-(7.41) to pick a K_1 , i.e. ι_1 (see Definition 3.4), which only depend on their values at $t = 0$;

$$(7.54) \quad \iota_1(t) \geq \iota_1(0)/2,$$

Lemma 7.9. *Suppose that $\varepsilon_1(r)/2 \leq \varepsilon_1 \leq \varepsilon_1(r)$ and let \mathcal{T} be as in Lemma 7.7. Pick $\iota_1 > 0$ such that*

$$(7.55) \quad |\mathcal{N}(x(0, y_1)) - \mathcal{N}(x(0, y_2))| \leq \varepsilon_1/2, \quad \text{whenever } |x_1(0, y_1) - x(0, y_2)| \leq 2\iota_1$$

Then if $t \leq \mathcal{T}$ we have

$$(7.56) \quad |\mathcal{N}(x(t, y_1)) - \mathcal{N}(x(t, y_2))| \leq \varepsilon_1, \quad \text{whenever } |x_1(t, y_1) - x(t, y_2)| \leq \iota_1$$

Proof. (7.56) follows from (7.55) and (7.39)-(7.40). \square

Theorem 7.2 now follows directly from Lemma 7.9 and Lemma 7.8 since Lemma 7.9 allows us to pick a K_1 depending only on initial conditions and then Lemma 7.8 gives us $\mathcal{T} > 0$ which depends only on the initial conditions and K_1 such that, by Lemma 7.9, $1/\iota_1 \leq K_1$ for $t \leq \mathcal{T}$.

Note that there is an evolution equation also for θ but using it would require control of one more derivative of u :

$$(7.57) \quad D_t \theta_{ij} = -\gamma_i^\ell \gamma_j^a N^d \nabla_\ell \nabla_a u_d + N^a N^b \nabla_a u_b \theta_{ij} + 2(\theta_{ia} N_j + \theta_{ja} N_i) g^{ab} N^c \nabla_b u_c$$

We can control the size of θ through (7.43) but we can not control it in terms of initial data without going to energies with one more derivatives. This is why we need to estimate all the Sobolev constants in terms of K_1 instead of K , since (7.38)-(7.41) will allow us to control the time evolution of K_1 .

8. APPENDIX: SOBOLEV LEMMAS AND INTERPOLATION INEQUALITIES

Let us now state some Sobolev's lemmas and interpolation inequalities. Most of the results here are standard in \mathbb{R}^n , but we must control how it depends on the metric. There are two convenient ways to do this. The first is to use the fact that our set expressed in the x coordinates $\mathcal{D}_t \subset \mathbf{R}^n$ inherits the metric in \mathbf{R}^n and the surface $\partial\mathcal{D}_t$ can be expressed locally as a graph over \mathbf{R}^{n-1} .

Let $\mathcal{N}(\bar{x})$ be the unit normal at $\bar{x} \in \partial\mathcal{D}_t$ and suppose that

$$(8.1) \quad |\mathcal{N}(\bar{x}_1) - \mathcal{N}(\bar{x}_2)| \leq \varepsilon_1, \quad \text{whenever} \quad |\bar{x}_1 - \bar{x}_2| \leq \iota_1, \quad \bar{x}_1, \bar{x}_2 \in \partial\mathcal{D}_t.$$

By (8.1) we can write the surface as a graph within a ball of radius $\iota_1 = 1/K_1$ and for functions supported in such a ball we can thus use Sobolev's lemma in \mathbf{R}^{n-1} or \mathbf{R}^n . In general we make a partition of unity into functions supported in such balls and the Sobolev constant will thus depend only on K_1 .

When controlling how the metric changes with time we can use that our metrics γ on $\partial\Omega$ and g in Ω are equivalent to the same metrics at $t = 0$ in the y coordinates:

$$(8.2) \quad C_0^{-1} \gamma_{ij}^0(y) Z^i Z^j \leq \gamma_{ij}(t, y) Z^i Z^j \leq C_0 \gamma_{ij}^0(y) Z^i Z^j, \quad \text{if } Z \in T(\Omega),$$

$$(8.3) \quad C_0^{-1} g_{ij}^0(y) Z^i Z^j \leq g_{ij}(t, y) Z^i Z^j \leq C_0 g_{ij}^0(y) Z^i Z^j, \quad \text{if } Z \in T(\Omega),$$

and use Sobolev's lemma for the metrics γ_{ij}^0 respectively g_{ij}^0 . In this case, the Sobolev constants depend only on $\gamma_{ij}^0(y) = \gamma_{ij}(0, y)$ respectively $g_{ij}^0(y) = g_{ij}(0, y)$ and on C_0 .

Lemma 8.1. *If α is a $(0, r)$ tensor then with $a = k/m$ and a constant C that only depends on m and n :*

$$(8.4) \quad \|\bar{\nabla}^k \alpha\|_{L^s(\partial\Omega)} \leq C \|\alpha\|_{L^q(\partial\Omega)}^{1-a} \|\bar{\nabla}^m \alpha\|_{L^p(\partial\Omega)}^a, \quad \text{if } \frac{m}{s} = \frac{k}{p} + \frac{m-k}{q}, \quad 2 \leq p \leq s \leq q \leq \infty$$

Proof. Let us first prove (8.4) in case $m = 2$ and $k = 1$. We claim that

$$(8.5) \quad \|\bar{\nabla} \alpha\|_{L^s}^2 \leq C_s \|\alpha\| \|\bar{\nabla}^2 \alpha\|_{L^{s/2}}, \quad \text{if } s \geq 2 \quad \text{and} \quad C_s = s - 2 + \sqrt{n-1},$$

from which (8.4) follows in case $m = 2$ and $k = 1$. Then, the norm in the left of (8.4) to the power r is the limit as $\varepsilon \rightarrow 0$ of

$$\begin{aligned} \int_{\partial\Omega} (\langle \bar{\nabla} \alpha, \bar{\nabla} \alpha \rangle + \varepsilon)^{s/2-1} \langle \bar{\nabla} \alpha, \bar{\nabla} \alpha \rangle d\mu_\gamma &= - \int_{\partial\Omega} (\langle \bar{\nabla} \alpha, \bar{\nabla} \alpha \rangle + \varepsilon)^{s/2-1} \langle \alpha, \bar{\Delta} \alpha \rangle d\mu_\gamma \\ &\quad - \int_{\partial\Omega} 2(s/2-1) (\langle \bar{\nabla} \alpha, \bar{\nabla} \alpha \rangle + \varepsilon)^{s/2-2} \langle \bar{\nabla} \alpha, \bar{\nabla}^2 \alpha \rangle \cdot \langle \alpha, \bar{\nabla} \alpha \rangle d\mu_\gamma \end{aligned}$$

where we have integrated by parts. As $\varepsilon \rightarrow 0$ we see that

$$(8.6) \quad \|\bar{\nabla} \alpha\|_{L^s}^s \leq C_s \int \langle \bar{\nabla} \alpha, \bar{\nabla} \alpha \rangle^{s/2-1} |\alpha| \|\bar{\nabla}^2 \alpha\| d\mu_\gamma \leq C_s \|\bar{\nabla} \alpha\|_{L^s}^{s-2} \|\alpha\| \|\bar{\nabla}^2 \alpha\|_{L^{s/2}}$$

Dividing both sides by $\|\bar{\nabla} \alpha\|_{L^s}^{s-2}$ gives the desired inequality (8.4).

For fixed m, p and q let $s = s(k)$ be defined by (8.4) and set $M_k = \|\nabla^k \alpha\|_{L^{s(k)}}$. Then we have just proven that $M_k^2 \leq C_m M_{k-1} M_{k+1}$ for $1 \leq k \leq m-1$. Hence $N_k = C_m^{k^2} M_k$ satisfies $N_k^2 \leq N_{k-1} N_{k+1}$ and this logarithmic convexity implies that $N_k \leq N_0^{(m-k)/m} N_m^{k/m}$ which proves (8.4) in general. \square

Lemma 8.2. *Suppose that (8.1) and (8.2) hold with $\iota_1 \geq 1/K_1$. Then if α is a $(0, r)$ tensor:*

$$(8.7) \quad \|\alpha\|_{L^{(n-1)p/(n-1-kp)}(\partial\Omega)} \leq C(K_1) \sum_{\ell=0}^k \|\nabla^\ell \alpha\|_{L^p(\partial\Omega)}, \quad 1 \leq p < (n-1)/k$$

$$(8.8) \quad \|\alpha\|_{L^\infty(\partial\Omega)} \leq \delta \|\nabla^k \alpha\|_{L^p(\partial\Omega)} + C_\delta(K_1) \sum_{0 \leq \ell \leq k-1} \|\nabla^\ell \alpha\|_{L^p(\partial\Omega)} \quad k > (n-1)/p$$

for any $\delta > 0$.

Remark. For the boundary there are two possible interpretations of (8.7) and (8.8). One is to let the norm be given by the inner product $\langle \alpha, \alpha \rangle = \gamma^{IJ} \alpha_I \alpha_J$ and the covariant differentiation given by $\bar{\nabla}$ which corresponds to covariant differentiation on the boundary. The other interpretation is to let the inner product on the boundary be that of the interior $\langle \alpha, \alpha \rangle = g^{IJ} \alpha_I \alpha_J$ and the covariant differentiation be that of the interior ∇ . In fact, in both cases the proof reduces to $k = 1$ as before. If ϕ is a function then the lemma for ϕ follows from using covariant differentiation on the boundary. And applying this result to a norm gives

$$(8.9) \quad |\gamma_i^j \nabla_j \langle \alpha, \alpha \rangle| = 2 |\langle \alpha, \gamma_i^j \nabla_j \alpha \rangle| \leq |\alpha| |\gamma_i^j \nabla_j \alpha|$$

which is bounded by $|\alpha| |\bar{\nabla} \alpha|$ respectively $|\alpha| |\nabla \alpha|$

Proof of Lemma 8.2. We may assume that $p > n$ and hence $k \leq 1$ in (8.8) and $k = 1$ in (8.7). In fact, the general case follows from first using (8.8) respectively (8.7) in this case and then repeatedly using (8.7). Secondly, the case $r > 0$ can be reduced to the case of functions $r = 0$ by applying it to the norms $\phi = |\alpha|$. Hence we may assume that α is a function and $k = 1$.

Using the partition of unity $\{\chi_i\}$ in Lemma 3.4 we write $\phi = \sum_i \phi_i$, where $\phi_i = \chi_i \phi$. The support of each ϕ_i is then contained in a set S_i where the surface can be written as a graph $x_n = f_i(x')$, with $|\partial f_i| \leq \varepsilon_1 \leq 1$, as in (3.20). Then $dx' \leq dS \leq C dx'$ and $|\partial_{x'} \phi|/C \leq |\bar{\nabla} \phi| \leq |\partial_{x'} \phi|$, where $C = (1 + \varepsilon_1)^{1/2} \leq 2$ so apart from a constant factor Sobolev's Lemma on S_i reduces to Sobolev's lemma in \mathbf{R}^{n-1} . Using Minkowski's inequality, Sobolev's lemma in \mathbf{R}^{n-1} and Minkowski's inequality again:

$$(8.10) \quad \int_{\partial\Omega} \left(\sum |\phi_i| \right)^q dS \leq 2 \sum \int_{B(4r_0, x_i)} |\phi_i|^q dx' \leq 2C \sum \left(\int_{B(4r_0, x_i)} |\nabla \phi_i|^p dx' \right)^{q/p} \\ \leq 8C \left(\int_{\partial\Omega} \left(\sum |\nabla \phi_i|^q \right)^{p/q} dS \right)^{q/p}$$

since $q > p$. Here

$$(8.11) \quad \left(\sum |\nabla \phi_i|^q \right)^{p/q} = \left(\sum (|\nabla \chi_i| |\phi| + |\chi_i| |\nabla \phi|)^q \right)^{p/q} \leq CK_1^p (32)^{(n-1)p/q} (|\phi| r_0^{-1} + |\nabla \phi|)^p$$

which proves (8.7). (8.8) with δ replaced by a constant follows in the same way. Finally, we get (8.8) by considering (8.8) with δ replaced by a constant and $k = 1$ applied to α replaced by $|\alpha|^2$. In fact then we get $\|\alpha\|_{L^\infty}^2 \leq C \|\alpha\|_{L^q} \|\nabla \alpha\|_{L^q} + C \|\alpha\|_{L^q}^2$ for some $(n-1)/k < q < p$. Using Hölder's inequality we can estimate the first term by $C \|\alpha\|_{L^{pq/(p-q)}} \|\nabla \alpha\|_{L^p} \leq \delta \|\nabla \alpha\|_{L^p}^2 + C^2 \delta^{-1} \|\alpha\|_{L^{pq/(p-q)}}^2$, where the last term is bounded by $C_\delta \|\alpha\|_{L^\infty}^{1-(p-q)/q} \|\alpha\|_{L^p}^{(p-q)/q}$. \square

Lemma 8.3. *With notation as in Lemma 8.1 and Lemma 8.2 we have*

$$(8.12) \quad \sum_{j=0}^k \|\nabla^j \alpha\|_{L^s(\Omega)} \leq C \|\alpha\|_{L^q(\Omega)}^{1-a} \left(\sum_{i=0}^m \|\nabla^i \alpha\|_{L^p(\Omega)} K_1^{m-i} \right)^a,$$

Proof. As in the proof of (8.5), the general case of (8.12) will follow from the special case $m = 2$ and $k = 1$. If we integrate by parts as in the proof of (8.4) we also get a boundary term

$$\int_{\Omega} |\nabla \alpha|^s d\mu \leq C \int_{\Omega} |\nabla \alpha|^{s-2} |\alpha| |\nabla^2 \alpha| d\mu + C \int_{\partial\Omega} |\nabla \alpha|^{s-1} |\alpha| d\mu_{\gamma}.$$

If α has compact support in Ω then the boundary term cancels so by the proof of (8.4)

$$(8.13) \quad \|\nabla \alpha\|_{L^s(\Omega)}^2 \leq C \|\alpha\|_{L^q(\Omega)} \|\nabla^2 \alpha\|_{L^p(\Omega)}$$

We will prove that (8.13) is also true if α has compact support in a neighborhood of the boundary $\iota_1 < \text{dist}(y, \partial\Omega) \leq 0$. We have

$$\begin{aligned} \int_{\partial\Omega} |\nabla \alpha|^{s-1} |\alpha| d\mu_{\gamma} &\leq \left(\int_{\partial\Omega} |\nabla \alpha|^{(s-1)t} d\mu_{\gamma} \right)^{1/t} \left(\int_{\partial\Omega} |\alpha|^{t/(t-1)} d\mu_{\gamma} \right)^{(t-1)/t} \\ &\leq C \left(\int_{\Omega} |\nabla_N |\nabla \alpha|^{(s-1)t}| d\mu \right)^{1/t} \left(\int_{\Omega} |\nabla_N |\alpha|^{t/(t-1)}| d\mu \right)^{(t-1)/t} \\ &\leq C \left(\int_{\Omega} |\nabla \alpha|^{(s-1)t-1} |\nabla^2 \alpha| d\mu \right)^{1/t} \left(\int_{\Omega} |\alpha|^{1/(t-1)} |\nabla \alpha| d\mu \right)^{(t-1)/t} \end{aligned}$$

Now we want to use Hölder's inequality again on each factor with $\|\nabla^2 \alpha\|_{L^p}$ and $\|\alpha\|_{L^q}$ and $\|\nabla \alpha\|_{L^s}$ where $1/q + 1/p = 2/s$. Let $1/q' = 1 - 1/q$, $1/p' = 1 - 1/p$ and $1/s' = 1 - 1/s$. We will show that we can pick t so that $s = p'((s-1)t - 1)$ and $s' = (t-1)q$. We need to show that the two expressions for t are the same, i.e. that $(s - s/p + 1)/(s-1) = t = (s-1 + s/q)/(s-1)$ which is equivalent to $1/p + 1/q = 2/s$. The boundary term can hence be bounded by $\|\nabla \alpha\|_{L^s(\Omega)}^{s-2/t} \|\nabla^2 \alpha\|_{L^p(\Omega)}^{1/t} \|\alpha\|_{L^q(\Omega)}^{1/t}$. On the other hand the interior term can be estimated as in the proof of (8.5) so we get

$$\|\nabla \alpha\|_{L^s(\Omega)}^s \leq C \|\nabla \alpha\|_{L^s(\Omega)}^{s-2} \|\alpha\|_{L^q(\Omega)} \|\nabla^2 \alpha\|_{L^p(\Omega)} + C \|\nabla \alpha\|_{L^s(\Omega)}^{s-2/t} \|\nabla^2 \alpha\|_{L^p(\Omega)}^{1/t} \|\alpha\|_{L^q(\Omega)}^{1/t}$$

from which (8.13) follows also in case α is supported in the neighborhood $\iota_1 < \text{dist}(y, \partial\Omega) \leq 0$. Let $\{\chi_i\}$ be the partition of unity in Lemma 3.4. Now, since $|\nabla^\ell \chi_i| \leq C \iota_1^{-\ell}$ it follows that $\|\nabla^2(\chi_i \alpha)\|_{L^p}$ are bounded by the sum in the right hand side of (8.12) if $m = 2$ and $k = 1$. Since also $\|\alpha\|_{L^s}^2 \leq \|\alpha\|^2\|_{L^{s/2}} \leq \|\alpha\|_{L^q} \|\alpha\|_{L^p}$ by Hölder's inequality, (8.12) follows in case $m = 2$ and $k = 1$.

The general case of (8.12) follows from the special case as in the proof of (8.5) with the only exception that now $M_k = \sum_{i=0}^k \|\nabla^i \alpha\|_{L^{s(k)}}$. So far we have only proven that $M_1 \leq CM_0 M_2$, but the general case of $M_k^2 \leq CM_{k-1} M_{k+1}$ follows by induction from the previous case applied to $M'_k = \sum_{i=0}^k \|\nabla^i \nabla \alpha\|_{L^{s(k)}}$; $(M'_{k-1})^2 \leq CM'_{k-2} M'_k$ and Hölder's inequality $\|\alpha\|_{L^s} \leq \|\alpha|^{1-a} |\alpha|^a\|_{L^s} \leq \|\alpha\|_{L^q}^{1-a} \|\alpha\|_{L^p}^a$ again. \square

Lemma 8.4. *Suppose that $\iota_1 \geq 1/K_1$ and α is a $(0, r)$ tensor. Then*

$$(8.14) \quad \|\alpha\|_{L^{np/(n-kp)}(\Omega)} \leq C \sum_{\ell=0}^k K_1^{k-\ell} \|\nabla^\ell \alpha\|_{L^p(\Omega)}, \quad 1 \leq p < n/k$$

$$(8.15) \quad \|\alpha\|_{L^\infty(\Omega)} \leq C \sum_{0 \leq \ell \leq k} K_1^{n/p-\ell} \|\nabla^\ell \alpha\|_{L^p(\Omega)}, \quad k > n/p$$

Proof. As in the proof of Lemma 8.2 we may assume that α is a function and $k = 1$. We now want to extend the functions to outside Ω and then use Sobolev's lemma in \mathbf{R}^n . We can extend the function by writing the surface as a graph $x_n = f(x')$, $(x', x_n) \in \mathbf{R}^n$, as in the proof of Lemma 8.2. Let $\{\chi_i\}$ be the partition of unity in Lemma 3.4 and set $\phi_i = \chi_i \phi$. In a neighborhood of $\text{supp}(\chi_i)$ we can then write $\partial \mathcal{D}_t$ as a graph after a rotation:

$$x^n = f(x'), \quad (x', x^n) \in \mathbf{R}^n, \quad |\partial f| \leq 1$$

We now define

$$(8.16) \quad \hat{\phi}_i(x) = \begin{cases} \phi_i(x), & \text{when } x \in \Omega \\ \phi(\hat{x}), & \text{when } x \notin \Omega \end{cases}, \quad \text{where } \hat{x} = (\hat{x}', \hat{x}^n) = (x', x^n - 2(x^n - f(x')))$$

In proving the estimates (8.14)-(8.15) we may assume that $\phi \in C^\infty(\bar{\Omega})$ since this is dense in $W^{1,p}(\Omega)$, see [Ev]. Then by Sobolev's Lemma in \mathbf{R}^n :

$$\|\hat{\phi}_i\|_{L^q(\mathbf{R}^n)} \leq C \|\nabla \hat{\phi}_i\|_{L^p(\mathbf{R}^n)} \leq C \|\nabla \phi_i\|_{L^p(\Omega)} + C \|\nabla \hat{\phi}_i\|_{L^p(\mathbb{R}^n \setminus \Omega)} \leq C' \|\nabla \phi_i\|_{L^p(\Omega)}$$

since $|\partial \hat{x}^i / \partial x^j| \leq C$. Since $|\nabla \chi_i| \leq CK_1$ this proves (8.14) and (8.15) follows in a similar manner. \square

Lemma 8.5. *Suppose that $q = 0$ on $\partial \Omega$ then*

$$(8.17) \quad \|q\|_{L^2(\Omega)} \leq C(\text{Vol } \Omega)^{1/n} \|\nabla q\|_{L^2(\Omega)}, \quad \|\nabla q\|_{L^2(\Omega)} \leq C(\text{Vol } \Omega)^{1/2n} \|\Delta q\|_{L^2(\Omega)},$$

Proof. The first inequality is Faber-Krahn's theorem. Its proof uses a symmetrization argument, see [SY]. The second follows from the first and integration by parts. \square

We state two more lemmas:

Lemma 8.6. *Furthermore if the metric satisfies*

$$(8.18) \quad C_0^{-1} g_{ij}^0(y) Z^i Z^j \leq g_{ij}(t, y) Z^i Z^j \leq C_0 g_{ij}^0(y) Z^i Z^j, \quad \text{if } Z \in T(\Omega),$$

where g^0 is a positive definite metric, then with a constant depending only on g^0 and c_0 :

$$(8.19) \quad \|\partial_t^k \alpha\|_{L^s(\Omega \times [0, T])} \leq C \|\alpha\|_{L^q(\Omega \times [0, T])}^{1-a} \|\partial_t^m \alpha\|_{L^p(\Omega \times [0, T])}^a,$$

provided that $\partial_t^j \alpha(0, \cdot) = 0$, for $j = 0, \dots, m-1$.

Proof. It remains to prove (8.19) which is similar to the proof of (8.12). Suppose now that $\alpha(0, \cdot) = \partial_t \alpha(0, \cdot) = 0$. By (8.18) can bound the norm and the measure from above and below by a measure which is independent of t so as before it follows that

$$\begin{aligned} \int_0^T \int_\Omega |\partial_t \alpha|^s d\mu dt &\leq C \int_0^T \int_\Omega |\partial_t \alpha|^{s-2} |\alpha| |\partial_t^2 \alpha| d\mu dt + C \int_\Omega |\partial_t \alpha|^{s-1} |\alpha| d\mu(T) \\ \int_\Omega |\partial_t \alpha|^{s-1} |\alpha| d\mu(T) &\leq \left(\int_0^T \int_\Omega |\partial_t \alpha|^s d\mu dt \right)^{1-2/ts} \left(\int_0^T \int_\Omega |\partial_t^2 \alpha|^p d\mu dt \right)^{1/tp} \left(\int_0^T \int_\Omega |\alpha|^q d\mu dt \right)^{1/tq} \end{aligned}$$

from which (8.19) follows as before. \square

Using Lemma 8.2 and the proof of Lemma 5.4 we can get a slightly improved version of Lemma 5.4:

Lemma 8.7. *Let α be $(0, r)$ tensor and assume that $|\theta|_{L^\infty(\partial\Omega)} + 1/\iota_0 \leq K$ and $\text{Vol}(\Omega) \leq V$. Then there is $C = C(K, V, r, n)$ such that*

$$(8.20) \quad \|\alpha\|_{L^{(n-1)p/(n-p)}(\partial\Omega)} \leq C\|\nabla\alpha\|_{L^p(\Omega)} + C\|\alpha\|_{L^p(\Omega)}, \quad 1 \leq p < n$$

$$(8.21) \quad \|\nabla^2\alpha\|_{L^2(\Omega)} \leq C\left(\|\Pi\nabla^2\alpha\|_{L^{2(n-1)/n}(\partial\Omega)} + \|\Delta\alpha\|_{L^2(\Omega)} + \|\nabla\alpha\|_{L^2(\Omega)}\right)$$

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