Lecture 20: 7.3 Stokes' theorem. Let S be a surface with unit normal $\mathbf{n}$ and positively oriented boundary C, i.e. if you walk in the direction of the curve on the side of the normal then the surface should be on your left. **Stokes' theorem** says

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

if $\mathbf{F}$ is a smooth vector field on S.
If S is a domain in the x-y plane then Stoke's theorem reduces to Green's theorem. In fact $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{s} = \int_C P dx + Q dy$, if $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}$ and $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$, if $\mathbf{n}=\mathbf{k}$.

Ex. Find the integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$ and the orientation of C corresponds to a counterclockwise motion in the x-y plane.
Sol. Let $\mathbf{F} = -y^3\mathbf{i}+x^3\mathbf{j}-z^3\mathbf{k}$. The integral is by Stokes theorem equal to the surface integral of $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}$ over some surface S with the boundary C and with unit normal positively oriented with respect to the orientation of the boundary. We have $\mathbf{curl}\,\mathbf{F} = ... = (3x^2+3y^2)\mathbf{k}$. We take S to be the region in the plane $h(x,y,z) = x+y+z = 1$ with boundary C. A unit normal to S is given by $\mathbf{n} = \nabla h/|\nabla h| = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$ and it has the correct orientation since $\mathbf{n}\cdot\mathbf{k} = 1/\sqrt{3} > 0$.

We therefore get

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S 3(x^2+y^2)/\sqrt{3}\,dS$$

Writing $dS = dxdy/|\mathbf{n}\cdot\mathbf{k}| = \sqrt{3}dxdy$ we get

$$\iint_{x^2+y^2\le 1} 3(x^2+y^2)\,dxdy = \int_0^{2\pi}\int_0^1 3r^2\,rdr\,d\theta = \int_0^{2\pi} \frac{3}{4}r^4\Big|_0^1 d\theta = 2\pi\frac{3}{4} = \frac{3\pi}{2}$$

Sol. 2. Directly calculating the line integral. Parameterizing the curve C we can write $x=\cos t$, $y=\sin t$ and $z = 1-x-y = 1-\cos t-\sin t$, $0\le t\le 2\pi$ and write

$$\int_C -y^3dx + x^3dy - z^3dz = \int_0^{2\pi}\Big(-y^3\frac{dx}{dt} + x^3\frac{dy}{dt} - z^3\frac{dz}{dt}\Big)\,dt$$
$$= \int_0^{2\pi}\big(\sin^4 t + \cos^4 t + (1-\cos t-\sin t)^3(\sin t-\cos t)\,dt = \text{...a lot more work}$$

Interpretation of curl. Furthermore, Stokes Theorem can alternatively be used to define the curl: The component of $\mathbf{curl}\,\mathbf{F}$ in the direction of a unit vector $\mathbf{n}$ is defined to be the limit as $\varepsilon \to 0$ of the line integral of $\mathbf{F}$ around a small circle C_ε of radius ε perpendicular to $\mathbf{n}$, divided by the area of the disc S_ε enclosed by C_ε:

$$\int_{C_\varepsilon} \mathbf{F}\cdot d\mathbf{s} = \iint_{S_\varepsilon} \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\ \ \text{Area}(S_\varepsilon)$$

where $\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}$ is evaluated at some point on S_ε. It follows that

$$\mathbf{curl}\,\mathbf{F}\cdot\mathbf{n} = \lim_{\varepsilon\to 0} \frac{\int_{C_\varepsilon} \mathbf{F}\cdot d\mathbf{s}}{\text{Area}(S_\varepsilon)}$$

Ex. Show that $\int_C ye^z dx + xe^z dy + xye^z dz = 0$ for a closed curve C.
Sol. $\mathbf{F} = \nabla(xye^z)$ so $\mathbf{curl}\,\mathbf{F} = 0$ and by Stokes's theorem the integral vanishes.

Ex. Find $\int_{C_a} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F} = (-y\mathbf{i}+x\mathbf{j})/(x^2+y^2)$ and C_a is the circle $x^2+y^2 = a^2$ in the x-y plane going counterclockwise.
Sol. $\mathbf{curl}\,\mathbf{F} = ... = 0$. Hence one would have thought that by Stokes theorem the line integral would vanish. **Wrong!** because $\mathbf{F}$ is not continuous.

However, if we parameterize $x = a\cos t$ and $y = a\sin t, 0 \le t < 2\pi$, we get

$$\int_{C_a} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \Big(\frac{-y}{x^2+y^2}\frac{dx}{dt} + \frac{x}{x^2+y^2}\frac{dy}{dt}\Big)\,dt$$
$$= \int_0^{2\pi} \Big(\frac{-a\sin t(-a\sin t)}{a^2} + \frac{a\cos t(a\cos t)}{a^2}\Big)\,dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi$$

The reason Stokes' theorem failed to hold in this case was that the vector field $\mathbf{F}$ is singular when $(x,y) = (0,0)$, i.e. along the z-axis.

Proof of Stokes' theorem for a graph. We have seen that Stokes' theorem for a surface S in the $x-y$ plane reduces to Green's theorem. We will now show that Stokes' theorem for a surface S that can be written as a graph $z = f(x, y)$, $(x, y) \in D$, over a region D in the plane, also reduces to Green's theorem. If $\mathbf{T}$ is the tangent vector to the boundary curve C the Stokes' theorem can be written:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \mathbf{curl}\,\mathbf{F} \cdot \mathbf{n}\, dS$$

where ds is the arc length and dS the surface area element. The surface integral can then be written as an integral over D and the integral over the boundary curve can be written as an integral over the projection of the curve in the $x-y$ plane. Then one can use Green's theorem in the plane to show that these things are equal.

Since the surface can be written $h(x, y, z) = z - f(x, y)$ a normal is given by $\mathbf{N} = \nabla h = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}$ and the unit normal is given by $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$. The surface measure is $dS = dxdy/|\mathbf{k} \cdot \mathbf{n}|$, where $\mathbf{k}\cdot\mathbf{n} = \mathbf{k}\cdot\mathbf{N}/|\mathbf{N}| = 1/|\mathbf{N}|$, so $dS = |\mathbf{N}|\, dxdy$ and

$$\iint_S \mathbf{G} \cdot \mathbf{n}\, dS = \iint_D -G_1 f_x - G_2 f_y + G_3\, dxdy, \qquad \text{if} \quad \mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$$

If we apply to $\mathbf{F}$ this to $\mathbf{G} = \mathbf{curl}\,\mathbf{F}$ we get

$$\iint_S \mathbf{curl}\,\mathbf{F}\cdot\mathbf{n}\, dS = \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, dxdy.$$

If we parameterize the boundary $x = x(t)$, $y = y(t)$ and $z = f(x, y)$ we have

by the chain rule, and $$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt},$$

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b \Big(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\Big)\, dt = \int_a^b \Big(\big(F_1 + f_x F_3\big)\frac{dx}{dt} + \big(F_2 + f_y F_3\big)\frac{dy}{dt}\Big)\, dt$$

This can now be considered as a line integral in the plane:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D} P dx + Q\, dy, \qquad \text{where}$$
$$P(x, y) = F_1(x, y, f(x, y)) + f_x(x, y) F_3(x, y, f(x, y)),$$
$$Q(x, y) = F_2(x, y, f(x, y)) + f_y(x, y) F_3(x, y, f(x, y))$$

We can therefore apply Greens formula in the plane.

$$\frac{\partial P}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} f_y + f_x \frac{\partial F_3}{\partial y} + f_x f_y \frac{\partial F_3}{\partial z} + f_{xy} F_3$$
$$\frac{\partial Q}{\partial x} = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} f_x + f_y \frac{\partial F_3}{\partial x} + f_x f_y \frac{\partial F_3}{\partial z} + f_{xy} F_3$$

so by Green's theorem

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D} P dx + Q\, dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dxdy$$
$$= \iint_D -\Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big) f_x - \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big) f_y + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, dxdy,$$

Some more calculations of surface integrals.
Ex. Let S be the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ with $0 \le z \le 1$.
A parametrization of the surface is given by
$\mathbf{X}(u, v) = (\cos u - v \sin u)\mathbf{i} + (\sin u + v \cos u)\mathbf{j} + v\mathbf{k}, \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 1.$
a) Find the area element dS expressed in terms of the parametrization $du\,dv$.
b) Find the surface integral $\iint_S z\,dS$.

Sol. a) $\mathbf{X}_u = (-\sin u - v\cos u)\mathbf{i} + (\cos u - v \sin u)\mathbf{j}$ and $\mathbf{X}_v = -\sin u\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$;

$$\mathbf{X}_u \times \mathbf{X}_v = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u - v\cos u & \cos u - v\sin u & 0 \\ -\sin u & \cos u & 1 \end{bmatrix} = (\cos u - v\sin u)\mathbf{i} + (\sin u + v\cos u)\mathbf{j} - v\mathbf{k}$$

Hence $dS = |\mathbf{X}_u \times \mathbf{X}_v|\, dudv = \sqrt{1 + 2v^2}\, dudv$.

$$b)\ \iint_S z\,dS = \int_0^1 \int_0^{2\pi} v\sqrt{1+2v^2}\, dudv = 2\pi \int_0^1 v\sqrt{1+2v^2}\, dv = \pi \frac{(1+2v^2)^{3/2}}{3}\Bigg|_0^1 = \pi \frac{3^{3/2}-1}{3}$$