

**Section 1.7: Cylindrical and Spherical Coordinates.** Recall that in the plane it is sometimes useful to introduce polar coordinates. There are two possible natural and useful generalizations of this to space:

**Cylindrical coordinates**  $(r, \theta, z)$  of a point  $P(x, y, z)$  are obtained by using polar coordinates  $(r, \theta)$  of the projection in the  $x$ - $y$  plane and leaving  $z$  unchanged:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

To convert from rectangular to cylindrical coordinate we use:

$$r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z.$$

Note that the surfaces  $r = c$  are cylinders of all points at distance  $c$  from the  $z$ -axis. The surfaces  $\theta = c$  are half planes from the  $z$ -axis at an angle  $c$  with the  $x$ - $z$  plane. The surface  $z = c$  are the planes parallel to and at a distance  $c$  from the  $x$ - $y$  plane.

**Spherical coordinates**  $(\rho, \theta, \phi)$  of a point  $P(x, y, z)$  are obtained by first using polar coordinates of the projection in the  $x$ - $y$  plane. We use the angle  $\theta$  in the  $x$ - $y$  plane but instead of the distance  $r$  to the  $z$  axis use the distance to the origin  $\rho$ :

$$\rho^2 = x^2 + y^2 + z^2$$

Then  $\rho^2 = r^2 + z^2$  and we introduce another angle  $\phi$  to describe the height above the  $x$ - $y$  coordinate plane:  $z = \rho \cos \phi$  and  $r = \rho \sin \phi$ . We obtain the coordinates:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

where we must have

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

Note that the surfaces  $\rho = c$  are spheres of radius  $c$ .

The surfaces  $\theta = c$  are half planes from the  $z$ -axis at an angle  $c$  with the  $x$ - $z$  plane.

The surfaces  $\phi = c$  are half cones at an angle  $c$  with the positive  $z$ -axis.

**Ex.** Write down the equations in both cylindrical and spherical coordinates.

(a)  $z^2 = x^2 + y^2, z \geq 0$ , (b)  $z^2 = x^2 + y^2 + 1$ , (c)  $x^2 + y^2 + z^2 = 4$ .

**Sol.** (a)  $z^2 = r^2$  in cylindrical coordinates and  $\phi = \pi/4$  in spherical coordinates.

(b)  $z^2 = r^2 + 1$  in cylindrical coordinates and  $\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi + 1$  in spherical.

(c)  $z^2 + r^2 = 4$  in cylindrical coordinates and  $\rho^2 = 4$  in spherical coordinates.

**Ex.** Write down in cylindrical and spherical coordinates the inequalities for the solid region  $\{(x, y, z); x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$ .

**Sol.** In cylindrical coordinates:  $\{(r, \theta, z); 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq z \leq \sqrt{1 - r^2}\}$  and in spherical coordinates it is  $\{(\rho, \theta, \phi); 0 \leq \rho \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/2\}$ .

## 2.1 Functions and graphs.

A function  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  send each point  $\mathbf{x} \in \mathbf{R}^n$  to a specific point  $f(\mathbf{x}) \in \mathbf{R}^m$ .

If  $m = 1$  it is called a scalar valued function and if  $m > 1$  a vector valued function.

**Ex** The temperature at each point in space is a scalar function (or field)  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ .

**Ex** The velocity of a fluid is a vector that depends on the point in space as well as the time so it is a vector function (or field)  $f : \mathbf{R}^4 \rightarrow \mathbf{R}$ .

The **graph** of a function  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  is the set of all points  $(x, y, z)$  where  $z = f(x, y)$ .

A **level set** of a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is a set of all points  $(x_1, \dots, x_n)$  such that  $f(x_1, \dots, x_n) = c$ . If  $n = 2$  it is called a level curve  $f(x, y) = c$  and if  $n = 3$  it is called a level surface  $f(x, y, z) = 0$ .

**Ex** Draw the level curve  $f(x, y) = x^2 + y^2 = 1$ .

**Sol** The level curve is a circle.

**Ex** Draw the level surface  $f(x, y, z) = x^2 + y^2 = 1$ .

**Sol** The level surface is a cylinder since  $z$  is arbitrary.

A **section** of a graph of a function  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  is obtained by intersecting the graph with vertical planes, e.g. setting  $x = 0$  produces the section  $z = f(0, y)$  which is the graph of a function of one variable.

One can also think of a level set of a function  $f(x, y) = c$  raised to the graph  $f(x, y) = c$  and  $z = c$  as the intersection of the graph with the horizontal planes  $z = c$ .

Level sets and sections are useful tools to sketch a graph of a function  $z = f(x, y)$ .

**Ex** Sketch the elliptic paraboloid  $z = x^2 + y^2$ .

**Sol** (1) Draw the intersection with the  $y - z$  plane when  $x = 0$ ,  $z = y^2$ .

(2) Draw the level curves  $z = 1, 2, 3, 4$  and raise them to the graph.

(2b) Alternatively, draw the intersection with the  $x - z$  plane when  $y = 0$   $z = x^2$ .

**Ex** Sketch the hyperbolic paraboloid (or saddle)  $z = y^2 - x^2$ .

**Sol** One draws the intersection with the  $y - z$  plane, when  $x = 0$  then  $z = y^2$ , and the intersection with the  $x - z$  plane, when  $y = 0$  then  $z = -x^2$ .

## 2.2 Limits.

**Ex** Let  $f(x) = \begin{cases} x, & x > 0, \\ 0, & x < 0 \end{cases}$

What is the limit of  $f(x)$  as  $x \rightarrow 0$ ?  $\lim_{x \rightarrow 0} f(x) = 0!$

**Ex** Let  $f(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$

What is the limit of  $f(x)$  as  $x \rightarrow 0$ ?  $f(x)$  does not have a limit as  $x \rightarrow 0$  since its different from the different sides.

**Intuitive definition of limit** Suppose that  $\mathbf{f} : \mathbf{R}^n \rightarrow \mathbf{R}^m$ . The equation

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

means that we can make  $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\|$  arbitrarily small (i.e. near zero) by keeping  $\|\mathbf{x} - \mathbf{a}\|$  sufficiently small (but not zero). The precise statement is: For every  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$  whenever  $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ .

**Ex.** Does  $f(x, y) = \frac{x^2}{x^2 + y^2}$  have a limit when  $(x, y) \rightarrow (0, 0)$ ?

**Sol.** Note that  $\lim_{x \rightarrow 0} f(x, 0) = 1$  and  $\lim_{y \rightarrow 0} f(0, y) = 0$ . Hence the limit of  $f(x, y)$  is different in different directions. Therefore the limit does not exist.

**Def** A function  $\mathbf{f} : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is called **continuous** at  $\mathbf{a}$  if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

**Ex** The function defined by  $f(x) = 1$  for  $x \geq 0$  and  $f(x) = 0$  for  $x < 0$  is not continuous at 0.