Lecture 5: 2.3 More models.

Model III: Mixing of chemicals. Suppose a time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min, and that the well-stirred mixture is draining from the tank at the same rate. Find the amount of salt $Q(t)$ in the tank at any time t and the limiting amount Q_L as $t \to \infty$.

The rate of change of the amount of salt is equal to the rate of salt in minus the rate of salt out. The rate of salt in is $\frac{1}{4}$ lb/gal times the rate of water in r gal/min and the rate of salt out is the concentration of salt in the tank, $Q(t)/100$ lb/gal times the rate of water out r gal/min:

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{r\,Q}{100} \tag{2.3.1}$$

We want to solve this differential equation with initial condition $Q(0) = Q_0$. Multiplying both sides of

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

by the integrating factor

$$e^{rt/100}\Big(\frac{dQ}{dt} + \frac{r}{100}\Big) = e^{rt/100}\frac{r}{4}$$

makes the left the derivative of a product:

$$\frac{d}{dt}\Big(Qe^{rt/100}\Big) = e^{rt/100}\frac{r}{4} \tag{2.3.2}$$

If we take the antiderivative we obtain

$$Q(t)e^{rt/100} = 25e^{rt/100} + C$$

i.e.

$$Q(t) = 25 + Ce^{-rt/100}$$

And solving for the initial condition $Q_0 = Q(0) = 25 + C$ so $C = Q_0 - 25$ and hence

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100} \tag{2.3.3}$$

It in particular follows from this that

$$\lim_{t\to\infty} Q(t) = 25$$

no matter what Q_0 is. This is however not so surprising since a limiting concentration in the container of 25 lb/100 gal, i.e. $\frac{1}{4}$ lb/gal, is exactly the concentration of salt in the incoming water. Our physical intuition tells us that eventually the concentration in the container ought to approach that of the incoming flow.

Model IV: Compound Interest. Suppose that a sum of money is deposited in a bank account that pays interest at annual rate r. The value $S(t)$ of the investment at time t depends on how often the interest is compounded as well as the rate. If the interest is compounded once a year then then the change is

$$S(k+1) - S(k) = rS(k)$$

or

$$S(k+1) = S(k)(1+r) = S(k-1)(1+r)^2$$

and if we repeat it t times we get

$$S(t) = (1+r)^t S(0) \tag{2.3.5}$$

On the other hand if the interest is assumed to be compounded continuously then we get the differential equation:

$$\frac{d}{dt}S(t) = rS(t) \tag{2.3.6}$$

If we instead assume (2.3.6) we get

$$S(t) = S(0)e^{rt} = S(0)\big(e^r\big)^t \tag{2.3.7}$$

As it turns out when r is small and t is bounded then by using Taylor series

$$e^r \sim 1 + r$$

which is why (2.3.7) is a good approximation to (2.3.6).

Ex Ms Doe retired yesterday. Her IRA account has a principal of $\$450,000$ which gives an annual interested rate of 5.25% compounded continuously. Her budget calls for annual expenses of $\$25,000$, with projected inflation of 2.5%. The differential equation giving her savings is:

$$\frac{dy}{dt} = 0.0525y - 20000e^{0.025t}, \qquad y(0) = 450000$$

which has the solution

$$y = 730000e^{0.025t} - 280000e^{0.0525t}$$

Model V: Cooling. A hot object with temperature T is left to cool down in a room with temperature A. Newton's law of cooling states that the change in temperature is negatively proportional to the temperature difference between the object and the surroundings:

$$\frac{dT}{dt} = -k(T - A)$$

2.6 More Exact Equations.
Ex Solve the differential equation

$$\frac{x}{(x^2+y^2)^{3/2}} + \frac{y}{(x^2+y^2)^{3/2}}\frac{dy}{dx} = 0$$

If we can find $\psi(x,y)$ such that

$$\psi_x = \frac{x}{(x^2+y^2)^{3/2}}, \quad \text{and} \quad \psi_y = \frac{y}{(x^2+y^2)^{3/2}} \tag{2.6.3}$$

then this equation can be written

$$\frac{d}{dx}\psi\big(x,y(x)\big) = \psi_x(x,y(x)) + \psi_y(x,y(x))\frac{dy}{dx} = 0$$

which has the solution

$$\psi(x,y) = 0$$

from which we can solve for y as a function of x.

It is possible to find ψ satisfying (2.6.3) because

$$\frac{d}{dy}\frac{x}{(x^2+y^2)^{3/2}} = \frac{d}{dx}\frac{y}{(x^2+y^2)^{3/2}}$$

First we solve the first equation in (2.6.3)

$$\frac{\partial}{\partial x}\psi = \frac{x}{(x^2+y^2)^{3/2}}$$

which has the solution

$$\psi = -(x^2+y^2)^{1/2} + h(y)$$

for some function $h(y)$ which has to satisfy the second equation of (2.6.3)

$$\frac{\partial}{\partial y}\psi = \frac{\partial}{\partial y}\Big(-(x^2+y^2)^{1/2} + h(y)\Big)\frac{y}{(x^2+y^2)^{3/2}} + h'(y) = \frac{y}{(x^2+y^2)^{3/2}}$$

if $h(y) = 0$. Hence the solution is given by

$$\psi(x,y) = -(x^2+y^2)^{1/2} = C$$

from which we can solve for y as a function of x.

Exact Equations with integrating factor. Even if (2.6.1) in not exact it may become exact if we multiply by an integrating factor

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0 \tag{2.6.2}$$

This is exact if we can find a μ such that

$$M\mu_y - N\mu_x + (M_y - N_x) = 0,$$

one can e.g. solve this in special cases with μ independent of y if $(M_y - N_x)/N$ is independent of y or with μ independent of x if $(M_y - N_x)/M$ is independent of x.

See example in the book.