

**Lecture 6: 2.4 Tangent Plane and Differential.**

We did the definition of differentiable map between surfaces in section 2.3 We did Proposition 1 and Proposition 2 with proofs.

We also should have introduced the normal in the end of the section.

We also talked about contractions, see next page below, in preparation for the proof of the inverse function theorem next time. (An alternative source of this is Rudin's analysis book used for 140.)

## Appendix B: The inverse function theorem.

### Contractions.

A map  $T : W \rightarrow W$  is called a *contraction*, if for  $x, y \in W$ :

$$(B1) \quad \|T(x) - T(y)\| \leq K\|x - y\|, \quad K < 1$$

A point  $x \in W$  is called a *fixed point* if  $T(x) = x$ . We have:

**Lemma 2.** *Let  $T : W_0 \rightarrow W_0$  be a contraction of a complete normed space  $W_0$ . Then  $T$  has a unique fixed point  $x \in W_0$ . In fact for any  $x_0 \in W_0$ ,  $x_k = T^k(x_0) = T \circ \dots \circ T(x_0)$  ( $k$  times) converges to  $x$ ;  $\|x - x_k\| \rightarrow 0$ , as  $k \rightarrow \infty$ .*

*Proof.* Using (B1) repeatedly we get

$$(B2) \quad \|x_{k+1} - x_k\| = \|T(x_k) - T(x_{k-1})\| \leq K\|x_k - x_{k-1}\| \leq \dots \leq K^k\|x_1 - x_0\|$$

Here  $\|x_1 - x_0\| = \|T(x_0) - x_0\| = C$  is a fixed constant. For  $m > k$  we write  $x_m - x_k = (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \dots + (x_{k+1} - x_k)$  and estimate the norm of each term by (B2):

$$(B3) \quad \|x_m - x_k\| \leq \|x_m - x_{m-1}\| + \dots + \|x_{k+1} - x_k\| \leq (K^{m-1} + \dots + K^{k-1})C$$

This is a geometric sum and since  $K < 1$  the infinite sum converges;  $\sum_{\ell=k-1}^{m-1} K^\ell \leq \sum_{\ell=k-1}^{\infty} K^\ell = K^{k-1} \sum_{n=0}^{\infty} K^n = K^{k-1}/(1-K)$ . Hence

$$(B4) \quad \|x_m - x_k\| \leq \varepsilon(N) = \frac{CK^{N-1}}{1-K}, \quad \text{if } m, k \geq N,$$

where  $\varepsilon(N) \rightarrow 0$  as  $N \rightarrow \infty$ , i.e.  $x_k$  is a Cauchy sequence.

The uniqueness follows from (B1); if  $T(x) = x$  and  $T(y) = y$  then  $\|x - y\| = \|T(x) - T(y)\| \leq K\|x - y\|$  and since  $K < 1$  it follows that  $\|x - y\| = 0$  so  $x = y$ .  $\square$

**Ex.** Find an approximation for  $\sqrt{2}$ . Let

$$(B5) \quad g(x) = \frac{x^2 + 2}{2x}$$

Then  $\sqrt{2}$  is the only positive fixed point for  $g(x)$ ;  $g(\sqrt{2}) = \sqrt{2}$ . We claim that  $x \rightarrow g(x)$  is a contraction of the set  $W_0 = \{x; x \geq 1\}$ :

$$(B6) \quad |g(x) - g(y)| \leq \frac{1}{2}|x - y|, \quad \text{if } x, y \geq 1 \quad \text{and} \quad g(x) \geq 1, \quad \text{if } x \geq 1.$$

By the above lemma, if we set  $x_0 = 2$  and  $x_{n+1} = g(x_n)$ , for  $n \geq 0$  then  $x_n \rightarrow \sqrt{2}$ , as  $n \rightarrow \infty$ :

$$(B7) \quad x_0 = 2, \quad x_1 = 1.5, \quad x_2 = 1.41667\dots, \quad x_3 = 1.41422\dots, \dots$$

To prove (B6) we note that  $|g'(s)| = |1/2 - 1/s^2| \leq 1/2$ , if  $|s| \geq 1$  and hence

$$(B8) \quad |g(x) - g(y)| = \left| \int_y^x g'(s) ds \right| \leq \int_y^x |g'(s)| ds \leq \frac{|x - y|}{2}, \quad \text{if } x \geq y \geq 1.$$

Moreover, if  $x \geq 0$  then  $g(x) \geq 1$  is equivalent to  $x^2 + 2 - 2x = (x - 1)^2 + 1 > 0$ .