Lecture 4. Then we went back to Section 2.2 about regular surfaces:

We proved Proposition 1 that graph is a regular surface.

We went over definition 2, what is meant by a regular value of a function.

We also started to prove Proposition 2, that if a is a regular value then $f(x, y, z) = a$ is a regular surface. This however required the use of the inverse function theorem from Appendix B that we just had time to mention briefly. We also did the example showing that if $f(x, y, z) = z^2 - x^2 - y^2 = a$, then $a = 0$ is not a regular value.

We also mentioned Proposition 3 that if S is a regular surface and $p \in S$ that close to p S can be written as graph over one of the coordinate planes.

They author suggests that one omits the proofs in Section 2.2 at the first reading so we will come back to the proofs and the proof of the Inverse Function theorem later on.

Lecture 7: 2.3 Differentiable map on Surface.

We did Proposition 1 with proof, Definition 1 and Examples 1 and 3.

Lecture 8: 2.4 Tangent Plane and Differential.

We did Proposition 1 and Proposition 2 with proofs.

We also introduced the normal on page 47.

Lecture 9: 2.4 The differential of a map between surfaces.

We did the proof of Proposition 2.

We also talked about contractions, see next page below, in preparation for the proof of the inverse function theorem next time. (An alternative source of this is Rudin's analysis book used for 140.)

Appendix B: The inverse function theorem.

Contractions.
A map $T : W \to W$ is called a *contraction*, if for $x, y \in W$:

(B1) $$\|T(x) - T(y)\| \le K\|x - y\|, \qquad K < 1$$

A point $x \in W$ is called a *fixed point* if $T(x) = x$. We have:

Lemma 2. *Let $T : W_0 \to W_0$ be a contraction of a complete normed space W_0. Then T has a unique fixed point $x \in W_0$. In fact for any $x_0 \in W_0$, $x_k = T^k(x_0) = T \circ \cdots \circ T(x_0)$ (k times) converges to x; $\|x - x_k\| \to 0$, as $k \to \infty$.*

Proof. Using (B1) repeatedly we get

(B2) $$\|x_{k+1} - x_k\| = \|T(x_k) - T(x_{k-1})\| \le K\|x_k - x_{k-1}\| \le \cdots \le K^k\|x_1 - x_0\|$$

Here $\|x_1 - x_0\| = \|T(x_0) - x_0\| = C$ is a fixed constant. For $m > k$ we write $x_m - x_k = (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + ... + (x_{k+1} - x_k)$ and estimate the norm of each term by (B2):

(B3) $$\|x_m - x_k\| \le \|x_m - x_{m-1}\| + \cdots + \|x_{k+1} - x_k\| \le (K^{m-1} + \cdots + K^{k-1})C$$

This is a geometric sum and since $K < 1$ the infinite sum converges; $\sum_{\ell=k-1}^{m-1} K^\ell \le \sum_{\ell=k-1}^{\infty} K^\ell = K^{k-1}\sum_{n=0}^{\infty} K^n = K^{k-1}/(1-K)$. Hence

(B4) $$\|x_m - x_k\| \le \varepsilon(N) = \frac{CK^{N-1}}{1-K}, \qquad \text{if} \quad m, k \ge N,$$

where $\varepsilon(N) \to 0$ as $N \to \infty$, i.e. x_k is a Cauchy sequence.

The uniqueness follows from (B1); if $T(x) = x$ and $T(y) = y$ then $\|x - y\| = \|T(x) - T(y)\| \le K\|x - y\|$ and since $K < 1$ it follows that $\|x - y\| = 0$ so $x = y$. $\square$

Ex. Find an approximation for $\sqrt{2}$. Let

(B5) $$g(x) = \frac{x^2 + 2}{2x}$$

Then $\sqrt{2}$ is the only positive fixed point for $g(x)$; $g(\sqrt{2}) = \sqrt{2}$. We claim that $x \to g(x)$ is a contraction of the set $W_0 = \{x; x \ge 1\}$:

(B6) $$|g(x) - g(y)| \le \frac{1}{2}|x - y|, \quad \text{if} \quad x, y \ge 1 \qquad \text{and} \qquad g(x) \ge 1, \quad \text{if} \quad x \ge 1.$$

By the above lemma, if we set $x_0 = 2$ and $x_{n+1} = g(x_n)$, for $n \ge 0$ then $x_n \to \sqrt{2}$, as $n \to \infty$:

(B7) $$x_0 = 2, \quad x_1 = 1.5, \quad x_2 = 1.41667..., \quad x_3 = 1.41422..., \cdots$$

To prove (B6) we note that $|g'(s)| = |1/2 - 1/s^2| \le 1/2$, if $|s| \ge 1$ and hence

(B8) $$|g(x) - g(y)| = \Big|\int_y^x g'(s)\,ds\Big| \le \int_y^x |g'(s)|ds \le \frac{|x-y|}{2}, \quad \text{if} \quad x \ge y \ge 1.$$

Moreover, if $x \ge 0$ then $g(x) \ge 1$ is equivalent to $x^2 + 2 - 2x = (x-1)^2 + 1 > 0$.

Lecture 10: Appendix B: The Inverse and Implicit Function Theorems.

Contractions.
A map $T: W \to W$ is called a *contraction*, if for $x, y \in W$:

$$\|T(x) - T(y)\| \le K\|x - y\|, \qquad K < 1 \tag{B1}$$

A point $x \in W$ is called a *fixed point* if $T(x) = x$. We have:

Lemma 2. *Let $T: W_0 \to W_0$ be a contraction of a complete normed space W_0. Then T has a unique fixed point $x \in W_0$. In fact for any $x_0 \in W_0$, $x_k = T^k(x_0) = T \circ \cdots \circ T(x_0)$ (k times) converges to x; $\|x - x_k\| \to 0$, as $k \to \infty$.*

Theorem 1. *Suppose that $F: \mathbf{R}^n \to \mathbf{R}^n$ is C^1. Let $F(x_0) = y_0$ and suppose that*

$$dF_{x_0} = \frac{\partial F}{\partial x}(x_0) \tag{B9}$$

is invertible. Then for y close to y_0 there is an unique x close to x_0 such that

$$F(x) = y \tag{B10}$$

Furthermore $x = x(y)$ is a C^1 function of y close to y_0.

By Taylor's formula, if $F \in C^2$,

$$y - y_0 = F(x) - F(x_0) = dF_{x_0}(x - x_0) + O\big(|x - x_0|^2\big) \tag{B11}$$

where the derivative $dF_{x_0}: \mathbf{R}^n \to \mathbf{R}^n$ is the linear map that best approximates the function close to x_0 and $O\big(|x - x_0|^2\big)$ means terms that are bounded by a constant times $|x - x_0|^2$ and hence much smaller than $|x - x_0|$, when $|x - x_0|$ is small. Therefore, to get a first approximation we must be able to invert the linear linear map, and we get that $x - x_0 = (dF_{x_0})^{-1}(y - y_0) + O(|y - y_0|^2)$.

The proof of Theorem 1 uses the contraction mapping theorem. First by a translation replacing $F(x)$ by $F(x + x_0) - y_0$ we can reduce to the case when $x_0 = y_0 = 0$. Furthermore by multiplying both sides of (B10) by the matrix $(dF_0)^{-1}$ and making a change of variables replacing y by $(dF_0)^{-1}y$ we may assume that the equation (B10) takes the form

$$y = x + \phi(x) \tag{B12}$$

where $\phi(x)$ is small; $\phi(0) = 0$ and $d\phi_0 = 0$. We seek a solution in the form

$$x = y + \psi(y) \tag{B13}$$

Then for $\phi(y)$ we obtain the equation $\psi(y) = -\phi\big(y + \psi(y)\big)$ Consequently, the function ψ being sought is a fixed point of the mapping T defined by the formula

$$(T\psi)(y) = -\phi\big(y + \psi(y)\big) \tag{B14}$$

Problem 1: Show that T is a contraction in some norm for y sufficiently small. You have to use that since ϕ is continuously differentiable and $d\phi_0 = 0$ there is a neighborhood $\delta > 0$ such that

$\|d\phi_z\| = \sup_{|x|\le 1} |d\phi_z(x)|/|x| < 1/2$, when $|z| < \delta$. Let $W = \{\psi \in C^1(\{|y| \le \delta/2\});\ |\psi(y)| \le |y|\}$. With $z(t) = y + \psi_1(y) + t(\psi_2(y) - \psi_1(y))$, $0 \le t \le 1$, the line segment between $y + \psi_1(y)$ and $y + \psi_2(y)$;

$$T(\psi_1)(y) - T(\psi_2)(y) = \phi\big(y + \psi_2(y)\big) - \phi\big(y + \psi_1(y)\big) = \int_0^1 \frac{d}{dt}\,\phi\big(z(t)\big)\,dt = \int_0^t d\phi_{z(t)}(\psi_2(y) - \psi_1(y))dt.$$

If $\psi_1, \psi_2 \in W$ and $|y| \le \delta/2$ then $|z(t)| \le \delta$, for $0 \le t \le 1$ and hence

$$|T(\psi_1)(y) - T(\psi_2)(y)| \le \sup_{0\le t\le 1} \|d\phi_{z(t)}\|\,|\psi_2(y) - \psi_1(y)| \le \frac{1}{2}|\psi_2(y) - \psi_1(y)|, \quad \text{if} \quad |y| < \delta/2.$$

In particular if we take $\psi_2(y) = -y$ we obtain

$$|T(\psi_1)(y)| \le \frac{1}{2}|\psi_1(y) + y| \le |y|, \quad \text{if} \quad |y| < \delta/2,$$

i.e. $T(\psi) \in W$ if $\psi \in W$. If $|y| < \delta/2$ and we set $\psi_0(y) = 0$ and $\psi_{n+1}(y) = T(\psi_n)(y)$, for $n \ge 0$, then by the contraction lemma $\psi_n(y) \to \psi(y)$ in W, where $T(\psi)(y) = \psi(y)$.

Theorem 2. *Suppose that $G : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n$ is C^1. Let $G(x_0, y_0) = c_0$ and suppose that*

$$\frac{\partial G}{\partial y}(x_0, y_0) \tag{B15}$$

is invertible. Then for x close to x_0 there is a unique $y = g(x)$ close to y_0 such that

$$G(x, g(x)) = 0 \tag{B16}$$

Furthermore $y = g(x)$ is a C^1 function of x close to y.

Problem 2: Show that Theorem 2 follows from Theorem 1, by considering $F(x, y) = \big(x, G(x, y)\big)$.
Problem 3: Suppose that $G(x_0, y_0, z_0) = 0$, and **grad** $G(x_0, y_0, z_0) \ne 0$. Use Theorem 2 to deduce that close to (x_0, y_0, z_0) the equation $G(x, y, z) = c_0$ is a surface, i.e. show that one of the variables say z (if $\partial G/\partial z \ne 0$ can by written as a graph $z = g(x, y)$ so that $G(x, y, g(x, y)) = 0$.

Lecture 11: 2.5 The First Fundamental Form.

We did Definition 1 of the first fundamental form, Example 1, Example 2, Example 4.

We also did Definition 2 or the area formula in terms of a parametrization and the coefficients of the first fundamental form. We should also have done Example 5.

2.6 Orientation. We did the definition 1 of orientation and Proposition 1.