Lecture 1: 1.1 Introduction and overview.

Math 439 Differential Geometry and 441 Calculus on Manifolds can be seen as continuations of Vector Calculus. In 439 we will learn about the Differential Geometry of Curves and Surfaces in space. The word geometry, comes from Greek Geo=earth and metria=measure. Geometry is the part of mathematics concerned with questions of size, shape and position of objects in space. Differential geometry uses the methods of differential and integral calculus to study the geometry. In 441 we will study Vector Calculus on Manifolds (which locally are hyper surfaces) and how it applies to physics. Let us now continue with briefly describing the content of 439 and how it related to Vector Calculus.

First we will see that a curve in space is determined by its initial point and direction and two scalars called the curvature and torsion at each point along the curve, that measures how fast the curve pulls away from the tangent line close to a point. What do you remember about curves from Vector Calculus? Arc length. Curvature. What is the curvature of a plane curve? At each point along a curve we are given the tangent line but we can also fit a circle that is tangent to the curve to second order and that best approximates the curve close to a point. This circle has a certain radius R and the inverse of the radius is called the curvature $\kappa = 1/R$. A plane curve is determined by giving the initial position and tangent line and the curvature everywhere along the curve. A space curve doesn't only go in a plane but also twists around and that is measured by the torsion. Locally it approximately lies in the plane containing the circle that best approximates it but that plan twists around.

Then we properly define what is meant by a regular surface, to be something that close to each point looks like a deformed plane. What is your picture of a surface from Vector Calculus? Something given as a level surface, e.g $z^2 = x^2 + y^2 + a^2$. What if $a = 0$ is this set a surface when $z = x = y = 0$? We call something a regular surface if things like this doesn't happen. I more useful description is something given as graph or as parameterized surface, e.g. the sphere $x^2 + y^2 + z^2 = a^2$ can be parameterized by spherical coordinates. It can also locally be written as graphs of the coordinate planes $z = \pm\sqrt{a^2 - x^2 - y^2}$. Note however, that you need several parameterizations to cover the whole surface. We show that one can do calculus for functions defined on surfaces.

We also introduce the first fundamental form which is used to measure lengths and areas on the surface itself. This is done as follows. We can measure lengths and areas in space and this gives us a measure of lengths and areas on curves and surfaces by measuring the lengths of tangent vectors.

Next we define the the second fundamental form and the principal curvatures of surface, i.e. the maximum and minimum curvatures of the curves of intersection of the surface with planes through the normal. The second fundamental form is the quadric form that best approximates a surface close to a point if the coordinates are chosen so that the linear part vanishes. These measure how fast the surface pulls away from the tangent plane in a neighborhood of a point on it.

We will show that a surface is determined by its first and second fundamental forms.

We will also prove Gauss Theorema Egregium, which states that the Gaussian curvature, i.e. the product of the principal curvatures, is invariant under isometries, i.e. maps that bend the surface without stretching it.

We will also study geodesics, which are the closest curves between points on a surface.

Gauss and Riemann asked how much of the geometry of a surface is independent of how it bends in space and can be described from creatures that live on the surface and can measure lengths on the surface but are unaware of space outside. This later lead to Einstein general theory of Relativity.

1.2 Parameterized Curves.

We say that a real valued function is (infinitely) differentiable or smooth if it has, at all points, derivatives of all orders (which are automatically continuous).

Def A *parameterized differentiable curve* is a map $\alpha : I \to \mathbf{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbf{R}$ to three dimensional space $\mathbf{R}^3$.

Let $\alpha(t) = (x(t), y(t), z(t))$ denote the point in space corresponding to parameter value $t \in I$.

That α is differentiable means that each of the functions $x(t), y(t), z(t)$ are differentiable.

$\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the *tangent vector* (or the *velocity vector* if we think of the curve as the path of a particle)

The image set $\alpha(I) \subset \mathbf{R}^3$ is called the *trace* of the curve α.

Ex The two parameterized curves in the plane $\alpha(t) = (\cos t, \sin t)$, $t \in (0, 2\pi)$, and $\beta(t) = (\cos 2t, \sin 2t)$, where $t \in (0, \pi)$, have the same trace, namely the unit circle. Note that the velocity vector of the first curve has the double length of the first.

Ex A parameterized curve in space is given by $\alpha(t) = (\cos t, \sin t, t)$. Its trace is a subset of the cylinder $x^2 + y^2 = 1$.

Let us start to review some vector calculus. The length of a vector $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$ is given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

The dot product of two vectors $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{R}^3$ and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{R}^3$ are given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = |\mathbf{u}|\,|\mathbf{v}| \cos\theta,$$

where $0 \le \theta \le \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The following formula for the derivative of a dot product will turn out to be useful:

$$\frac{d}{dt}\big(\mathbf{u}(t) \cdot \mathbf{v}(t)\big) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$

1.3 Arc Length.

Def A parameterized differentiable curve is said to be *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

The arc length of a regular curve from a point $t_0 \in I$ is by definition

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt, \qquad \text{where} \quad |\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

is the length of the tangent vector $\alpha'(t)$. We have $ds(t)/dt = |\alpha'(t)|$. This says that the length along the curve is the integral of its speed along the curve.

It can happen that the parameter t is already the arc length in which case $|\alpha'(t)| = 1$. In fact, if this is not the case one can always make a change of parameter along a regular curve so it is parameterized by arc length. In fact since $s'(t) > 0$ it is invertible with the inverse $t(s)$, which satisfies $t'(s) = 1/s'(t)$. The curve $\beta(s) = \alpha(t(s))$ is the same curve parameterized by the arc length, in fact $|\beta'(s)| = |\alpha'(t(s))| t'(s) = s'(t) t'(s) = 1$.

Ex Find the arc length of the circle $\alpha(t) = (\cos t, \sin t)$ and $t \in (0, 2\pi)$.

The *acceleration vector* of a curve is

$$\alpha''(t) = (x''(t), y''(t), z''(t))$$

Note that the curve α is a constant speed straight line if and only if $\alpha'' = 0$. This follows from just integrating each coordinate function.

What direction does $\alpha''(t)$ point in? If it always point in the direction of $\alpha'(t)$ for and t, then in fact the trace of α is just a line. The interesting thing is that if the arc length is the parameter, then $\alpha''(t)$ is perpendicular to $\alpha'(t)$. This follows by differentiating $|\alpha'(t)|^2 = \alpha'(t) \cdot \alpha'(t) = 1$;

$$0 = \frac{d}{dt}\big(\alpha'(t) \cdot \alpha'(t)\big) = 2\alpha'(t) \cdot \alpha''(t)$$

Note also that the shortest distance between two points is a line. If the points are $\mathbf{p}$ and $\mathbf{q}$ then the distance is $|\mathbf{p} - \mathbf{q}$ and if $\mathbf{u} = (\mathbf{p} - \mathbf{q})/|\mathbf{p} - \mathbf{q}|$ then

$$\int_a^b \alpha'(t) \cdot \mathbf{u}\, dt = ... = |\mathbf{p} - \mathbf{q}|$$

On the other hand

$$\int_a^b \alpha'(t) \cdot \mathbf{u}\, dt \leq ... \leq \int_a^b |\alpha'(t)|\, dt$$