Lecture 18: 6.5-6 Impulse Functions and Convolutions. We want to solve

$$ay'' + by' + cy = g(t)$$

where $g(t) = 0$, when $|\,t - t_0| \geq \tau$ but $g(t)$ is very large when $|\,t - t_0| < \tau$, for some very small number $\tau > 0$. In case of a weight hanging on a spring we can think of $g(t)$ as a large push during a small time period. We may not know exactly how $g(t)$ looks but just know the total impulse

$$I = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

We may for example take $g(t)$ equal to

$$d_\tau(t) = \begin{cases} 0, & |\,t| \leq \tau, \\ 1/(2\tau), & |\,t| > \tau. \end{cases}$$

Then the total impulse is then

$$\int_{-\infty}^{+\infty} d_\tau(t)\,dt = \frac{1}{2\tau}\int_{-\tau}^{+\tau} d\tau = 1$$

The impulse or delta function is defined to be the limit

$$\delta(t) = \lim_{\tau\to 0} d_\tau(t)$$

We have $d_\tau(t) \to 0$, when $t \neq 0$ and $d_\tau(0) \to \infty$, as $\tau \to 0$. We would like

$$\delta(t) = 0, \quad \text{when } t \neq 0, \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

However even if we define $\delta(0) = \infty$ its not clear what the integral should mean since $\delta(t)$ is ∞ on an interval of length 0 and the area $0 \cdot \infty$ is not well defined. The delta function only makes sense below an integral sign and then the integral has to be interpreted as the limit, as $\tau \to 0$, of the integrals when $\delta(t)$ is replaced by $\delta_\tau(t)$. The delta function integrated with a continuous function $\phi(t)$ is defined to be

$$\int_{-\infty}^{+\infty} \delta(t)\phi(t)\,dt = \lim_{\tau\to 0}\int_{-\infty}^{+\infty} d_\tau(t)\phi(t) \;= \lim_{\tau\to 0}\frac{1}{2\tau}\int_{-\tau}^{+\tau}\phi(t)\,dt = \phi(0).$$

In fact, by the mean value theorem $\int_{-\tau}^{+\tau}\phi(t)\,dt = 2\tau\phi(\xi_\tau)$, for some point ξ_τ such that $-\tau \leq \xi_\tau \leq \tau$, i.e. there is point such that the area below the graph of the function is the value of the function at that point times the length of the interval.

Even though the limit of $d_\tau(t)$ as $\tau \to 0$ does not make sense as an integrable function, the limit of the solution to the the equation

$$ay_\tau'' + by_\tau' + cy_\tau = d_\tau(t - t_0), \qquad t_0 > 0$$

will exist pointwise. Moreover the Laplace transform will have a limit as $\tau \to 0$:

$$\mathcal{L}\{\delta_\tau(t - t_0)\} = \int_0^\infty \delta_\tau(t - t_0)e^{-st}\,dt = \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt = \frac{-1}{2s\tau}e^{-st}\Big|_{t=t_0-\tau}^{t_0+\tau}$$
$$= e^{-t_0 s}\frac{e^{s\tau} - e^{-s\tau}}{2s\tau} \to e^{-t_0 s},$$

since by l'Hopital's rule $(e^x - e^{-x})/(2x) \to 1$.

Ex Solve the differential equation

$$y'' + y = \delta(t-2), \qquad y(0) = y'(0) = 0$$

Taking the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$, we get

$$(s^2+1)Y(s) = e^{-2s}$$

and hence

$$Y(s) = \frac{1}{s^2+1}e^{-2s}$$

Recall that $\mathcal{L}\{\sin t\} = 1/(s^2+1)$ and for any function $f(t) = \mathcal{L}^{-1}\{F(s)\}$ $\mathcal{L}^{-1}\{F(s)e^{-cs}\} = u_c(t)f(t-c)$, where $u_c(t)$ is the step function. Hence

$$y(t) = u_2(t)\sin(t-2)$$

Note that $y(t)$ for $t < 2$ and its a solution to the homogeneous equation for $t > 2$.

6.6 Convolutions. To help us olve

$$ay'' + by' + cy = g(t), \qquad y(0) = y'(0) = 0$$

with the Laplace transform

$$Y(s) = \frac{1}{as^2+bs+c}G(s)$$

we have the following theorem:
Th Suppose that $H(s) = F(s)G(s)$ where $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$. Then

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

The right is called the convolution if f and g and denoted by $f * g(t)$.
Pf Multiplying together and changing variables

$$\begin{aligned} F(s)G(s) &= \int_0^\infty f(\sigma)e^{-s\sigma}\,d\sigma \int_0^\infty g(\tau)e^{-s\tau}\,d\tau = \int_0^\infty \int_0^\infty f(\sigma)g(\tau)e^{-s(\sigma+\tau)}\,d\sigma\,d\tau \\ &= \int_0^\infty \int_\tau^\infty f(t-\tau)g(\tau)\,dr\,e^{-st}\,dt = \int_0^\infty \int_0^t f(t-\tau)g(\tau)\,d\tau \;\; e^{-st}\,dt. \end{aligned}$$

Ex Find the solution to

$$y'' + y = tu_0(t), \qquad y(0) = y'(0) = 0$$

Using the above formula and integration by parts we get

$$y(t) = \int_0^t \sin(t-\tau)\tau\,d\tau = t - \sin(t).$$