Lecture 15: 6.1-6.2 The Laplace transform.

Def The **Laplace transform** of the function $f(t)$ is the function $F(s)$ given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt, \tag{6.1.1}$$

if f is (piece wise) continuous and the limit exist. That f is piecewise continuous means that f is continuous apart from that it can jump at a finite number of points.

Ex 0 The integral below is convergent if $\alpha > 1$ and divergent if $\alpha \le 1$. As $T \to \infty$:

$$\int_0^T \frac{dt}{(1+t)^\alpha} = \frac{1}{1-\alpha}\frac{1}{(1+t)^{\alpha-1}}\Big|_0^T = \frac{1}{1-\alpha}\Big(\frac{1}{(1+T)^{\alpha-1}} - 1\Big) \to \begin{cases} \infty, & \text{if } \alpha < 1 \\ \frac{1}{\alpha-1}, & \text{if } \alpha > 1 \end{cases}$$

The Laplace transform $\mathcal{L}: f(t) \to F(s)$ is invertible $\mathcal{L}^{-1}: F(s) \to f(t)$, i.e. there is only one function that has a given Laplace transform. We will see that one can use the Laplace transform to transform constant coefficient linear differential equations into algebraic equations, involving the characteristic polynomial, that one can easily solve and then inverse transform to get the solution of the differential equation.

Ex 1 If $f(t) = e^{at}$ then the Laplace transform exist and is equal to

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \frac{1}{s-a}, \qquad \text{when } \ s > a$$

Pf As $T \to \infty$;

$$\int_0^T e^{(a-s)t}\, dt = \frac{1}{a-s} e^{(a-s)t}\Big|_{t=0}^T = \frac{1}{a-s} e^{(a-s)T} - \frac{1}{a-s} \to 0 - \frac{1}{a-s}, \quad \text{if } \ s > a.$$

The key to using the Laplace transform to solve differential equations is the formula:

Th 1 $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$, or $\displaystyle\int_0^\infty e^{-st} f'(t)\, dt = s\int_0^\infty e^{-st} f(t)\, dt - f(0)$

valid if the integral on the right is convergent and $\lim_{T\to\infty} e^{-sT} f(T) = 0$.
Pf Recall integration by parts (proved by $fg\big|_0^T = \int_0^T (fg)'\, dt = \int_0^T (fg' + f'g)\, dt$.)

$$\int_0^T f(t) g'(t) = f(t)g(t)\Big|_{t=0}^T - \int_0^T f'(t) g(t)\, dt \tag{6.1.2}$$

$\mathcal{L}\{f'(t)\}$ is the limit as $T \to \infty$ of

$$\int_0^T e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_{t=0}^T + s\int_0^T e^{-st} f(t) dt$$
$$= e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t) dt \to s\int_0^\infty e^{-st} f(t) dt - f(0), \qquad T \to \infty$$

Ex Find the solution of $y' + 2y = 0$, $y(0) = 3$ using the Laplace transform.
Sol Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation using the formula in Th 1 we get :

$$\mathcal{L}\{y'(t) + 2y(t)\} = \mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = sY(s) - y(0) + 2Y(s) = 0$$

and since $y(0) = 3$ we get

$$Y(s) = \frac{3}{s+2}$$

We can find the function that has this as its Laplace transform in Ex 1 with $a = -2$ by taking out the factor 3:

$$y(t) = 3e^{-2t}$$

which we know is the solution.
The formula for the second order derivative is:

Th 2 $$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Pf The proof resembles the proof of Th 2 but use integration by parts (6.1.2) twice.

Ex Solve the equations $y'' - y' - 2y = 0$,with initial conditions $y(0) = 1$, $y'(0) = 0$.
Sol Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation using the formulas in Th 1-2 we get:

$$0 = \mathcal{L}\{y''(t) - y'(t) - 2y(t)\} = \mathcal{L}\{y''(t)\} - \mathcal{L}\{y'(t)\} - 2\mathcal{L}\{y(t)\}$$
$$= s^2Y(s) - sy(0) - y'(0) - \big(sY(s) - y(0)\big) - 2Y(s) = (s^2 - s - 2)Y(s) - (s-1)y(0) - y'(0)$$

Hence we also use the initial conditions and factorize the denominator:

$$Y(s) = \frac{(s-1)y(0) + y'(0)}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

We have now found the Laplace transform $Y(s)$ of the solution, but what function $y(t)$ has this as its Laplace transform? Using partial fractions we can write

(6.1.3) $$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

for some constants A and B to be determined. Writing the right hand side on a common denominator we get

$$\frac{A}{s-2} + \frac{B}{s+1} = \frac{A(s+1) + B(s-2)}{(s-2)(s+1)} = \frac{(A+B)s + A - 2B}{(s-2)(s+1)}$$

Hence $A+B=1$ and $A-2B=-1$, and subtracting the second equation from the first gives $3B=2$ so $B=2/3$ and plugging this into the first equation gives $A=1/3$. Hence

$$Y(s) = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$$

By Ex. 1 this is the Laplace transform of

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$