Lecture 4: 2.2 Separable equations. We can not solve a general first order equation:

$$\frac{dy}{dx} = f(x,y) \tag{2.2.1}$$

but there is another special case that we can deal with called separable equations. First we note that there are many ways to write (2.2.1) in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

If we can find a way such that M only depends on x and N only depends on y, i.e.

$$M(x) + N(y)\frac{dy}{dx} = 0$$

then the equation is called **separable**. In that case we formally multiply with dx

$$M(x)\,dx + N(y)\,dy = 0$$

and take the anti derivative:

$$\int M(x)\,dx + \int N(y)\,dy = 0$$

The linear equation with constant coefficients (2.1.1) was in fact separable and to explain the method let us do a couple of more examples:

$$\frac{dy}{dx} = -\frac{x}{y}$$

We separate the variables i.e. multiply both sides by $y\,dx$

$$ydy = -xdx$$

and integrating this gives

$$y^2/2 = -x^2 + C$$

or

$$x^2 + y^2 = K$$

for some constant K. The solution curves are hence circles.

$$\frac{dy}{dx} = \frac{y}{x}$$

Multiplying both sides by dx/y gives

$$\frac{dy}{y} = \frac{dx}{x}$$

and integration gives

$$\ln|\,y| = \ln|\,x| + C$$

and exponentiating both sides gives

$$|\,y| = e^C|\,x|$$

i.e.

$$y = Kx$$

for some constant K. The solution curves are lines through the origin.

A difficult problem is to in general find the solution curves for

$$\frac{dy}{dx} = \frac{cx+dy}{ax+by},$$

where a, b, c, d are constants.

Model I modified Falling body. As it turns out, a more realistic model of the air resistance for a falling body is that instead of $-\gamma v$ the force is $-kv^2$:

$$m\frac{dv}{dt} = mg - kv^2$$

As for the simpler linear model discussed earlier the velocity v_∞ when the right hand side vanishes $mg - kv_\infty^2 = 0$ corresponds to a stable equilibrium. In fact for some realistic values of the parameters the ode becomes

$$\frac{dv}{dt} = 9.8 - 9.8 \cdot 10^{-4} v^2, \qquad v(0) = 0$$

which can be solved using separation of variables:

$$\frac{dv}{100^2 - v^2} = 9.8 \cdot 10^{-4} dt$$

and partial fractions

$$\frac{1}{200}\Big(\frac{dv}{100 - v} + \frac{dv}{100 + v}\Big) = 9.8 \cdot 10^{-4} dt$$

If we integrate this we get

$$\frac{1}{200}\Big(\ln|100 + v| - \ln|100 - v|\Big) = 9.8 \cdot 10^{-4}\, t + C$$

where $C = 0$ since all the other terms vanish when we put in $t = 0$ and use that $v(0) = 0$. Hence

$$\ln\left|\frac{100 + v}{100 - v}\right| = 0.196\, t$$

$$v = \frac{100 e^{0.196t} - 1}{e^{0.196t} + 1}$$

Hence

$$\lim_{t\to\infty} v(t) = 100 = v_\infty$$

2.6 Exact Equations. The equation

$$2x + y^2 + 2xyy' = 0$$

is neither linear nor separable but still it can be reduced to the form

$$\frac{d}{dx}\psi(x, y(x)) = 0$$

if ψ is chosen correctly. In that case by the chain rule we have

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

Hence if we can find ψ so that

$$\frac{\partial \psi}{\partial x} = 2x + y^2, \qquad \frac{\partial \psi}{\partial y} = 2xy$$

we have

$$\frac{d}{dx}\psi(x, y(x)) = 2x + y^2 + 2xy\frac{dy}{dx} = 0$$

which has the solutions

$$\psi(x, y) = c$$

for some constant c. One can check that

$$\psi(x, y) = x^2 + xy^2.$$

works. But how did we find it? First we solve

$$\frac{\partial \psi}{\partial x} = 2x + y^2$$

by integrating with respect to x when y is thought of as constant which gives

$$\psi(x, y) = x^2 + y^2 x + f(y)$$

where $f(y)$ is an arbitrary function of y, since the derivative of $f(y)$ with respect to x vanishes. Next we plug this into the second equation to get

$$\frac{\partial}{\partial y}\big(x^2 + xy^2 + f(y)\big) = y^2 + f'(y) = y^2$$

if we choose $f(y) = 0$.

Consider a general first order equation

$$M(x, y) + N(x, y)y' = 0$$

If we can find $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M, \qquad \frac{\partial \psi}{\partial y} = N$$

then

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

which has the solutions

$$\psi(x, y) = c$$

for some constant c.

Theorem Suppose that

$$M_y(x, y) = N_x(x, y)$$

for all (x, y). Then there is a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = M, \qquad \frac{\partial \psi}{\partial y} = N.$$

Its easy to see that this is a necessary condition, In fact if this is the case we must have that

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$$

which are equal.