Lecture 9: 3.2 Fundamental Solutions of linear homogeneous equations. Most of what we will do in this chapter concerns linear second order differential equations with constant coefficients. However, the results in this section also holds for variable coefficients. Let us first recall the **existence theorem**:

Th 1 Suppose that p, q and g are continuous on an open interval $I = (a, b)$ containing t_0. Then the differential equation

$$y'' + p\,(t)\,y' + q(t)\,y = g(t) \tag{3.2.1}$$

has a unique solution on I satisfying

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{3.2.2}$$

The proof basically reduces to the same proof as for the first order differential equation by rewriting it as a system. If we put

$$z = y', \qquad z' = y'' = -p\,(t)y' - q(t)y + g(t) = -p\,(t)z - q(t)y + g(t)$$

we get a first order system for (y, z):

$$\begin{array}{lll} y' = z, & & y(0) = y_0 \\ z' = f(t, y, z) & , & z(0) = y_1 \end{array}$$

where

$$f(t, y, z) = -p\,(t)z - q(t)y - g(t),$$

which can be solved by writing it in integral form

$$y(t) = y_0 + \int_{t_0}^{t} z(s)\,ds,$$

$$z(t) = z_0 + \int_{t_0}^{t} f(s, y(s), z(s))\,ds$$

One can then use Piccard iteration, i.e. plugging in a guess for $y(t)$ and $z(t)$ in the right and getting a better guess in the left, the first guess simply being y_0 and z_0.

Let us now introduce the operator notation;

$$L[\phi] = \phi'' + p\phi' + q\phi,$$

i.e. for each function ϕ, $L[\phi]$ is the function who's value at at a point t is given by

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

L is called an operator because it maps functions to functions.

We also proved the **superposition principle** for linear equations in last lecture:

Th 2 If y_1 and y_2 are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then so is

$$c_1y_1 + c_2y_2 \tag{3.2.3}$$

Basically, this section is about finding a condition on y_1 and y_2 that guarantee that all solutions are of the form $y = c_1y_1 + c_2y_2$ for some constants c_1 and c_2. We therefore define the **Wronskian determinant** of two solutions by:

$$W = y_1y_2' - y_1'y_2 \tag{3.2.4}$$

Th 3 If $W(t_0) \neq 0$ then one can find c_1 and c_2 such that (3.2.3) satisfies (3.2.2).
Proof of Th 3 We must solve the following system for the unknowns c_1 and c_2:

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0' \end{aligned} \tag{3.2.5}$$

Each equation represent a line in the (c_1, c_2) plane and the system can be solved as long as the lines intersect and are not parallel. This happens when the vectors $(y_1(t_0), y_2(t_0))$ and $(y_1'(t_0), y_2'(t_0))$ are not parallel, which is the case if (3.2.4) is different from 0 at t_0. However, you can also see that the Wronskian comes in by trying to solve the system. If we multiply the first equation by $y_2'(t_0)$ and the second by $y_2(t_0)$ and subtract we get

$$c_1\left(y_2'y_1 - y_2y_1'\right) = y_2'y_0 - y_2y_0'$$

and in general this can only be solved for c_1 if $W = y_2'y_1 - y_2y_1' \neq 0$. Similarly for c_2
Th 4 If $W(t_0) \neq 0$ for some t_0 then all solutions are of the form $y = c_1y_1 + c_2y_2$.
Proof This follows from Theorem 3 and and the uniqueness in Theorem 1.
Definition y_1 and y_2 are called a **fundamental set of solutions** if all solution can be written as $c_1y_1 + c_2y_2$.
Ex Consider the equation $ay'' + by' + cy = 0$. Let r_1 and r_2 be the roots of the characteristic equation $ar^2 + br + c = 0$. Then $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$ are both solutions. The Wronskian is $W = y_2'y_1 - y_2y_1' = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0$ if and only if $r_1 \neq r_2$. Hence y_1 and y_2 form a fundamental set of solutions if and only if $r_1 \neq r_2$.

It turns out that we can always find two different solutions.
Th 5 Let y_1 and y_2 be two solutions of $L[y] = 0$ satisfying the initial conditions

$$y(t_0) = 1, \quad y_1'(t_0) = 0, \qquad y_2(t_0) = 0, \quad y_2'(t_0) = 1$$

Then $W = y_2'y_1 - y_2y_1' = 1 \neq 0$.

If the roots r_1 and r_2 are complex then e^{r_1t} and e^{r_2t} are in fact complex solutions if we define the expontial function for complex numbers right. We want to investigate what it means to be a **complex solution**. Suppose that p and q are real valued and let

$$y(t) = u(t) + i\,v(t)$$

be a complex function, where u and v are real and $i = \sqrt{-1}$. The derivative of a complex function is given by

$$y'(t) = u'(t) + i\,v'(t)$$

and similar for second order derivatives so

$$L[y] = L[u] + i\,L[v]$$

It follows that if $L[y] = 0$ then $L[u] = 0$ and $L[v] = 0$.