Lecture 2.

1.4 The vector product in $\mathbf{R}^3$. The definition of and notation for the vector product in the text is unusual. We define the vector product of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ to be the unique vector $\mathbf{u} \wedge \mathbf{v}$ such that for all vectors $\mathbf{w} = (w_1, w_2, w_3)$ we have

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) \equiv \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} w_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} w_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} w_3,$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ac - bd$$

This means that

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_3,$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis.

There are certain properties of the vector product that follows directly from the definition:

(1) It is anti-symmetric: $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

(2) It is separately linear in each argument $(a\mathbf{u} + b\mathbf{w}) \wedge \mathbf{v} = a\mathbf{u} \wedge \mathbf{v} + b\mathbf{w} \wedge \mathbf{v}$.

(3) $\mathbf{u} \wedge \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

(4) $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{v} = 0$.

Moreover

(5) $|\mathbf{u} \wedge \mathbf{v}| = |\mathbf{u}|\, |\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(6) $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \wedge \mathbf{v}\}$ form a positively oriented basis if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

1.5 The Frenet formulas.

Ex Consider a circle with radius R parameterized by arc length:
$\alpha(s) = x_0 + Re_1 \cos(s/R) + Re_2 \sin(s/R)$, where e_1 and e_2 are orthogonal unit vectors.
Then tangent is $\alpha'(s) = -e_1 \sin(s/R) + e_2 \cos(s/R)$ and
the second derivative $\alpha''(s) = (-e_1/R)\cos(s/R) - (e_2/R)\sin(s/R) = (x_0 - x(s))/R^2$
is directed towards the center of the circle and $\|\alpha''\| = 1/R$.

Def If $\alpha(s)$ is parameterized by arc length, i.e. $|\alpha'(s)| = 1$ then $k(s) = |\alpha''(s)|$ is called the curvature and $\mathbf{n}(s) = \alpha''(s)/|\alpha''(s)|$ is called the principal normal of the curve at $\alpha(s)$ at s. The plane spanned by the tangent $\mathbf{t}(s) = \alpha'(s)$ and $\mathbf{n}(s)$ is called the osculating plane. We have

$$\mathbf{t}'(s) = \kappa\, \mathbf{n}(s).$$

Recall that $\mathbf{t}(s) = \alpha'(s)$ and $\mathbf{n}(s)$ are perpendicular. In fact for any any vector field along a curve $\mathbf{v}(s)$ satisfying $|\mathbf{v}(s)| = 1$ we have

$$\mathbf{v}'(s) \cdot \mathbf{v}(s) = 0$$

which follows from applying Leibnitz rule to $\mathbf{v}(s) \cdot \mathbf{v}(s) = 1$.

However, unless the curve is plane curve it can move out of the osculating plane and to describe the motion in that direction we introduce the binormal vector to be

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$$

which again is a vector of length $|\mathbf{b}(s)| = 1$, since its a vector product of two perpendicular vectors with length one. It follows that $\mathbf{b}'(s)$ is perpendicular to $\mathbf{b}(s)$. Moreover, its derivative satisfies

$$\mathbf{b}'(s) = \mathbf{t}'(s) \wedge \mathbf{n}(s) + \mathbf{t}(s) \wedge \mathbf{n}'(s) = \mathbf{t}(s) \wedge \mathbf{n}'(s)$$

and hence its also perpendicular to $\mathbf{t}(s)$. We conclude that $\mathbf{b}'(s)$ has to be parallel to $\mathbf{n}(s)$ and we can write

$$\mathbf{b}'(s) = \tau(s)\mathbf{n}(s),$$

for some scalar $\tau(s)$ called the torsion. Finally $\mathbf{n}(s) = \mathbf{b}(s) \wedge \mathbf{t}(s)$, so

$$\mathbf{n}'(s) = \mathbf{b}'(s) \wedge \mathbf{t}(s) + \mathbf{b}(s) \wedge \mathbf{t}'(s) = \tau(s)\mathbf{n}(s) \wedge \mathbf{t}(s) + \kappa(s)\mathbf{b}(s) \wedge \mathbf{n}(s) = -\tau(s)\mathbf{b}(s) - k(s)\mathbf{t}(s)$$

$$\begin{aligned} \mathbf{t}'(s) &= \qquad\quad \kappa(s)\,\mathbf{n}(s) \\ \mathbf{n}'(s) &= -\kappa(s)\,\mathbf{t}(s) \qquad\qquad - \tau(s)\,\mathbf{b}(s) \\ \mathbf{b}(s) &= \qquad\quad \tau(s)\,\mathbf{n}(s) \end{aligned}$$

Given $\kappa(s)$ and $\tau(s)$ this system of differential equations has a unique solution for given initial conditions. The interpretation of this is as follows: the curvature $\kappa(s)$ measures how much the curve curves in its osculating plane and the torsion $\tau(s)$ measures how much the osculating plane turns as we go along the curve. These two quantities completely characterize the curve. In particular if it is a curve in the plane then the torsion is zero and the curve is completely determined by its curvature at each point and initial tangent.