Lecture 17: 6.3-4 Step functions and discontinuous forcing functions. The step function is defined to be

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

Ex 1 The function $h(t) = u_3(t) - u_5(t)$ is given by

$$h(t) = \begin{cases} 0, & t < 3, \\ 1, & 3 \leq t < 5, \\ 0, & t \geq 5. \end{cases}$$

Ex 2 The function

$$f(t) = \begin{cases} 0, & t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$$

can be written with step functions $f(t) = u_2(t) - 2u_3(t) + u_4(t)$.
Ex 3 For $c \geq 0$ we have

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st}\,dt = \int_c^\infty e^{-st}\,dt = -\frac{e^{-st}}{s}\Big|_{t=s}^{\infty} = \frac{e^{-cs}}{s}$$

We often want to consider the truncated translate of a function $f(t)$ defined to be

$$u_c(t)f(t-c) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c \end{cases}$$

The reason we truncated be 0 for $t \leq c$ is that the Laplace transform only used the values of $f(t)$ for $t \geq 0$.
Th1

$$\mathcal{L}\{u_t(t)f(t-c)\} = e^{-cs}F(s), \qquad \text{where} \quad F(s) = \mathcal{L}\{f(t)\}$$

Pf See book.

We proved last time **Th2**

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \qquad \text{where} \quad F(s) = \mathcal{L}\{f(t)\}$$

Ex 3 Solve

$$y'' + y = u_1(t) - u_2(t), \qquad y(0) = y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and recall the formula $\mathcal{L}\{y''(t)\} = Y(s) - sy(0) - y'(0)$. Taking the Laplace transform of both sides of the equation gives

$$s^2Y(s) + Y(s) = \frac{e^{-s} - e^{-2s}}{s}$$

or

$$Y(s) = \frac{1}{(s^2+1)s}\left(e^{-s} - e^{-2s}\right)$$

Using partial fractions

$$\frac{1}{(s^2+1)s} = \frac{As+B}{s^2+1} + \frac{C}{s} = \frac{(As+B)s + C(s^2+1)}{(s^2+1)s}$$

gives $A + C = 0$, $B = 0$ and $C = 1$ so $A = -1$. Hence

$$Y(s) = \frac{1}{s}\left(e^{-s} - e^{-2s}\right) - \frac{s}{s^2+1}\left(e^{-s} - e^{-2s}\right)$$

Also recalling that

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

we get

$$y(t) = u_1(t) - u_2(t) - u_1(t)\cos(t-1) + u_2(t)\cos(t-2)$$

Ex 4 Solve

$$y'' + 4y = g(t), \qquad y(0) = y'(0) = 0$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le t < 10, \\ 1, & t \ge 10 \end{cases}$$

We can write

$$g(t) = \frac{t-5}{5}u_5(t) - \frac{t-10}{5}u_{10}(t)$$

Note that $g(t) = u_5(t)f(t-5)/5 - u_{10}(t)f(t-10)/5$ where $f(t) = t$ and since

$$\mathcal{L}\{t\} = -\frac{d}{ds}\mathcal{L}\{1\} = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}.$$

we have

$$\mathcal{L}\{g(t)\} = \frac{1}{5}\frac{e^{-5s}}{s^2} - \frac{1}{5}\frac{e^{-10s}}{s^2}$$

Hence with $Y(s) = \mathcal{L}\{y(t)\}$ we have

$$(s^2+4)Y(s) = \frac{1}{5}\frac{e^{-5s}}{s^2} - \frac{1}{5}\frac{e^{-10s}}{s^2}$$

For rest see example 2 in section 6.4 in the book.