Lecture 2: Derivation of the geometric formulas for the dot and cross products from the algebraic expressions. Let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$$

Here we take the first equality as the definition and we must prove the second equality. We have

$$\|\mathbf{a}-\mathbf{b}\|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\,\mathbf{a}\cdot\mathbf{b}$$

Let $\mathbf{a}$ and $\mathbf{b}$ start from the origin and consider the triangle with these as two of its edges. The third edge is then given by the vector $\mathbf{a}-\mathbf{b}$. By the law of cosines:

$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$$

Comparing the two expressions gives the geometric formula for the dot product. (Actually, we do not need to use the full strength of the law of cosine but only the Phytagorean theorem which is the case $\cos\theta = 0$ above, to conclude that $\mathbf{a}\cdot\mathbf{b} = 0$ if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. It then follows that $\mathbf{b} - P_{\mathbf{a}}\mathbf{b}$ is perpendicular to $\mathbf{a}$, if $P_{\mathbf{a}}\mathbf{b} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}/\|\mathbf{a}\|^2$. Hence $P_{\mathbf{a}}\mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$. Then $\|P_{\mathbf{a}}\mathbf{b}\| = |\mathbf{a}\cdot\mathbf{b}|/\|\mathbf{a}\|$. But the length of the projection is $\|\mathbf{b}\|\cos\theta$.)

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \le \theta \le \pi$), then we want to prove that

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta.$$

First we note that it follows (after a lot of work) from the algebraic expression for the cross product that

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2$$

that follows from expanding out both sides. Since $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$ it follows that

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2(1-\cos^2\theta) = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\sin^2\theta.$$

Section 1.5 Equations of planes. $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where (x_0, y_0, z_0) is a point in the plane and $\mathbf{n} = (a, b, c)$ is normal to the plane. In fact, if (x, y, z) is a point in the plane then the vector $(x-x_0, y-y_0, z-z_0)$ is in the plane so its perpendicular to the normal: $(x-x_0, y-y_0, z-z_0)\cdot(a, b, c) = 0$.

Ex Find the equation of a plane passing through $(1,0,0)$, $(2,2,3)$,$(0,2,4)$.
Sol $\mathbf{a} = (2,2,3) - (1,0,0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = (0,2,4) - (1,0,0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ are parallel to the plane and a normal to the plane is given by

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{k}$$
$$= \big(2\cdot4 - 3\cdot2\big)\mathbf{i} - \big(1\cdot4 - 3\cdot(-1)\big)\mathbf{j} + \big(1\cdot2 - 2\cdot(-1)\big)\mathbf{k} = 2\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$$

Let $(x_0, y_0, z_0) = (1,0,0)$. Hence the equation of the plane is $2(x-1) - 7y + 4z = 0$. Note that $\mathbf{a}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} = 0$ as it should be.
Parametric equations of a plane If (x_0, y_0, z_0) is a point in the plane and $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are two vectors in the plane then any point in the plane is given by $(x, y, z) = (x_0, y_0, z_0) + (a_1, a_2, a_3)\,t + (b_1, b_2, b_3)\,s$, $-\infty < s, t < \infty$.

1.6 Matrix algebra. An $m \times n$ matrix $A = [a_{ij}]$ is a collection of $m \cdot n$ numbers

$$\begin{array}{ccc} & j \text{ th } \textbf{column} & \\ A = & \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ & & \vdots & & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ & & \vdots & & \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} & i \text{ th } \textbf{row} \end{array}$$

A special case are the $1 \times n$ matrices which are row vector $(x_1, ..., x_n)$ or the $n \times 1$ matrices which are column vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Another important case are the square $n \times n$ matrices, e.g. a 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$ or a 3×3 matrix $\begin{bmatrix} 1 & 2 & 7 \\ 5 & 4 & 1 \\ 3 & 9 & 2 \end{bmatrix}$.
The **matrix multiplication** of an $m \times n$ matrix $A = [a_{ij}]$ by an $n \times k$ matrix $B = [b_{ij}]$ is the $m \times k$ matrix $C = [c_{ij}]$ whose entries are given by $c_{ij} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj}$. The c_{ij} entry is the dot product between the i th row of A and the j th column of B

$$\begin{array}{cc} & \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} \\ \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} & \begin{bmatrix} c_{ij} \end{bmatrix} \end{array}$$

Any $m \times n$ matrix A determines a **mapping** $\mathbf{R}^n \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^m$ defined by

$$A\mathbf{x} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \vdots & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + & \dots & + a_{1n}x_n \\ & \vdots & \\ a_{m1}x_1 + & \dots & + a_{mn}x_n \end{bmatrix}.$$

The map is seen to be linear:

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}, \qquad A(\alpha\mathbf{x}) = \alpha A\mathbf{x}$$

Conversely, a linear map is exactly one given by matrix multiplication.

If B is an $n \times k$ matrix then multiplication first by B and then A

$$\mathbf{x} \xrightarrow{\text{multiply by } B} B\mathbf{x} \xrightarrow{\text{multiply by } A} A(B\mathbf{x})$$

defines a map $\mathbf{R}^k \ni \mathbf{x} \to A(B\mathbf{x}) \in \mathbf{R}^m$. The matrix product AB is constructed so that multiplying by the matrix AB

$$\mathbf{x} \xrightarrow{\quad\text{multiply by } AB\quad} (AB)\mathbf{x}$$

is the same as first multiplying by B and then by A, i.e. $(AB)\mathbf{x} = A(B\mathbf{x})$.

Let us conclude the discussion by some examples.
Ex 1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ **rotates** vectors an angle $\pi/2$ counterclockwise.
Ex 2 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ **scales** vectors by a factor 3.
Ex 3 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ 3x_1 \end{bmatrix}$ scales and rotates vectors.

For square matrices we can define an inverse as follows. First let the identity matrix I be the $n \times n$ matrix with $1's$ in the diagonal and $0's$ elsewhere, e.g. if $n = 3$, then $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Multiplying by the identity matrix doesn't change anything $I\mathbf{X} = \mathbf{X}$.

Now, given an $n \times n$ matrix A, then a matrix B such that $AB = I$ and $BA = I$ is called an inverse of A and denoted by $A^{-1} = B$, if it exist. The multiplication by the inverse A^{-1} then corresponds to the inverse operation of multiplying by A and $A^{-1}A = I$ says that the composition of the two operations does not change anything. However, the inverse might not exist. If $n = 1$ then A is just a number a and if $a = 0$ then there is no number b such that $ab = 1$. However, even for matrices A with nonzero elements the inverse might not exist. E.g. in two dimension if $A = P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the projection onto the x_1 axis then (x_1, x_2) all gets maps to the same $(x_1, 0)$ so there is not a unique inverse point.