Lecture 7: 2.5 Autonomous equations and population dynamics. A differential equation is called **autonomous** if it has the form

$$\frac{dy}{dt} = f(y), \qquad y(0) = y_0 \tag{2.5.1}$$

The simplest model of variation of the population of a spices is **exponential growth** that the rate of increase is proportional to the current population:

$$\frac{dy}{dt} = ry$$

where the constant of proportionality r is called the **rate of growth**. The solution

$$y = y_0 e^{rt}$$

grows exponentially and this can not continue forever due to limitations in food etc. To take into account that the growth rate depends on the population we modify it:

$$\frac{dy}{dt} = h(y)y$$

We now want to choose $h(y)$ so $h(y) \sim r$ when y is small and so that $h(y) < 0$ when y is very large. The simplest example of this is the so called **logistic equation**:

$$\frac{dy}{dt} = r\Big(1 - \frac{y}{K}\Big)y, \qquad y(0) = y_0 \tag{2.5.2}$$

Note that $y = 0$ and $y = K$ are constant solutions since the right of (2.5.2) vanishes.

More generally for (2.5.1) the zeros of $f(y)=0$, called **critical points** or **equilibrium points** correspond to constant solutions, called **equilibrium solutions**.

In this case the solution can be explicitly calculated as we shall see but we can draw a lot of conclusions just from the general form of the function $f(y)$ in the right of (2.5.2). If we draw a plot of this function we see that it is strictly positive between the two zeros 0 and K and strictly negative outside this interval, in particular when $y > K$. From this we first draw the conclusion:

If $0 < y < K$ then $dy/dt > 0$ so y is an increasing function of t in this interval, and if $y > K$ then $dy/dt < 0$ so y is a decreasing function of t in this interval.

An important question arises: Can the increasing integral curves coming from $y < K$ enter the region with $y > K$? The answer is no, because they would then immediately become decreasing which would be a contradiction since one can prove that they are continuous. It still remains to rule out that the curves from the region where $y < K$ can not reach $y = K$ in finite time. The reason this can not happen is that in that case there would be two different solution curves going through that point since also the constant solution goes through that point and that would contradict the uniqueness theorem in the previous section. We therefore conclude:

Solution curves starting with $0 < y_0 < K$ will stay in $0 < y(t) < K$ for all positive t and curve starting in $y_0 > K$ will stay in $y(t) > K$ for all positive t.

Another question arises: Will the solutions starting in the interval $0 < y_0 < K$ approach K as $t \to \infty$? The answer is yes. Again we can argue by contradiction.

$y(t)$ is increasing in this interval so it is increasing towards some value $y_1 \leq K$. $y_1 < K$ then the derivative dy/dt would be strictly positive close to y_1 so the solution would be above some line with slope $f(y_1)$ that will intersect the $y = K$ at some finite time leading to a contradiction. We hence conclude:

The solution curves starting in $0 < y_0 < K$ will approach $y = K$ as $t \to \infty$, and the solution curves starting in $y_0 > K$ will also approach $y = K$.

More generally, an equilibrium solution of (2.5.1) is called **stable** (asymptotically stable) if every solutions to (2.5.1) starting close to it approach it as $t \to \infty$. Otherwise it is **unstable**. In this terminology the equilibrium $y = K$ is stable but the equilibrium $y = 0$ is unstable.

We can actually conclude just from the form of f close to an equilibrium z if the equilibrium is stable or unstable: If $f(z) = 0$ and $f'(z) > 0$ then $f(y) > 0$ for $y > z$ close to z and a solution starting at y increases so it can not approach z so its unstable. Similarly if $f'(z) < 0$, then $f(y) < 0$ for $y > z$ close to z and a solution starting at y decreases towards z, and if $y < z$ it increases, so it is stable.

One can also solve the logistic equation (2.5.3) explicitly by separation of variables

$$\frac{dy}{(1 - y/K)y} = r\,dt$$

and partial fractions:

$$\Big(\frac{1}{y} + \frac{1/K}{1 - y/K}\Big)dy = r\,dt$$

and

$$\ln|\,y| - \ln\Big|1 - \frac{y}{K}\Big| = rt + c$$

Hence

$$\frac{y}{1 - (y/K)} = Ce^{rt}$$

and

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

We notice that independently of what $y_0 > 0$ is we have

$$\lim_{t \to \infty} y = K,$$

Let us now look on a slightly different equation

$$\frac{dy}{dt} = -r\Big(1 - \frac{y}{T}\Big)y, \qquad y(0) = y_0 \tag{2.5.3}$$

We find the solution by replacing K by T and r by $-r$ in the above formula:

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

Since the sign of the right hand side changed the equilibrium $y = T$ is now unstable. When $0 < y_0 < T$ the solution converges to 0 as $t \to 0$ and when $y_0 > T$ the solution actually goes to infinity for a finite t^* give by that the denominator vanishes:

$$y_0 + (T - y_0)e^{rt^*} = 0$$

i.e.

$$t^* = \frac{1}{r}\ln\frac{y_0}{y_0 - T}.$$

As we have see it can happen for nonlinear equations that the solution blows up for finite time.