Lecture 4: 2.2 Limits.

Def. Suppose that $\mathbf{f} : \mathbf{R}^n \setminus \{\mathbf{a}\} \to \mathbf{R}^m$. We say that the limit of $\mathbf{f}(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is $\mathbf{L}$ and write

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{f}(\mathbf{x})-\mathbf{L}\| < \varepsilon$ whenever $0 < \|\mathbf{x}-\mathbf{a}\| < \delta$.

Def A function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is called **continuous** at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

Ex The function defined by $f(x) = 1$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$ is not continuous at 0.

Ex Let $f(x, y) = 3x - 5y$, show that $\lim\limits_{(x,y)\to(1,-1)} f(x, y) = 8$ using the definition. Note that $f(1, -1) = 8$. We need to show that for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$, depending on ϵ, such that for $\||(x, y) - (1, -1)\| = \sqrt{(x-1)^2 + (y+1)^2} < \delta$ we have $|f(x, y) - f(1, -1)| < \epsilon$. We start by assuming that we have such a δ and see what condition δ must satisfy. It then follows that $|x - 1| < \delta$ and $|y + 1| < \delta$. Hence by the triangle inequality $|f(x, y) - f(1, -1)| = |3(x - 1) - 5(y + 1)| \leq 3|x - 1| + 5|y - 1| < 8\delta \leq \epsilon$, if $\delta = \epsilon/8$.

Topology.

Def The set $B_r(\mathbf{a}) = \{\mathbf{x} \in \mathbf{R}^n;\ \|\mathbf{x} - \mathbf{a}\| < r\}$ is called an open ball of radius $r > 0$ centered at $\mathbf{a}$.

Def A set D is called open if for every $\mathbf{x} \in D$ there is an open ball centered at $\mathbf{x}$ contained in D, $B_r(\mathbf{x}) \subset D$.

Ex The interval $0 < x < 2$ is an open subset of the real line, whereas the intervals $0 \leq x < 2$ and $0 \leq x \leq 2$ are not since there is no open ball, i.e. open interval of the form $|x - a| < r$ centered at $a = 0$ which is contained in $0 \leq x < 2$.

2.3 Differentiation.

Given $f : \mathbf{R}^2 \to \mathbf{R}$ The **partial derivative** of f with respect x is defined by

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

if it exist, i.e. its the derivative of $f(x,y)$ with respect to x when y if fixed. The partial derivatives $\partial f/\partial y$ is defined similarly and the extension to functions of n variables is analogous.

What is the meaning of the derivative of a function $y = f(x)$ of one variable?
It is the slope of the tangent line $y = f(a) + f'(a)(x-a)$ and the tangent line is the linear function that best approximates $f(x)$ when x is close to a.
The fact that the derivative of a function of one variable exist means that

$$\frac{f(x) - f(a)}{x-a} - f'(a) = \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} \to 0, \qquad \text{as} \quad x \to a,$$

i.e. when x is close to a the distance between $f(x)$ and the tangent line is much smaller than the distance between x and a.

The **linear approximation** of $f(x,y)$ at (a,b) is

$$h(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

A function $f(x,y)$ is called **differentiable** at (a,b) if the partial derivatives exist and if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - h(x,y)}{\|(x,y) - (a,b)\|} = 0$$

The definition of differentiability is motivated by the idea that the **tangent plane** to the surface $z = f(x,y)$ at (a,b), i.e. the plane

$$z = h(x,y)$$

should give a good approximation to the function f close to (a,b). The definition means that the distance between $f(x,y)$ and the tangent plane $h(x,y)$ is much smaller than the distance between (x,y) and (a,b).

Even if the partial derivatives exist a function need not be differentiable:
Ex The function $f(x,y) = ||x| - |y|| - |x| - |y|$ is not differentiable at the origin $(0,0)$ even though the partial derivatives exist.