Lecture 20: 7.1 Systems of first order differential equations. A second order equation can always be written as a first order system:
Ex The equation of a spring $y'' + ky = 0$ can be written with $x_1 = y$ and $x_2 = y'$:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -kx_1 \end{aligned}$$

First order systems also show up naturally not coming from a higher order equation:
Ex Consider two interconnected tanks that contain water with a certain amount of salt Q_1 respectively Q_2 oz of salt. Suppose tank one contain 60 gal of water and tank two 100 gal. Suppose the water containing q_1 oz of salt per gal flows in to tank one at a rate of 3 gallons per min and q_2 oz of salt per gal flows in to tank two at a rate of 1 gallons per min. Suppose also that 4 gal per min flows out of tank one half of which flows in to tank two while the remainder leaves the system and 3 gal per min flows out of tank two, of which 1 gallon flows into tank one, and the rest leaves the system. The system of equations describing this is

$$\begin{aligned} Q_1' &= 3q_1 + Q_2/100 - 4Q_1/60 \\ Q_2' &= q_2 + 2Q_1/60 - 3Q_2/100 \end{aligned}$$

One could attempt to rewrite this as a second order equation for one unknown only $Q = aQ_1 + b\,Q_2$, but instead we will learn methods to directly solve systems.

A general first order 2×2 system of differential equations can be written

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2) \qquad & x_1'(t_0) &= x_1^0 \\ x_2' &= F_2(t, x_1, x_2)' \qquad & x_2'(t_0) &= x_2^0 \end{aligned}$$

or if we introduce vector notation

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \qquad \mathbf{x}_0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

we can write this in a more concise form:

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

The same methods uses for one equations works to show that we have existence for the 2×2 system. Using Euler's method:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{F}(t_n, \mathbf{x}_n)(t_{n+1} - t_n), \qquad t_n = t_0 + nh, \qquad n \geq 0,$$

gives and approximation for $\mathbf{x}(t_n) \approx \mathbf{x}_n$. Alternatively, as before we can also prove existence with successive approximation

$$\mathbf{x}_0(t) = \mathbf{x}_0, \qquad \mathbf{x}_{n+1}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{x}_n(s))\, ds, \quad n \geq 0.$$

7.2 (2x2) Linear systems with constant coefficients. We first consider a homogeneous 2×2 **constant coefficient linear system of differential equations**:

$$x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \tag{7.1.1}$$

Let us first consider a 2×2 **linear system of algebraic equations**

$$a_{11}x_1 + a_{12}x_2 = y_1 \\ a_{21}x_1 + a_{22}x_2 = y_2 \tag{7.1.2}$$

We will write this system in matrix form. Let A be the 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

i.e. a collection of 2×2 entries $A = (a_{ij})$, $i, j = 1, 2$, and let $\mathbf{x}$ be the 2 vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

We define the product of the 2×2 matrix A by the 2 vector $\mathbf{x}$ to be the 2 vector

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{21}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 \tag{7.1.3}$$

i.e. the vector whose first component is the dot product $(a_{11}, a_{12}) \cdot (x_1, x_2) = a_{11}x_1 + a_{12}x_2$ of the first row of A and $\mathbf{x}$ and whose second component is the dot product $(a_{21}, a_{22}) \cdot (x_1, x_2) = a_{11}x_1 + a_{12}x_2$ of the second row of A and $\mathbf{x}$. As indicated above; another way to see this matrix product is as a linear combination of the column vectors: $A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2]\,\mathbf{x} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2$. If

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

the algebraic system (7.1.2) can then be written

$$A\mathbf{x} = \mathbf{y}$$

and the system of differential equations (7.1.1) can be written

$$\mathbf{x}' = A\mathbf{x}$$

Any 2×2 matrix A determines a **linear map**

$$\mathbf{R}^2 \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^2$$

Conversely, every linear map is given by matrix multiplication. If B is another 2×2 matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then multiplication first by B and then A

$$\mathbf{x} \xrightarrow{\text{multiply by } B} B\mathbf{x} \xrightarrow{\text{multiply by } A} A(B\mathbf{x})$$

defines a linear map $\mathbf{R}^2 \ni \mathbf{x} \to A(B\mathbf{x}) \in \mathbf{R}^2$. This linear map corresponds to multiplying by some matrix. The matrix product AB is constructed so that multiplying by the matrix AB

$$\mathbf{x} \xrightarrow{\text{multiply by } AB} (AB)\mathbf{x}$$

is the same as first multiplying by B and then by A, i.e. $(AB)\mathbf{x} = A(B\mathbf{x})$.

If A and B are 2×2 matrices then the product AB is the 2×2 matrix

$$(7.1.4) \qquad AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

i.e. the entry in the ith row and j th column of AB is the dot product between the ith row of A and the jth column of B: $a_{i1}b_{1j} + a_{i2}b_{2j}$. One can see the matrix product in terms of the column picture, if $B = [\mathbf{b}_1\ \mathbf{b}_2]$ then $AB = [A\mathbf{b}_1\ A\mathbf{b}_2]$, i.e. if $\mathbf{b}_1$, $\mathbf{b}_1$ are the column vectors of B then $A\mathbf{b}_1$, $A\mathbf{b}_1$ are the column vectors of AB.

Ex 1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ **rotates** vectors an angle $\pi/2$ counterclockwise.

Ex 2 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ **scales** vectors by a factor 3.

Ex 3 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ 3x_1 \end{bmatrix}$ scales and rotates.

These maps are all invertible. However a projection is not:

Ex 4 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

Usually $AB \neq BA$. The 2×2 identity matrix I is given by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying with it is similar to multiplying by 1:

$$I\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}$$

An 2×2 matrix is called **invertible** if there is an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

It turns out that the algebraic system (7.1.2) can be solved only if the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is nonvanishing. In that case A has an inverse given by

$$(7.1.5) \qquad A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} \dfrac{a_{22}}{\det A} & -\dfrac{a_{12}}{\det A} \\ -\dfrac{a_{21}}{\det A} & \dfrac{a_{11}}{\det A} \end{bmatrix}$$

In fact by (7.1.4)

$$A^{-1}A = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & 0 \\ 0 & -a_{21}a_{12} + a_{11}a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Furthermore, if $\det A \neq 0$ then the linear algebraic system has a unique solution:

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \quad \Leftrightarrow \quad \mathbf{x} = I\mathbf{x} = A^{-1}\mathbf{b}$$

If $\det A \neq 0$ then the homogeneous problem $\mathbf{b} = \mathbf{0}$, only has the trivial solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$: But if $\det A = 0$, then the homogeneous problem has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ and the inhomogeneous problem, $\mathbf{b} \neq \mathbf{0}$, might not have any solution.

Ex 1 Solve the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_2 + 4x_2 &= 6 \end{aligned}$$

Sol The system can be written in matrix form

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

By (7.1.5)

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 4 - 3 \cdot 2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 2 & -1/2 \end{bmatrix}$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Ex 2 Solve the system

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 4x_2 &= 6 \end{aligned}$$

Sol Since the determinant of the system vanishes

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0$$

the inverse does not exist. Subtracting twice the first equation from the second gives the equivalent system:

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 0 &= -4 \end{aligned}$$

The second equation can not hold so the system has no solutions.

Ex 3 Solve the system

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 4x_2 &= 0 \end{aligned}$$

Sol Subtracting twice the first equation from the second gives the equivalent system:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

The second equation always hold and the first equations has a whole line of solutions, if we put x_2 equal to a parameter α we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \text{for any} \quad \alpha$$