Lecture 8: Section 3.4. We are now about to take derivatives of a vector field. Since the vector field have three components and each depends on three variables we are faced with trying to interpret the meaning of nine different derivatives. However, it turns out that certain combinations of these derivatives have a clear geometric and physical meaning. One is the **divergence** of a vector field which is a scalar field and the other is the **curl** of a vector field which is a vector field. The divergence tells us to what extent the field is spreading the particles out, "diverging" (Ex. 1). The curl tells us how the vector field "swirls" particles around (Ex. 2)

The divergence is defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial y}, \qquad \text{if} \qquad \mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \tag{3}$$

Ex. 3 If $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ then $\operatorname{div} \mathbf{F} = 2$. **Ex. 4** If $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ then $\operatorname{div} \mathbf{F} = 0$.

From these two examples it appears that the description of divergence as how much the flow lines go apart is correct. However, there is more to it:

Ex. 5 Let $\mathbf{F} = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$. The $\operatorname{div} \mathbf{F} = 0$.

The flow lines in Ex. 5 are the same as in Ex. 1 since only the magnitude of the vector field changed, i.e., by Ex. 1, lines that go out from the origin. The magnitude of the vector field in Ex. 5 however decreases as we go out, and that compensates for that the flow lines go apart, and it makes the divergence smaller.

Physical interpretation of the divergence. The divergence of the velocity vector field of a fluid is the rate of expansion of the fluid per unit volume. If $\mathbf{V}$ is the velocity vector field of the fluid and $\mathbf{R}$ is the position of a fluid particle then

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}(\mathbf{R}) \tag{4}$$

If we follow the positions of all fluid particles within a small domain we get a domain $\mathcal{D}_t$ depending on time t. The rate of change of the volume $\operatorname{Vol}(\mathcal{D}_t)$ of this domain is

$$\frac{d \operatorname{Vol}(\mathcal{D}_t)}{dt} = \operatorname{Vol}(\mathcal{D}_t)\ \operatorname{div} \mathbf{V}, \tag{5}$$

i.e. the divergence is the **rate of expansion** of the fluid volume per unit volume. An incompressible liquid is divergence free $\operatorname{div} \mathbf{V} = 0$ whereas a gas is compressible and the divergence is nonvanishing.

If the fluid expands then the average density must decrease and fluid must flow out of any fixed region. If we instead consider the amount of fluid $\mathcal{M}_t$ in a small fixed domain $\mathcal{D}$ and define the **flow rate density** by

$$\mathbf{F} = \mu \mathbf{V}, \tag{6}$$

where μ is the density, then the **rate of increase** of the amount of fluid in $\mathcal{D}$ is

$$\frac{d\mathcal{M}_t}{dt} = -\operatorname{Vol}(\mathcal{D})\ \operatorname{div} \mathbf{F} \tag{7}$$

The rate of change of the amount of fluid in the domain is equal to the amount of fluid that goes out through the surface $\mathcal{S}$ of the domain per unit time.

Flux of a vector field. The amount of fluid that goes out through the surface $\mathcal{S}$ per unit time is is called the **flux** or **rate of flow** of the vector field $\mathbf{F}$ through $\mathcal{S}$. We will calculate the flux later using surface integrals and here we only give a brief description how it can be calculated. If $\triangle S$ is a small area of a piece of a plane with outward unit normal $\mathbf{n}$ then we claim that the flow rate out of $\triangle S$ is given by

$$\mathbf{F}\cdot\mathbf{n}\ \triangle S \tag{8}$$

In fact, in a small time $\triangle t$, the fluid particles that will reach $\triangle S$ are at most $\mathbf{V}\triangle t$ away, and all particles within reach form a sloped cylinder with $\triangle S$ as its base and height $\mathbf{V}\cdot\mathbf{n}\,\triangle t$. Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: $\mu\mathbf{V}\cdot\mathbf{n}\,\triangle t\triangle S$. (8) is the rate per unit time is. Hence

$$\text{Flux of } \mathbf{F} \text{ out through } \mathcal{S} = \text{Vol}(\mathcal{D})\ \text{div}\,\mathbf{F} \tag{9}$$

The material below only took half a lecture so I also did the Flow rate density and the Flux from the previous lecture.

We define the **curl** of a vector field $\mathbf{F}= F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ by

$$\mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\Big)\mathbf{i}+\Big(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\Big)\mathbf{j}+\Big(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\Big)\mathbf{k}.$$

One way to remember this formula is that it looks like a cross product:

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix}\mathbf{k},$$

or with the del or nabla notation:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z},$$

we symbolically write

$$\mathbf{curl}\,\mathbf{F} = \nabla\times\mathbf{F}.$$

In the same way we can symbolically write

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

and

$$\mathbf{grad}\ f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

Ex. 1 Find $\nabla\times\mathbf{F}$ if $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$. **Sol.**

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x & y \end{vmatrix}\mathbf{k} = \mathbf{0}.$$

Ex. 2 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -y & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ -y & x \end{vmatrix} \mathbf{k} = 2\mathbf{k}.$$

The curl tells us how the vector field "swirls" around. From section 3.2 we know that the flow lines for Ex. 1 are lines going out from the origin where the flow lines for Ex. 2 are circles around the origin. Ex. 2 represents the velocity vector field of a body rotating around the z axis at angular velocity 1. In fact, $\mathbf{R}(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j} + c\mathbf{k}$ represents the rotation of a particle at angular velocity 1 and $\mathbf{R}'(t) = \mathbf{F}(\mathbf{R}(t))$, if $\mathbf{F}$ is the vector field in Ex. 2. Curl is a vector; the magnitude tells us how much it curls and the direction tells us the axis around which it curls. From these examples we might suspect that curl is how much the flow lines curves around, but there is more to it as we shall see.

Ex. 3 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = (-y\mathbf{i} + x\mathbf{k})/(x^2 + y^2)$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]\mathbf{k} = \ldots = \mathbf{0}$$

It is easy to check that for Ex. 3 the flow lines are still circles around the z-axis (in fact the same as Ex. 2.) It appears that our description of the curl as how much the flow lines curves around was not quite sufficient. A more accurate description is that curl is how much the fluid swirls around at a microscopic level at each point. Curl is how much a small paddle wheel "swirls" around its own axis. In Ex. 2 when the paddle wheel have made a complete rotation around the z-axis it has made a complete rotation around its own axis. However, in Ex. 3 when the paddle wheel has made a complete rotation around the z-axis it has in fact not rotated around its own axis. In the first case if you are sitting on the paddle wheel facing away from the origin (or z-axis) you are going to face away from the origin the complete rotation around the origin so you will have made a turn as well. However, in the second case you are during the complete rotation around the origin facing in a fixed direction. The explanation for this is that in the second example the velocity vector field gets stronger as we get closer to the origin so the side of the paddle wheel close to the origin will have larger angular velocity. Consider the following example.

Ex. 4 Find $\nabla \times \mathbf{F}$ if $\mathbf{F} = F_3(y)\mathbf{k}$. **Sol.**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & F_3 \end{vmatrix} = \frac{\partial F_3}{\partial y}\mathbf{i}$$

The curl can hence be non-vanishing even though in this case the flow lines are straight lines parallel to the z-axis. The magnitude of the curl then is simply the derivative of F_3 in the y direction and that curl is non-vanishing simply means that the velocity vector field is larger on one side of the paddle wheel then on the other.

A vector field $\mathbf{F}$ is called **irrotational** if $\nabla\times\mathbf{F}=\mathbf{0}$. A gradient $\mathbf{F}=\nabla f$ is irrotational

$$\nabla \times \nabla f = 0 \tag{1}$$

i.e. the curl of the gradient of any scalar field is zero. f is called a **potential**. The converse is also true locally, i.e. an irrotational vector field is a gradient. A vector field $\mathbf{F}$ is called **divergence free** if $\nabla\cdot\mathbf{F}=0$. A curl $\mathbf{F}=\nabla\times\mathbf{G}$ is divergence free;

$$\nabla \cdot (\nabla \times \mathbf{G}) = 0 \tag{2}$$

i.e. the divergence of the curl of any vector field is zero. The converse is also true locally, i.e. a solenoidal vector field is a curl. The proofs of (1) and (2) use the equality of mixed partial derivatives. They should formally be compared to the identities for the cross product and dot products of vectors $\mathbf{v}\times\mathbf{v}=\mathbf{0}$ and $\mathbf{v}\cdot\big(\mathbf{v}\times\mathbf{w}\big)=0$.

Another operator that shows up a lot in the applications is the **Laplacian**

$$\triangle f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

If a vector field $\mathbf{F}$ is irrotational and divergence free then $\mathbf{F}=\nabla f$, where f satisfy

$$\triangle f = 0,$$

The water in the bath tub is typically both divergence free and irrotational except for in the center of the drain.