Lecture 7: 2.6 The implicit function theorem. A surface can be described as a graph:

$$z = f(x, y)$$

or as a level surface

$$F(x, y, z) = C$$

It is clear that a graph can always be written as a level surface with $F(x, y, z) = z - f(x, y)$. The question is if a level surface can always be written as graph, i.e can we always solve uniquely the equation $F(x, y, z) = 0$ for z as a function of (x, y). The example of a sphere

$$F(x, y, z) = x^2 + y^2 + z^2 = C$$

shows that this not the case since for $C > 0$ we have two possible values

$$z = \pm\sqrt{C - x^2 - y^2},$$

corresponding to the northern and southern hemisphere. Hence in this case we need two different functions to describe parts of the surface as graphs. Its easy imagine surface were you would need several functions. However, for the sphere, even with two functions its not quite good since the slope or derivative would be infinite on the equator $x^2 + y^2 = C$. To solve this problem one can instead close to the equator write the surface as graph of y over the $x - z$ plane, or x over the $y - z$ plane:

$$y = \pm\sqrt{C - x^2 - z^2}, \qquad x = \pm\sqrt{C - y^2 - z^2}.$$

Then the question is if in general and equation of the form $F(x, y, z) = 0$ close to any point with $F(x_0, y_0, z_0) = 0$ can always be written as graph over one of the coordinate planes

$$z = f(x, y), \qquad y = g(x, z), \quad \text{or} \quad x = h(y, z)$$

In the example above we see that if $C = 0$ the only solution to $F(x, y, z) = x^2 + y^2 + z^2 = 0$ is $(x, y, z) = (0, 0, 0)$ which is not a surface but only a point so it is not always true. However it is true if

$$\nabla F(x_0, y_0, z_0) = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)) \neq (0, 0, 0).$$

Since if the gradient is non vanishing then one of the derivatives e.g. F_z is non vanishing, this is a consequence of the implicit function theorem:

Implicit Function Theorem Suppose that $F(x_0, y_0, z_0) = 0$ and $F_z(x_0, y_0, z_0) \neq 0$. Then there is function $f(x, y)$ and a neighborhood U of (x_0, y_0, z_0) such that for $(x, y, z) \in U$ the equation $F(x, y, z) = 0$ is equivalent to $z = f(x, y)$.

Ex A special case is $F(x, y, z) = f(x, y) - az = 0$. It is clear that we need $F_z = a \neq 0$ in order to solve for z as a function of (x, y).

A related theorem is:

Inverse Function Theorem Let $F : \mathbf{R}^n \to \mathbf{R}^n$. Suppose that $F(\mathbf{x}_0) = \mathbf{y}_0$ and $\det\big(\mathbf{D}F(\mathbf{x}_0)\big) \neq 0$. Then there is a function G and a neighborhood U of (x_0, y_0) such that for $(\mathbf{x}, \mathbf{y}) \in U$ the equation $F(\mathbf{x}) = \mathbf{y}$ is equivalent to $\mathbf{x} = G(\mathbf{y})$

Ex A special case of this theorem is $f : \mathbf{R} \to \mathbf{R}$, in which case its easy to see from the graph that the equation $f(x) = y$ is invertible if $f'(x) \neq 0$.

If $f(x) = x^2 = y$ then $f'(0) = 0$. For $y < 0$ there are no solutions and two for $y > 0$.

If $f(x) = x^3 = y$ is invertible even though the condition in the theorem fails.

Idea of Proof If $F(\mathbf{x}) = A\mathbf{x}$ where A is $n \times n$ matrix then the equation $A\mathbf{x} = \mathbf{y}$ is invertible if and only if A is an invertible matrix, i.e. if $\det A \neq 0$.

Sections 3.1 Paths, 3.2 Arc length. (The material below only took half a lecture so in the future I might move it and do it together with path integrals in section 7.1).

If $\mathbf{c}$ is a vector valued function

$$\mathbf{c}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \qquad a \le t \le b$$

then we defined the **derivative** to be

$$\mathbf{c}'(t) = \lim_{\triangle t \to 0} \frac{\mathbf{c}(t+\triangle t) - \mathbf{c}(t)}{\triangle t} = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}.$$

We define the **arc length** of the curve to be

$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\, dt$$

However, since

$$|\mathbf{c}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

this can also be written:

$$L = \int_a^b |\mathbf{c}'(t)|\, dt$$

It is easy to see that it is the arc length if we approximate it by a Riemann sum:

$$L \sim \sum_{i=0}^{n-1} |\,\mathbf{c}'(t_i)|\triangle t$$

where $t_i = a + i\triangle t$, $\triangle t = (b-a)/n$ and use the definition of derivative:

$$\mathbf{c}'(t_i) = \lim_{h \to 0} \frac{\mathbf{c}(t_i + h) - \mathbf{c}(t_i)}{h} \sim \frac{\mathbf{c}(t_i + \triangle t) - \mathbf{c}(t_i)}{\triangle t},$$

if $\triangle t$ is small, we obtain

$$L \sim \sum_{i=0}^{n-1} |\,\mathbf{c}(t_i + \triangle t) - \mathbf{c}(t_i)|.$$

This is exactly the length of the polygon consisting of the line segments between the vertices $\mathbf{c}(t_i)$, $i = 0, ..., n$, which is a good approximation of the arc length.

Ex. Find the arc length of the helix $\mathbf{c}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$, when $0 \le t \le 2\pi$.
Sol. We have $\mathbf{c}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, so $|\mathbf{c}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. Hence $L = \int_0^{2\pi} |\,\mathbf{c}'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, 2\pi$.

The same curve can be represented by different **parameterizations**:
Ex. Find the arc length of the helix: $\mathbf{c}_2(u) = \cos u^2 \mathbf{i} + \sin u^2 \mathbf{j} + u^2 \mathbf{k}$, $0 \le u \le \sqrt{2\pi}$.
Sol. We have $\mathbf{c}_2'(u) = -2u \sin u^2\,\mathbf{i} + 2u\cos u^2\,\mathbf{j} + 2u\mathbf{k}$. Hence $|\mathbf{c}_2'(u)| = \sqrt{4u^2 \sin^2 u^2 + 4u^2 \cos^2 u^2 + 4u^2} = 2\sqrt{2}\, u$ and

$$L = \int_0^{\sqrt{2\pi}} |\mathbf{c}_2'(u)|\, du = \int_0^{\sqrt{2\pi}} 2\sqrt{2} u\, du = \begin{bmatrix} u^2 = t, \\ 2u\, du = dt \end{bmatrix} = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

Two different parameterizations, $\mathbf{c}(t) = \mathbf{c}_2(u)$, where $u = u(t)$, leads to the same arc length. By the chain rule $\mathbf{c}'(t) = d\mathbf{c}_2(u)/dt = \mathbf{c}_2'(u)u'(t)$ and if we change variables

$$\int |\,\mathbf{c}_2'(u)|\, du = \begin{bmatrix} u = u(t), \\ du = u'(t)\, dt \end{bmatrix} = \int |\,\mathbf{c}'(t)|\, dt.$$

Section 3.3. A **Vector field** $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ is a map that to each point (x, y, z) in space assigns a vector $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$. Graphically, a vector field is illustrated by, from a few points in space $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ drawing the arrow representing the vector $\mathbf{F}(\mathbf{R})$. There are many physical examples of vector fields, e.g. the gravitational field at each point in space by Newton's law tells a mass in which direction to accelerate and how much. Another example is the velocity vector field of air flow: Sand particles moving in a storm or dust moving in the room, where the vector field at a point in space $\mathbf{F}(\mathbf{R})$ gives the velocity of the particles at the point $\mathbf{R}$. Similarly, the velocity vector field of a fluid gives the velocity of the fluid (particle) at each point in space.
A **Flow line** for a vector field $\mathbf{F}$ is a path $\mathbf{R}(t)$ such that at each point along it

$$\mathbf{R}'(t) = \beta \mathbf{F}\big(\mathbf{R}(t)\big), \qquad \beta > 0 \tag{1}$$

The flow lines for the velocity vector field of a fluid are just the curves along which the fluid particles travel. From the graphic illustration of a vector field one can approximately draw the flow lines, by going in the direction of the vector field. If β is a positive scalar field then the flow lines for $\beta\mathbf{F}$ are the same as the flow lines for $\mathbf{F}$ its just that we travel along the path at different speed, so we can reduce to the case $\beta = 1$ by a change of parameter along the curve. The flow lines can not intersect; since the vector field gives the direction there is a unique flow line through each point. In fact through each initial point there is a unique solutions to the system of differential equations (1):

$$\frac{dx}{dt} = \beta F_1, \qquad \frac{dy}{dt} = \beta F_2, \qquad \frac{dz}{d}t = \beta F_3,$$

In order to solve this system one can formally eliminate βdt and think of one of the other variables as the parameter along the curve:

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} \tag{2}$$

Ex. 1 Find the flow lines for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.
Sol. $dx/x = dy/y = dz/0$. The last equation should be interpreted as $dz = 0$ so $z = C_1$. The second equation $dx/x = dy/y$ gives $\ln x = \ln y - C$ so if we exponentiate $y = xe^C = C_2 x$. The flow lines are therefore half lines going out from the z-axis
Ex. 2 Find the flow lines for the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.
Sol. $-dx/y = dy/x = dz/0$. Hence $dz = 0$ so $z = c_1$. Also $-dx/y = dy/x$ so $-xdx = ydy$ and $-x^2/2 = y^2/2 - c$, i.e. $x^2 + y^2 = c_2$. The flow lines are circles around the z-axis. One can check that the parametrization for the circle $x = r\cos t$, $y = r\sin t$, $z = c$ satisfies the equations of the flow line $dx/dt = -y$, $dy/dt = x$, $dz/dt = 0$.