Lecture 27: 7.9 Nonhomogeneous equations. There are several methods in the book but we will only go over using diagonalization and the exponential matrix.

Diagonalization Suppose that A is an $n\times n$ matrix with n linearly independent eigenvectors $A\boldsymbol{\xi}^{(k)} = \lambda_k \boldsymbol{\xi}^{(k)}$. Let $T = [\,\boldsymbol{\xi}^{(1)} \cdots \boldsymbol{\xi}^{(n)}\,]$ be the $n \times n$ matrix whose kth column is $\boldsymbol{\xi}^{(k)}$, for $k = 1, \dots n$. Then $AT = [\,A\boldsymbol{\xi}^{(1)} \cdots A\boldsymbol{\xi}^{(n)}\,] = [\,\lambda_1\boldsymbol{\xi}^{(1)} \cdots \lambda_n\boldsymbol{\xi}^{(n)}\,]$ is the matrix who's kth column vector is given by $\lambda_k\boldsymbol{\xi}^{(k)}$. Moreover if D is the $n \times n$ diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$ then we also have $TD = [\,\lambda_1\boldsymbol{\xi}^{(1)} \cdots \lambda_n\boldsymbol{\xi}^{(n)}\,]$. Hence $AT = TD$ so we conclude

$$T^{-1}AT = D.$$

Note that the differential equation $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$, where A is the matrix above can be solved by diagonalizing the whole system as follows: Let $\mathbf{y} = T^{-1}\mathbf{x}$. Then

$$\mathbf{y}' = T^{-1}\mathbf{x}' = T^{-1}\big(A\mathbf{x} + \mathbf{g}\big) = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g} = D\mathbf{y} + \mathbf{h}, \quad \text{where } \mathbf{h} = T^{-1}\mathbf{g}$$

i.e.

$$y_k' = \lambda_k y_k + h_k, \qquad k = 1, \dots, .$$

If we multiply by the integrating factor $e^{-\lambda_k t}$ and integrate we get

$$y_k(t) = y_k(0)e^{\lambda_k t} + e^{\lambda_k t}\int_0^t e^{-\lambda_k s}\, h_k(s)\, ds,$$

We then transform it back to $\mathbf{x} = T\mathbf{y}$.

Exponential matrix and integrating factor.

$$\frac{d}{dt}\big(e^{-At}\mathbf{x}(t)\big) = -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}' = -Ae^{-At}\mathbf{x}(t) + e^{-At}(A\mathbf{x} + \mathbf{g}) = e^{-At}\mathbf{g}(\mathbf{t})$$

and integrating both sides gives

$$e^{-At}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-As}\mathbf{g}(\mathbf{s})\, ds$$

and hence

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + e^{At}\int_0^t e^{-As}\mathbf{g}(s)\, ds = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-s)}\mathbf{g}(s)\, ds$$

where we used that

$$e^{At}e^{-At} = I = e^{-At}e^{At}.$$

This is just the fact that if we first solve the differential equation $\mathbf{x}' = A\mathbf{x}$ time t forward and then solve it backwards the same time we are back to where we started. It can also be prove by calculating that the derivative of the product above vanishes. We also used that

$$e^{At}e^{-As} = e^{A(t-s)}$$

Ex 1 Find a particular solution to the system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$, where $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ and $\mathbf{g} = 2\begin{bmatrix} e^{-t} \\ t \end{bmatrix}$.

Sol 1 By Ex 7.3.2 the eigenvalues and vectors are $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Let $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $T^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, $T^{-1}AT = D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{h} = T^{-1}\mathbf{g} = \begin{bmatrix} e^{-t} + t \\ e^{-t} - t \end{bmatrix}$. We change variables $\mathbf{x} = T\mathbf{y}$ to obtain $\mathbf{y}' = D\mathbf{y} + \mathbf{h}$ or

$$y_1' = -y_1 + e^{-t} + t, \qquad y_2' = 3y_2 + e^{-t} - t$$

Moving y_1 respectively y_2 to the left and multiplying by the integrating factors gives

$$(y_1 e^t)' = e^t(e^{-t} + t), \qquad (y_2 e^{-3t})' = e^{-3t}(e^{-t} - t)$$

Integration gives

$$y_1(t) = e^{-t}\int^t (1 + se^s)\, ds = e^{-t}(t + te^t - e^t)$$

$$y_2(t) = e^{3t}\int^t (e^{-4s} + se^{-3s})\, ds = e^{3t}(-\frac{e^{-4t}}{4} - \frac{te^{-3t}}{3} - \frac{e^{-3t}}{9})$$

We then have to transform back to $\mathbf{x} = T\mathbf{y}$:

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} te^{-t} + t - 1 \\ -e^{-t} - t/3 - 1/9 \end{bmatrix} = \begin{bmatrix} (t-1)e^{-t} + 2t/3 - 10/9 \\ (t+1)e^{-t} + 4t/3 - 8/9 \end{bmatrix}.$$

Sol. 2 Using the exponential matrix. We have

$$e^{At} = Te^{Dt}T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t} + e^{3t} & e^{-t} - e^{3t} \\ e^{-t} - e^{3t} & e^{-t} + e^{3t} \end{bmatrix}$$

A particular solution is

$$\mathbf{x}(t) = e^{At}\int^t e^{-As}\mathbf{g}(s)\, ds$$

Here

$$e^{-As}\mathbf{g}(s) = \begin{bmatrix} e^s + e^{-3s} & e^s - e^{-3s} \\ e^s - e^{-3s} & e^s + e^{-3s} \end{bmatrix}\begin{bmatrix} e^{-s} \\ s \end{bmatrix} = \begin{bmatrix} 1 + e^{-4s} + s(e^s - e^{-3s}) \\ 1 - e^{-4s} + s(e^s + e^{-3s}) \end{bmatrix}$$

so

$$\int^t e^{-As}\mathbf{g}(s)\, ds = \begin{bmatrix} \int^t 1 + e^{-4s} + s(e^s - e^{-3s})\, ds \\ \int^t 1 - e^{-4s} + s(e^s + e^{-3s})\, ds \end{bmatrix} = \begin{bmatrix} t - e^{-4t}/4 + te^t - e^t + te^{-3t}/3 - e^{-3t}/9) \\ t + e^{-4t}/4 + te^t - e^t - te^{-3t}/3 + e^{-3t}/9) \end{bmatrix}$$

and

$$\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix} e^{-t} + e^{3t} & e^{-t} - e^{3t} \\ e^{-t} - e^{3t} & e^{-t} + e^{3t} \end{bmatrix}\begin{bmatrix} t - e^{-4t}/4 + te^t - e^t + te^{-3t}/3 - e^{-3t}/9) \\ t + e^{-4t}/4 + te^t - e^t - te^{-3t}/3 + e^{-3t}/9) \end{bmatrix} = \ldots$$

Ok, this is this way is too lengthy so the other way above is better.

The case when we don't have a basis of eigenvectors.