Lecture 6:Weak limits. Any weak limit of a distribution is a distribution. We say that $f_n \to f$ weakly if

$$\int f_n \phi \, dx \to \int f \phi \, dx, \quad \phi \in C_0^\infty .$$

Moreover, it follows directly from the definitions that $\partial^\alpha f_n \to \partial^\alpha f$ if $f_n \to f$.

Any distribution f is the weak limit of a sequence of $f_n \in C_0^\infty$.
In fact, let $\phi \in C_0^\infty$ satisfy $\int \phi \, dx = 1$ and set $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$. The convolution $f_k(x) = \phi_{1/k} * f(x) = \int f(y) \phi_{1/k}(x-y) \, dy = \langle f, \phi_{1/k}(x - \cdot) \rangle$ is well defined as the distribution f acting on the test function $\phi_{1/k}(x-y)$, considered as a function of y. That $f_k \in C^\infty$ is seen by looking at difference quotients $(f_k(x+he_j) - f_k(x))/h$ and using that $(\phi_{1/k}(x+he_j) - \phi_{1/k}(x))/h \to \partial_j \phi_{1/k}(x)$ in C_0^∞. To show that $f_k \to f$:

$$\langle f_k, \psi \rangle = \int\!\!\int f(y) \phi_{1/k}(x-y) \, dy \, \psi(x) dx = \int f(y) \int \phi_{1/k}(x-y) \psi(x) \, dx \, dy = \langle f, \phi_{1/k} * \psi \rangle$$

It follows from the proof of Lemma 3.2 that $\phi_{1/k} * \psi \to \psi$, and hence $f_k \to f$ weakly.

Operations on Distributions.

If f is a distribution and u is a smooth function then the product uf is defined by $\langle uf,\, \phi\rangle = \langle f,\, u\phi\rangle$. Multiplication of distributions is however not always defined. E.g. we can't multiply $\delta(x)$ with itself. In fact, if $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n \to \delta(x)$ then $\phi_\varepsilon(x)^2 = \phi(x/\varepsilon)^2/\varepsilon^{2n} \to \delta(x)^2$, but $\int \phi_\varepsilon(x)^2\, dx = \int \phi(x)^2\, dx/\varepsilon^n \to \infty$, as $\varepsilon \to 0$. On the other hand in $\mathbf{R}^2$, $\delta(x_1)\delta(x_2) = \delta(x_1, x_2)$ is well defined.

Problem 6.1 Show that $u(t)\delta(t) = u(0)\delta(t)$, and $u(t)\delta'(t) = u(0)\delta'(t) - u'(0)\delta(t)$

The convolution of a distribution u with a function $F \in C_0^\infty$ is defined by

$$u * F(x) = \int u(y)\, F(x-y)\, dy = \langle u, F(x-\cdot)\rangle$$

or with $\check{F}(x) = F(-x)$:

$$\langle u * F, \phi\rangle = \int\int u(y)\, F(x-y)\, dy \phi(x)\, dx = \langle u, \check{F} * \phi\rangle.$$

If u has compact support this is well defined for any smooth F. Note that $\delta * \phi = \phi$.

A distribution can also be composed with a diffeomorphism (i.e. a smooth map with smooth inverse) using the formula

$$\int u(\chi(x))\phi(x)\, dx = \int u(y)\phi(\chi^{-1}(y)\, |\det \chi'|^{-1}\, dy$$

Problem 6.2: Suppose $f \in C^\infty(\mathbf{R})$, $f(0)=0$, $f(t) \neq 0$ when $t \neq 0$, $f'(0) \neq 0$. Show

$$\delta(f(t)) = \frac{1}{|f'(0)|}\delta(t).$$

Problem 6.3 Show that $\delta(ax) = \dfrac{1}{|a|^n}\delta(x)$, in $\mathbf{R}^n$ and that $\delta(Qx) = \delta(x)$, if $Q^T Q = I$

The Fourier transform of a distribution is defined through duality using (3.6)

$$< \hat{f}, \phi > = < f, \hat{\phi} >$$

In order for this to be well defined for all ϕ we must assume that f is a tempered distribution, i.e. an element of the dual space $\mathcal{S}'$, of $\mathcal{S}$ or a continuous linear functional on $\mathcal{S}$ with respect to the seminorms $\rho_{\alpha,\beta}(\phi) = \sup_x |x^\beta \partial^\alpha \phi(x)|$, since $\mathcal{F}: \mathcal{S} \to \mathcal{S}$. We can hence extend the Fourier transform to a map $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$. If f has compact support then $\hat{f}$ is the smooth function $\langle f(x), e^{-ix\cdot\xi}\rangle$. (A distribution has compact support in K if $\langle f, \phi\rangle = 0$, $\operatorname{supp}\phi \cap K = \emptyset$.) We have $\mathcal{F}\colon \delta_a(x) \to e^{-ia\cdot\xi}$

Problem 6.4 Compute the Fourier transform of $e^{-ax^2/2}$, for Re $a \geq 0$.

Problem 6.5 Compute the Fourier transform of the function $f = 1$.

Fundamental solutions.

The fundamental solution E of a partial differential operator $P(D)=\sum a_\alpha \partial^\alpha$ is defined by

$$P(D)E = \delta$$

Using the fundamental solution one can solve the equation

$$P(D)u = F,$$

In fact $u = E * F$ satisfies

$$P(D)(E * F) = (P(D)E) * F = \delta * F = F$$

That you can let the derivatives fall on either factor follows from writing out the convolution integral and differentiating below the integral sign.

Let us first derive the fundamental solution of $\triangle$. $\triangle$ is invariant under rotations:
Problem 6.6 Show that $\triangle\big(E(ax)\big) = a^2(\triangle E)(ax)$ and $\triangle\big(E(Qx)\big) = (\triangle E)(Qx)$, if Q is an orthogonal matrix: $Q^T Q = I$.
Since $\triangle$ is invariant under rotations we expect $E(x) = f(|x|)$, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.