

# The Riesz Representation Theorem

MA 466

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Let  $H$  be a Hilbert space over  $\mathbb{R}$  or  $\mathbb{C}$ , and  $T$  a bounded linear functional on  $H$  (a bounded operator from  $H$  to the field,  $\mathbb{R}$  or  $\mathbb{C}$ , over which  $H$  is defined). The following is called the Riesz Representation Theorem:

**Theorem 1** *If  $T$  is a bounded linear functional on a Hilbert space  $H$  then there exists some  $g \in H$  such that for every  $f \in H$  we have*

$$T(f) = \langle f, g \rangle .$$

*Moreover,  $\|T\| = \|g\|$  (here  $\|T\|$  denotes the operator norm of  $T$ , while  $\|g\|$  is the Hilbert space norm of  $g$ ).*

*Proof:* Let's assume that  $H$  is separable for now. It's not really much harder to prove for any Hilbert space, but the separable case makes nice use of the ideas we developed regarding Fourier analysis. Also, let's just work over  $\mathbb{R}$ .

Since  $H$  is separable we can choose an orthonormal basis  $\phi_j$ ,  $j \geq 1$ , for  $H$ . Let  $T$  be a bounded linear functional and set  $a_j = T(\phi_j)$ . Choose  $f \in H$ , let  $c_j = \langle f, \phi_j \rangle$ , and define  $f_n = \sum_{j=1}^n c_j \phi_j$ . Since the  $\phi_j$  form a basis we know that  $\|f - f_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $T$  is linear we have

$$T(f_n) = \sum_{j=1}^n a_j c_j. \tag{1}$$

Since  $T$  is bounded, say with norm  $\|T\| < \infty$ , we have

$$|T(f) - T(f_n)| \leq \|T\| \|f - f_n\|. \tag{2}$$

Because  $\|f - f_n\| \rightarrow 0$  as  $n \rightarrow \infty$  we conclude from equations (1) and (2) that

$$T(f) = \lim_{n \rightarrow \infty} T(f_n) = \sum_{j=1}^{\infty} a_j c_j. \tag{3}$$

In fact, the sequence  $a_j$  must itself be square-summable. To see this, first note that since  $|T(f)| \leq \|T\| \|f\|$  we have

$$\left| \sum_{j=1}^{\infty} c_j a_j \right| \leq \|T\| \left( \sum_{j=1}^{\infty} c_j^2 \right)^{1/2}. \tag{4}$$

Equation (4) must hold for any square-summable sequence  $c_j$  (since any such  $c_j$  corresponds to some element in  $H$ ). Fix a positive integer  $N$  and define a sequence  $c_j = a_j$  for  $j \leq N$ ,  $c_j = 0$  for  $j > N$ . Clearly such a sequence is square-summable and equation (4) then yields

$$\left| \sum_{j=1}^N a_j^2 \right| \leq \|T\| \left( \sum_{j=1}^N a_j^2 \right)^{1/2}$$

or

$$\left( \sum_{j=1}^N a_j^2 \right)^{1/2} \leq \|T\|. \quad (5)$$

Thus  $a_j$  is square-summable, since the sequence of partial sums is bounded above.

Since  $a_j$  is square-summable the function  $g = \sum_j a_j \phi_j$  is well-defined as an element of  $H$ , and  $T(f) = \sum_j a_j c_j = \langle f, g \rangle$ . Finally, equation (5) makes it clear that  $\|g\| \leq \|T\|$ . But from Cauchy-Schwarz we also have  $|T(f)| = |\langle f, g \rangle| \leq \|f\| \|g\|$  or  $\frac{|T(f)|}{\|f\|} \leq \|g\|$ , implying  $\|T\| \leq \|g\|$ , so  $\|T\| = \|g\|$ .

## An Application

This examples illustrates how functional analytic methods are used in PDE (though the example is for an ODE). Consider the ODE

$$-f''(x) + b(x)f(x) = q(x) \quad (6)$$

on the interval  $0 < x < 1$ , where  $b(x) \geq \delta > 0$  for some  $\delta$ ; assume the functions  $b$  and  $q$  are continuous on  $[0, 1]$ . We want a solution to equation (6) with  $f'(0) = f'(1) = 0$  (other boundary conditions are possible). If we multiply (6) by a  $C^1$  function  $\phi$  and integrate the first term,  $-f''\phi$  by parts from  $x = 0$  to  $x = 1$ , we obtain

$$\int_0^1 (f'(x)\phi'(x) + b(x)f(x)\phi(x)) dx = \int_0^1 q(x)\phi(x) dx. \quad (7)$$

Equation (7) must hold for any  $\phi \in C^1([0, 1])$ , if  $f$  is a  $C^2(0, 1)$  solution to equation (6) which is continuous on  $[0, 1]$ . Conversely, if for a given  $C^2$

function  $f$  we find that (7) holds for all  $\phi$ , then  $f$  must be a solution to equation (6), for if we “undo” the integration by parts in (7) we obtain

$$\phi(1)f'(1) - \phi(0)f'(0) + \phi(x)(-f''(x) + b(x)f(x)) = \phi(x)q(x)$$

for all  $\phi$ . A familiar PDE argument then shows that  $f'(0) = f'(1) = 0$  and equation (6) must hold.

We’re going to show that there is a unique solution to equation (7). Such a “solution” will not necessarily be twice-differentiable as required by equation (6), but it will satisfy equation (7). Equation (7) is often called the “weak” formulation of the problem.

Define an inner product

$$\langle g, h \rangle = \int_0^1 (g'(x)h'(x) + b(x)g(x)h(x)) dx$$

on the space  $C^1([0, 1])$ , and let  $H$  denote the completion of this space. This is essentially the procedure used on the third problem of the first exam; the presence of  $b(x)$  makes no difference. (Note we must use  $b \geq \delta > 0$  to assure that  $\langle, \rangle$  really is an inner product, so that  $\|g\| = \sqrt{\langle g, g \rangle} = 0$  iff  $g \equiv 0$ ). The space  $H$  is a Hilbert space, and can be interpreted (if need be) as a subspace of  $C([0, 1])$ .

Define a functional  $T : H \rightarrow \mathbb{R}$  by

$$T(\phi) = \int_0^1 q(x)\phi(x) dx$$

You can easily check that  $T$  is bounded on  $H$  (use Cauchy-Schwarz). From the Riesz Representation Theorem it then follows that there must exist some function  $f \in H$  such that

$$T(\phi) = \langle f, \phi \rangle$$

for all  $\phi \in H$ . This is exactly equation (7), the weak form of the ODE!

The function  $f$  that satisfies equation (7) lies in  $H$ . Given the conditions on  $b$  (in particular,  $b \geq \delta > 0$  and  $\|b\|_\infty < \infty$  since  $b \in C([0, 1])$ ) the function  $f$  lies in the same space defined in the third problem of the first exam. In particular,  $f$  is a continuous function. To show that  $f$  is actually twice-differentiable requires more work, along with assumptions on the function  $q$ .